Time Series Analysis

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Fourier series

- Are series of cosine and sine terms.
- Representing general periodic functions.
- Solving problems of ordinary and partial differential equations.
- Remember that the periodic function (wave) is one of which the entire set of function values repeats itself at regular intervals, the time between successive repetitions begin called the periodic time.

Orthogonality

... are these functions orthogonal ?

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mt)\sin(nt)dt = 0$$

... YES, and these relations are valid for any interval of length 2π.

Now we know that this is an orthogonal basis, but how can we obtain the coefficients for the basis functions?

Fourier series

- What we would like to obtain?
  - A set of unknown coefficients $\boldsymbol{a}_{\mathbf{n}}$ and $\boldsymbol{b}_{\mathbf{n}}$ of the following infinite Fourier series of the function f(t):
  - Using the orthogonality given early, we can find the following integrals:

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin(nt)\right]$$

$$\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \int_{-\pi}^{\pi}\left[\frac{1}{2}a_o + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin(nt)\right]\right]\cdot\cos(nt)dt = \pi a_n$$

$$\int_{-\pi}^{\pi} f(t)\sin(nt)dt = \int_{-\pi}^{\pi}\left[\frac{1}{2}a_o + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin(nt)\right]\right]\cdot\sin(nt)dt = \pi b_n$$

Fourier series

- Immediately gives the unknown coefficients as

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \qquad (n = 0,1,.....)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \qquad (n = 1,2,.....)$$

Problem 2

- Let us find the Fourier series for the function defined by the equations

  F(t) = -t $-\pi < t \leq 0$

  F(t) = t $0 < t \leq \pi$

- **Solution**

$$a_0 = \pi$$

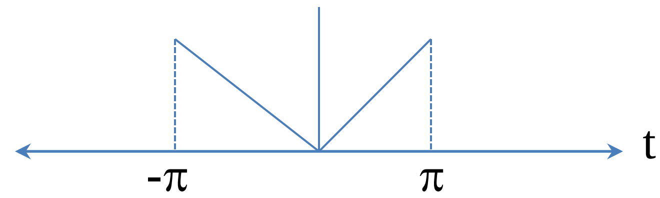
$$a_n = \frac{2}{\pi n}(\cos(n\pi - 1)$$

$$b_n = 0$$

S. of problem 2

f(t) = -t   -π < t ≤ 0

f(t) = t   0 < t ≤ π



- Solution

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} -tdt + \int_{0}^{\pi} tdt\right] =$$

$$= \frac{1}{\pi}\left(\left[\frac{1}{2}t^2\right]_{-\pi}^{0} + \left[\frac{1}{2}t^2\right]_{0}^{\pi}\right) = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\cos nt\cdot dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} t\cdot\cos nt\cdot dt + \int_{0}^{\pi} t\cdot\cos nt\cdot dt\right] =$$

$$\int_{0}^{\pi} t\cdot\cos nt\cdot dt = \left[t\cdot\sin nt\right]_0^{\pi} - \int_{0}^{\pi}\sin nt\cdot dt = \left[\frac{1}{n}\cos nt\right]_0^{\pi} = \cos n\pi - 1$$

$$b_n = 0$$

Note that: $\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) - (-\cos(x)) = x\sin(x) + \cos(x)$

Fourier transform

Complex Fourier Series

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left[a_n \cos(nt) + b_n \sin(nt)\right]$$

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2$$

$$\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$$

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left[a_n \frac{e^{\text{int}} + e^{-\text{int}}}{2} + b_n \frac{e^{\text{int}} - e^{-\text{int}}}{2i}\right]$$

Using 1/i=-i, then

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - ib_n}{2}\right)e^{\text{int}} + \left(\frac{a_n + ib_n}{2}\right)\right]e^{-\text{int}}$$

$$c_o = \frac{1}{2}a_o$$

$$c_n = \left(\frac{a_n - ib_n}{2}\right) \& \left(\frac{a_n + ib_n}{2}\right)$$

Complex Fourier Series

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{in\omega t}$$

$$c_n = \frac{1}{T} \int_{-\infty}^{\infty} f(t) e^{-in\omega t} dt$$

$$c_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-in\omega t} d\omega$$

$$f(t) = \int_{-\infty}^{\infty} c_n e^{-in\omega t} dt$$

$$|c_n| = \frac{\sqrt{a_n^2 + b_n^2}}{2} \quad \text{amplitude spectrum}$$

$$\phi = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \quad \text{phase spectrum}$$

The Fourier Transform Pair

$$\boxed{\begin{gathered} F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \\ f(t) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega \end{gathered}}$$

Forward transform

Inverse transform

Note the conventions concerning the sign of the exponents and the factor.

The Fourier Transform Pair

$$F(\omega) = R(\omega) + iI(\omega) = A(\omega)e^{i\Phi(\omega)}$$

$$A(\omega) = |F(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$$

$$\Phi(\omega) = \arg F(\omega) = \arctan \frac{I(\omega)}{R(\omega)}$$

$A(\omega)$ **Amplitude spectrum**

$\Phi(\omega)$ Phase spectrum

In most application it is the amplitude (or the power) spectrum that is of interest.

Discrete Fourier (series and transform)

Discrete Fourier Series

- For a series of N+1 discrete observations at equal spacing and with t=1, 2, …, N, the discrete summation formulas for the Fourier coefficients are:

$$a_o = \frac{1}{N}\sum_{j=1}^{N} f(t)$$

$$a_n = \frac{2}{N}\sum_{j=1}^{N} f(t)\cdot\cos(\frac{2n\pi}{N}j)$$

$$b_n = \frac{2}{N}\sum_{j=1}^{N} f(t)\cdot\sin(\frac{2n\pi}{N}j)$$

Discrete Fourier Transform

- The discrete Fourier transform is often abbreviated as **DFT** where the time series is f (1), f (2), ……, f (N) and N is the data points all at equal spacing[f (0)=f (N) for N+1 data points].

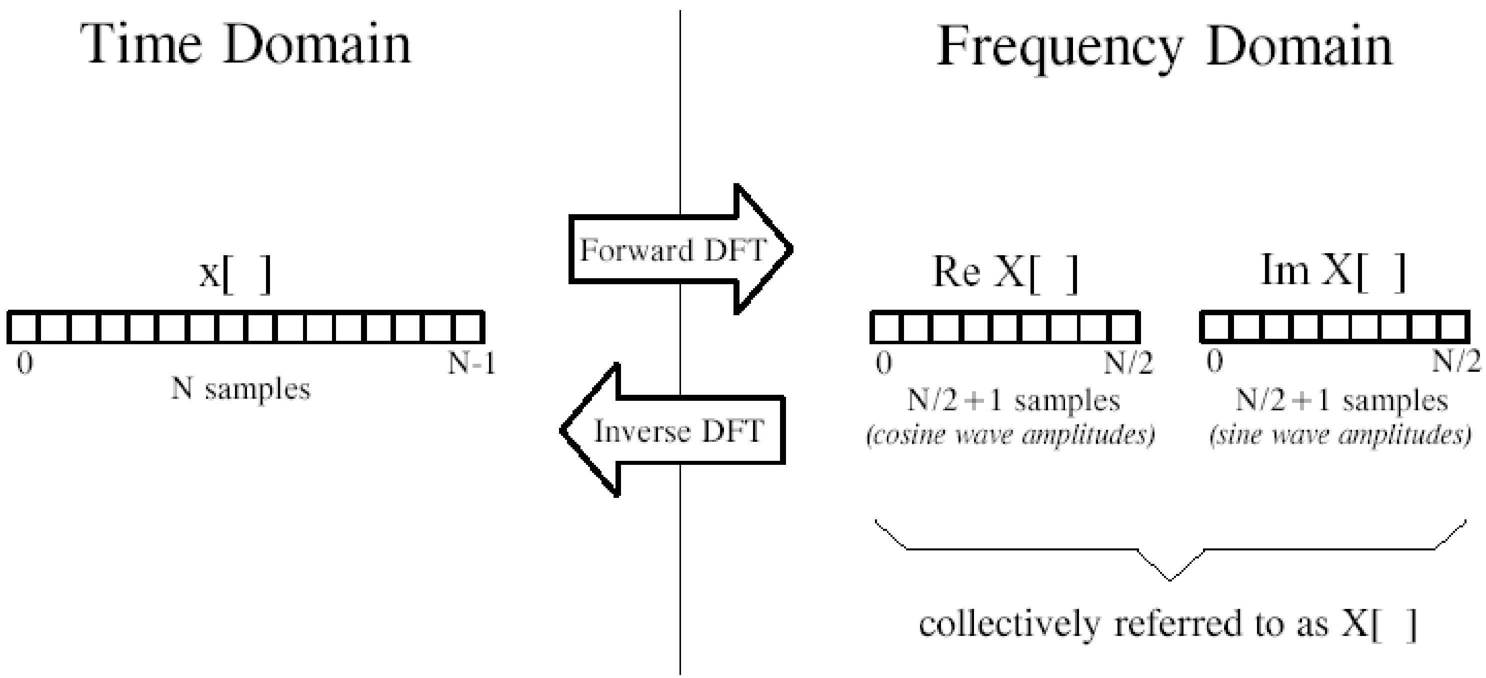
$$F(n) = \frac{1}{N}\sum_{j=1}^{N} f(t) \cdot e^{-i\left(\frac{2\pi n}{N} j\right)}$$

Note

$$\Delta t = \frac{1}{N_f \Delta f}$$

$$\Delta f = \frac{1}{N_t \Delta t}$$

DFT


Time Domain
Frequency Domain
Forward DFT
Inverse DFT
x[ ]
0
N-1
N samples
Re X[ ]
0
N/2
N/2 + 1 samples
(cosine wave amplitudes)
Im X[ ]
0
N/2
N/2 + 1 samples
(sine wave amplitudes)
collectively referred to as X[ ]

Exercise

- Compute the DFT for a cosine wave with At=1.0 seconds, N=32, period=16.0 seconds, amplitude=10, and phase=0.

  Note that the time series of cosine wave is given by

$$F(t) = A \cdot \cos(2\pi ft - \phi)$$

  Note that the original cosine wave is decomposed into two sine waves of a half amplitude having a positive and a negative frequencies as

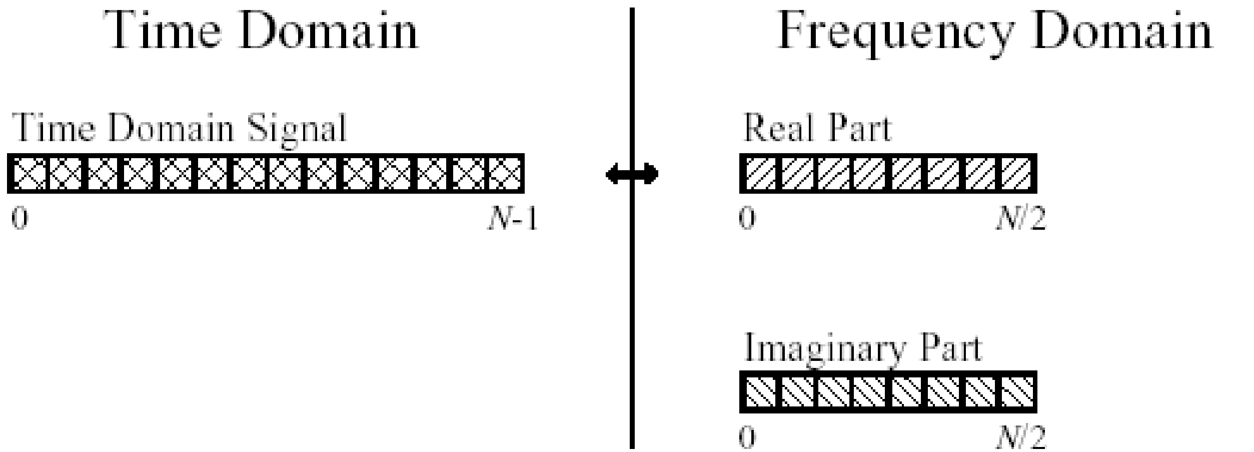
$$F(t) = \frac{A}{2}e^{i(2\pi ft)} + \frac{A}{2}e^{-i(2\pi ft)} = A \cdot \cos(2\pi ft - \phi)$$

Fast Fourier Transform (FFT)

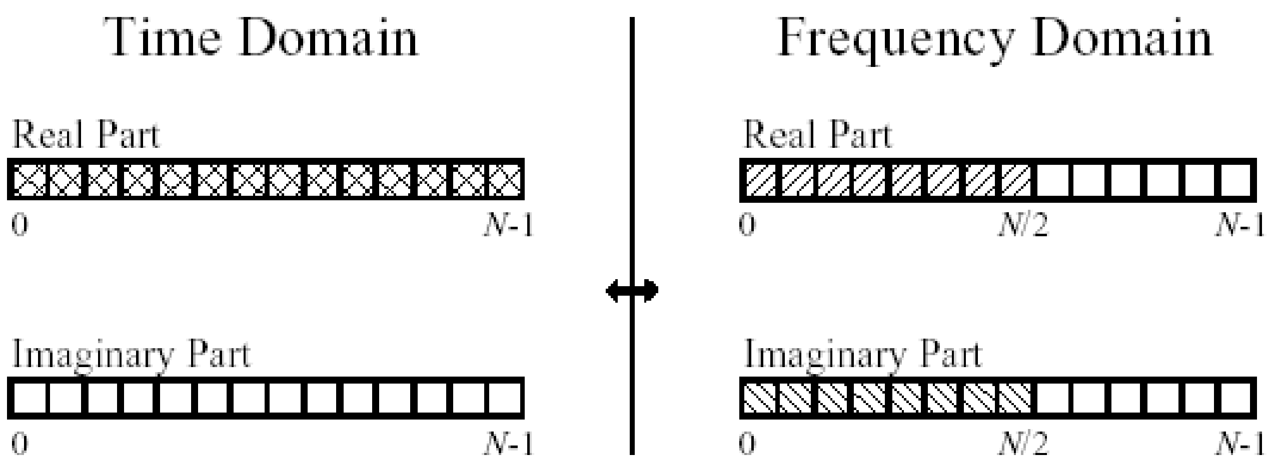
If number of data is = 2^n, where n is a positive integer number. Then we can use fast Fourier transform

FFT is incredibly more efficient, often reducing the computation time by *hundreds*

Real DFT

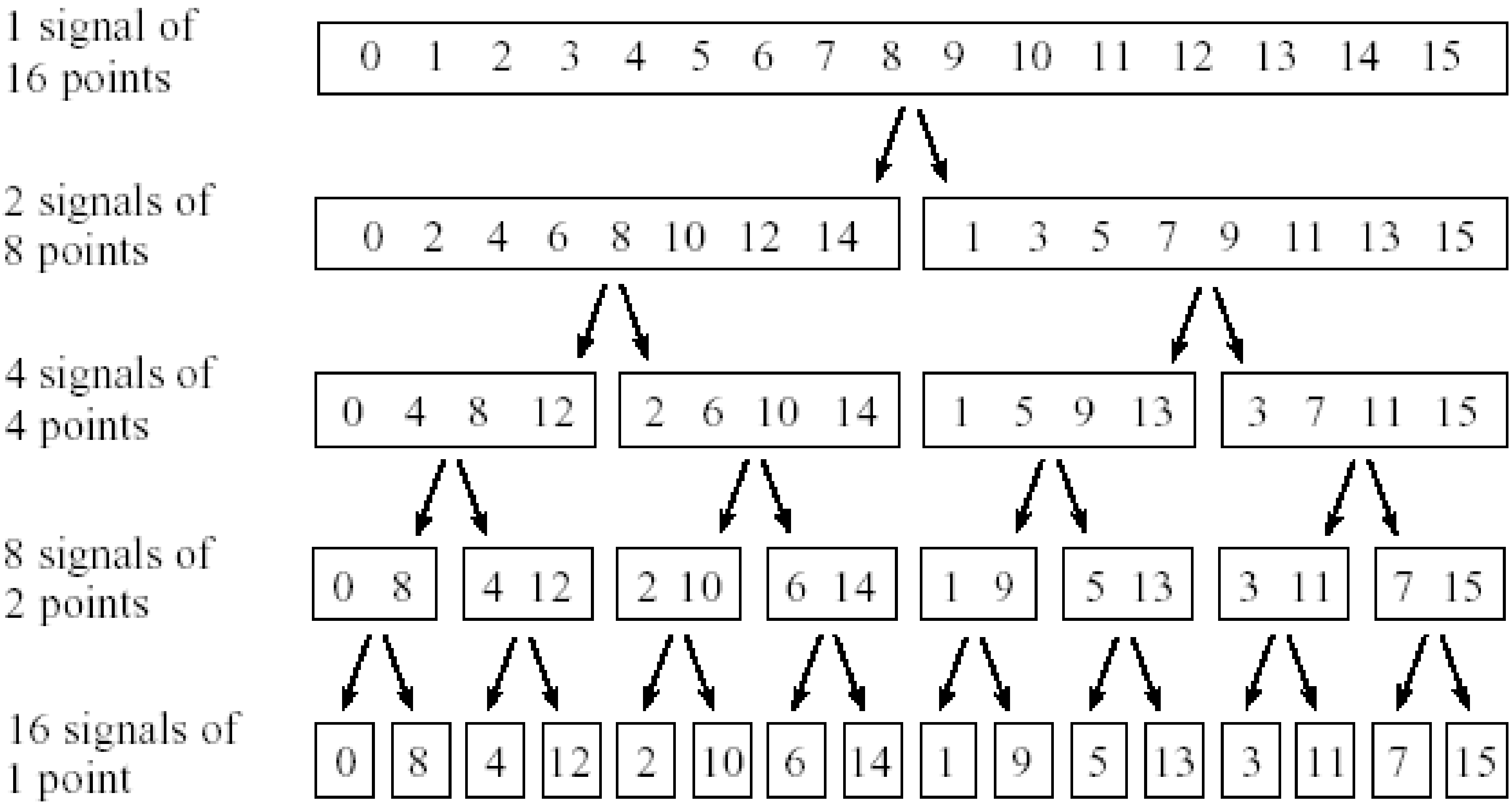


Complex DFT

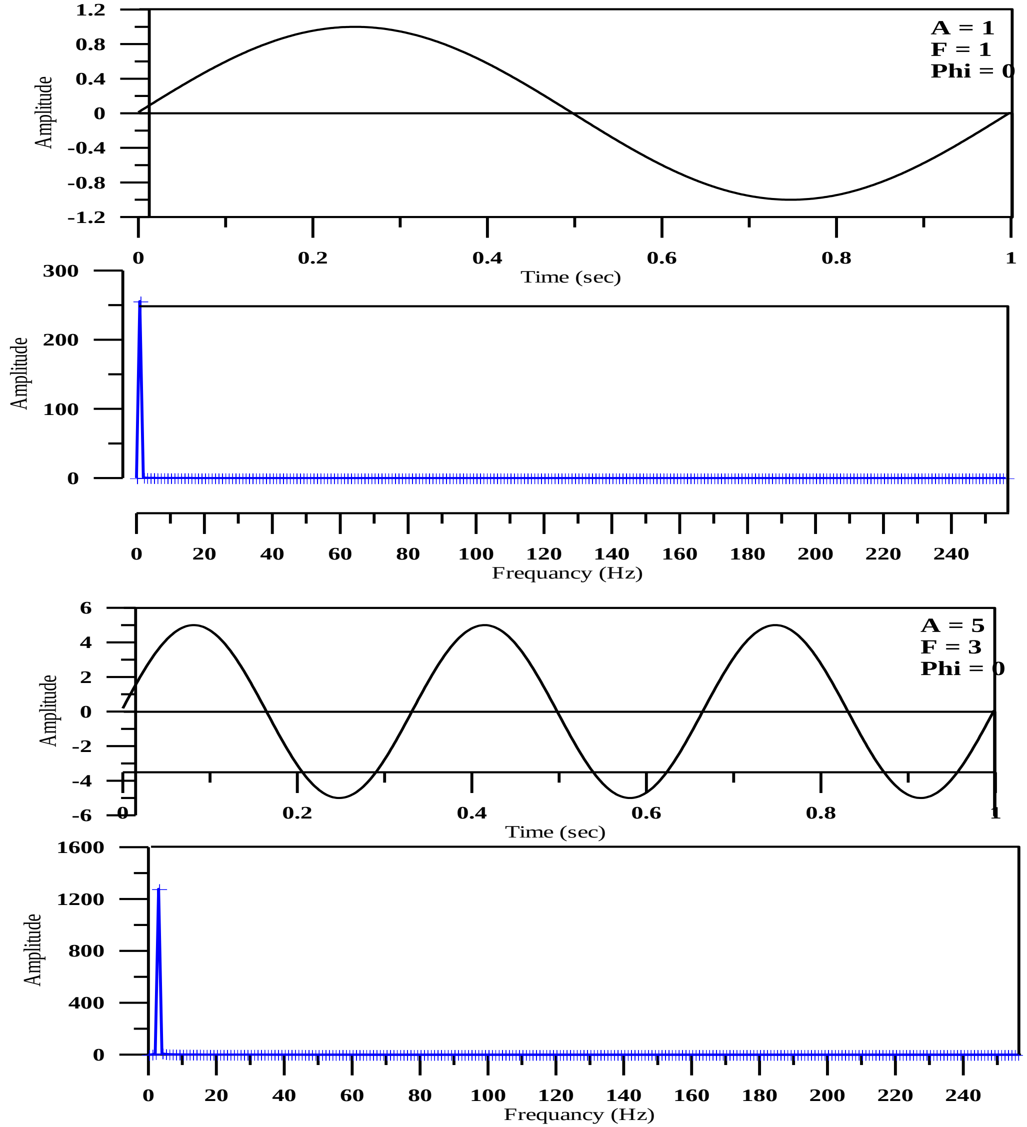


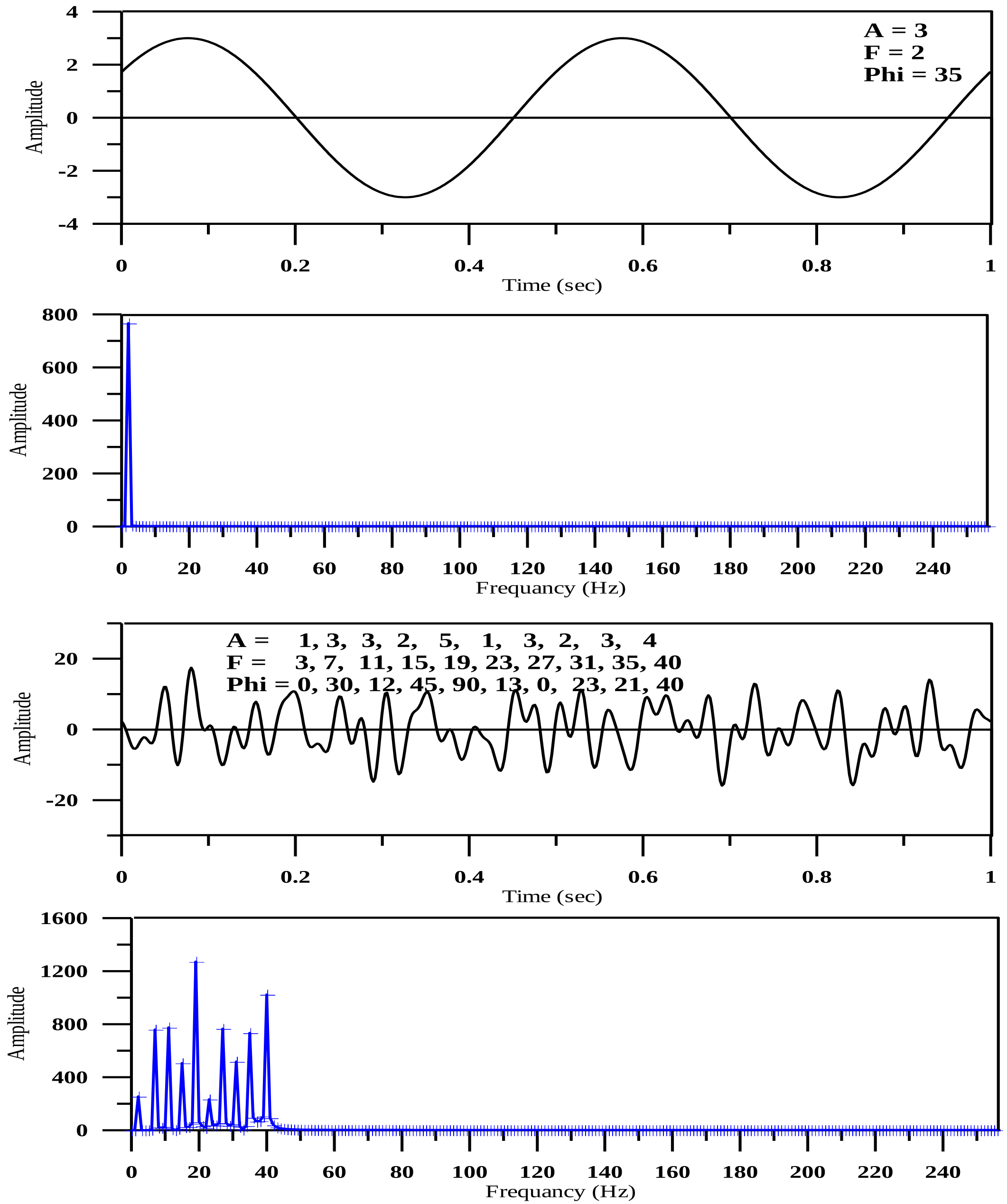
Practical solution of FFT

1. Transform the 1 signal of N points into N signals of 1 point

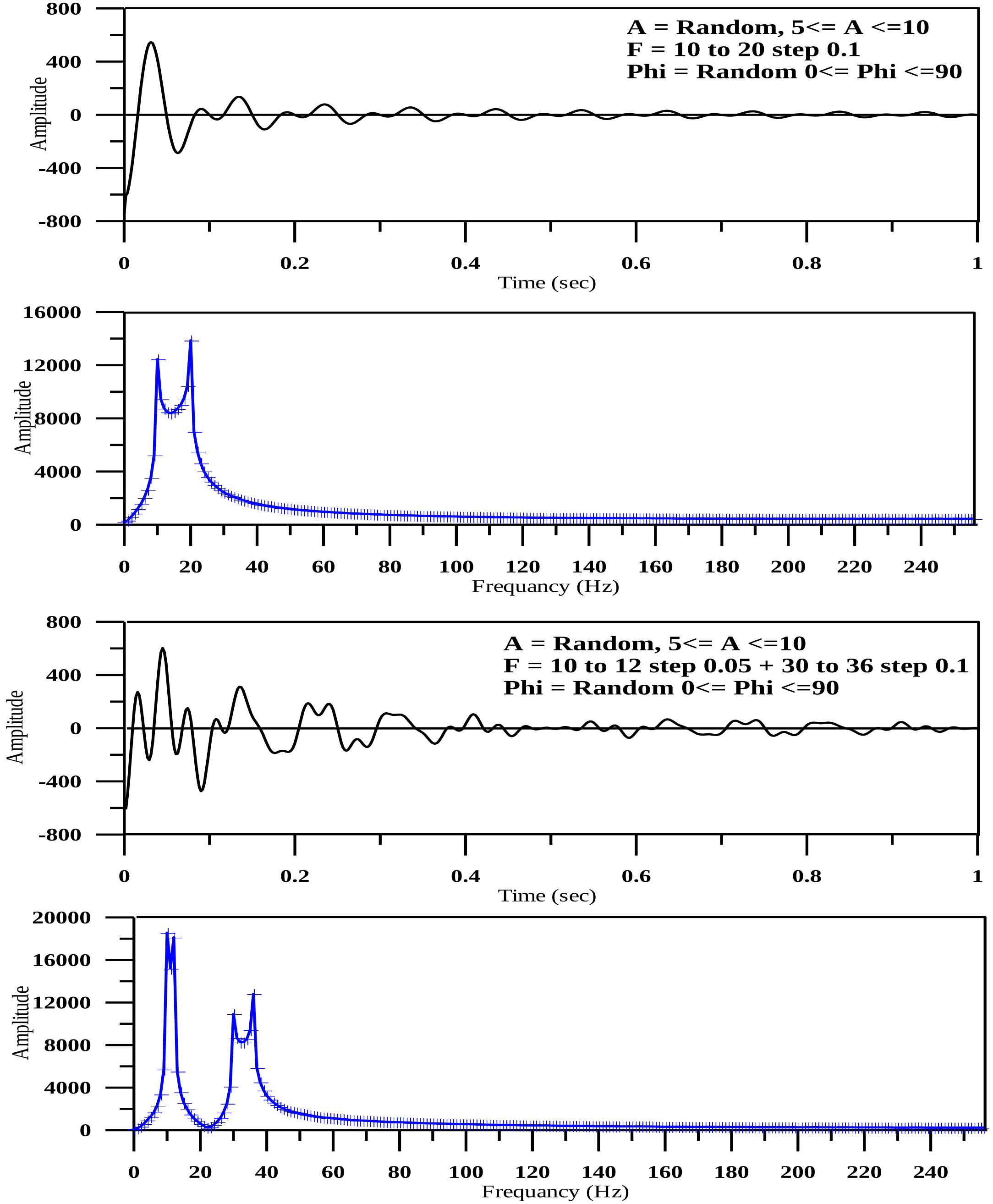


Synthetic Examples





A = 3
F = 2
Phi = 35
Amplitude
Time (sec)
Amplitude
Frequancy (Hz)
A = 1, 3, 3, 2, 5, 1, 3, 2, 3, 4
F = 3, 7, 11, 15, 19, 23, 27, 31, 35, 40
Phi = 0, 30, 12, 45, 90, 13, 0, 23, 21, 40
Amplitude
Time (sec)
Amplitude
Frequancy (Hz)


A = Random, 5<= A <=10
F = 10 to 20 step 0.1
Phi = Random 0<= Phi <=90
Amplitude
Time (sec)
Amplitude
Frequancy (Hz)
A = Random, 5<= A <=10
F = 10 to 12 step 0.05 + 30 to 36 step 0.1
Phi = Random 0<= Phi <=90
Amplitude
Time (sec)
Amplitude
Frequancy (Hz)