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The Approximation of Fourth Order Nonlinear Variational Inequalities

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ABSTRACT

In this paper we deal with existence and the approximation of the solution of the elliptic variational inequality from an abstract axiomatic point of view. We give an error estimate for the dierence of the two solutions in an appropriate norm. Also, we present some computational results.

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Introduction

An important and very useful class of nonlinear problems arising from mechanics, physics, engineering, etc. consists of the so-called variational in equalities. These are of two types, the elliptic variational inequalities and the parabolic variational inequalities (see, Duvant et al. and Lions [1,2]. In this paper we shall study the uniqueness, existence, and approximation of the solution of elliptic variational inequality, and also determine an error estimate for the dierence of the two solutions in an appropriate norm. Several authors have discussed the variational inequalities of elliptic type. In Brezis and Stampacchia have considered the elliptic variational inequality of the form:

Find $\phi \in K$ such that

$$(f - A\phi, \psi - \phi) \leq 0, \qquad \forall \ \psi \in K, \qquad (1)$$

where A is a monotone, semicontinuous operator from V to V' and K is a closed convex subset of V, a real Hilbert space, and $f \in V'$[3]. They have proved an abstract regularity theorem for this general problem and then apply it to some convex sets. In [4], Lions and Stampacchia have studied that if K is a closed convex set of V and $a(\phi, \psi)$ is coercive, that is

$$a(\phi, \phi) \geq \alpha \|\phi\|^2, \qquad \forall \ \phi \in V,$$

and some constant $\alpha > 0$, then the problem (1) has a unique solution. Some work on the approximation of the solutions of the elliptic variational inequalities has previously been done by Mosco and Aubin [5,6]. In Mosco has considered the approximation of the solutions of problem (1), taking into account perturbations of "a", the bilinear form, f, the element of V', and K, the closed convex set [5]. In Aubin also considered the approximation of the solutions of problem (1) [6].

The problem we consider lies in the following general setting. Let Ω be a bounded open domain in $\Re^n$ with boundary Γ Let V be the real Hilbert space with scalar product (.,.) and associated norm $\|\cdot\|$. Let V' be the dual of V, and the pairing between V and V' denoted by (.,.). We consider an operator $A : V = H_0^2(\Omega) \to V'$, which is linear, continuous and symmetric, and satisfies the coercivity condition, that is

$$(A\phi, \phi) = a(\phi, \phi) \geq \alpha \|\phi\|^2, \qquad \forall \ \phi \in V, \qquad (2)$$

for instance, we may take, $A = -\Delta.\Delta$. Let now consider the Sobolev space $H^2(\Omega)$ (which is the set of square integrable functions with square integrable rst derivatives) of real-valued function such that

$$\psi \in L^2(\Omega), \qquad \partial\psi/\partial x_i \in L^2(\Omega), \qquad i = 1, 2, ...n,$$

where $L^2(\Omega)$ is the space of (classes of) square summable function on Ω, and the derivative $\partial\psi/\partial x_i$ are taken in the sense of distributions on Ω. Dene

$$H_0^2(\Omega) = \{\psi | \psi \in H^2(\Omega), \psi = 0 \ \ on \ \ \Gamma\},$$

which is a closed vector subspace of $H^2(\Omega)$ and f an element of V'. Finally, let K be a convex subset of V and is dened as follows

$$K = \{\psi | \psi \ \ H_0^2(\Omega), \ \ \psi \geq 0 \ \ on \ \ \Omega\}. \qquad (3)$$

We want to approximate the solution to problem of the following:

The Continuous Problem

Consider the elliptic variational inequality of the following form

$$\phi \in K \quad (A\phi, \psi - \phi) + \mu(\phi, \psi - \phi) \geq (f, \psi - \phi), \quad \forall \ \psi \in K, \qquad (4)$$

where $\mu(> 0)$ is constant. The problem in which we are interested is finding the solution of (4) in the closed convex set K. In Butt has proved the existence and uniqueness of the solution of the penalized dierential equation of the form

$$A\phi_\epsilon + \mu\phi_\epsilon + 1/\epsilon \, \phi_\epsilon^- = f, \qquad \mu > 0,$$

(where $\phi_\epsilon^- = -sup(-\phi_\epsilon, 0))$, and showed that, its solution

ϕ_ϵ tends to the solution of the variational inequality (4) when tends to zero [7].. Now we shall prove the existence and uniqueness of the solution of (4) in the form of the following theorem:

Theorem A: Let A be a linear continuous and symmetric operator satisfying (2), and let K be a closed convex set of V. There exists a unique solution of (4). The map: $f \to \phi$ is continuous from V' 0 into V.

Proof: To prove the uniqueness of the problem, let ϕ and ψ be two possible solutions of (4); if we take $\psi = \varphi$ in (4) (respectively, $\varphi = \phi$ in the equation analogous to (4) with respect to ψ), we have

$$\phi \in K \quad (A\phi, \psi - \phi) + \mu(\phi, \psi - \phi) \geq (f, \psi - \phi), \quad \forall\ \psi \in K, \tag{5}$$

and

$$\phi \in K \quad (A\psi, \phi - \psi) + \mu(\psi, \phi - \psi) \geq (f, \phi - \psi), \quad \forall\ \psi \in K, \tag{6}$$

now, by adding (5) and (6), we obtain:

$$(A(\phi - \psi), \psi - \phi) + \mu(\phi - \psi, \psi - \phi) \geq 0,$$

or

$$(A(\psi - \phi), \psi - \phi) + \mu(\psi - \phi, \psi - \phi) \leq 0,$$

by using (2), we have

$$\alpha\|\psi - \phi\|^2 + \mu\|\psi - \phi\|^2 \leq (A(\psi - \phi), \psi - \phi) + \mu(\psi - \phi, \psi - \phi) \leq 0,$$

which implies that

$$C\|\psi - \phi\|^2 \leq 0,$$

where C is constant, and since $C > 0$, therefore, we obtain $\psi = \phi$. Let us denote $A \in L(V; V')$ the map dened, and by setting $\|A\| = M$ and $\|\mu\| = N$. Let Υ be the canonical isomorphism from V' onto V defined, for $f \in V'$, *by*

$$(f, \varphi) = ((\Upsilon f, \varphi)) \qquad for \quad \varphi \in V, \tag{7}$$

then

$$\|\Upsilon\|_{L(V,V')} = \|\Upsilon^{-1}\|_{L(V,V')} = 1.$$

Before proving the existence part of the problem, we prove a lemma.

Lemma A: Let λ be a number such that $0 < \lambda < \frac{2\gamma}{Q}$.

Then there exists θ with $0 < \theta < 1$ such that

$$|((\phi, \varphi)) - \lambda(A\phi, \varphi) + \mu(\phi, \varphi))| \leq \theta\|\phi\|\|\varphi\|, \qquad \forall\ \phi, \varphi \in V. \tag{8}$$

Proof: Infact

$$\begin{aligned} &|((\phi, \varphi)) - \lambda(A\phi, \varphi) + \mu(\phi, \varphi))| \\ &\qquad = |((\phi - \lambda\Upsilon(A\phi + \mu\phi), \varphi))| \leq \|\phi - \lambda\Upsilon(A\phi + \mu\phi)\|\|\varphi\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\phi - \lambda\Upsilon(A\phi + \mu\phi)\|^2 &= \|\phi\|^2 + \lambda^2\|\Upsilon(A\phi + \mu\phi)\|^2 - 2\lambda\Upsilon(A\phi + \mu\phi)\phi \\ &= \|\phi\|^2 + \lambda^2[\|\Upsilon A\phi\|^2 + \|\Upsilon\mu\phi\|^2 \\ &+ 2\mu(A\phi, \varphi)] - 2\lambda\Upsilon(A\phi + \mu\phi)\phi \\ &= \|\phi\|^2 + \lambda^2\|A\|^2\|\phi\|^2 + \lambda^2\|\mu\|^2\|\phi\|^2 \\ &+ 2\lambda^2\mu\alpha\|\phi\|^2 - 2\lambda(\alpha\|\phi\|^2 + \mu\|\phi\|^2) \\ &= (1 + \lambda^2(M^2 + N^2) + 2\lambda^2\mu\alpha - 2\lambda\alpha - 2\lambda\mu)\|\phi\|^2 \\ &= (1 + \lambda^2(M^2 + N^2 + 2\mu\alpha) - 2\lambda(\alpha + \mu))\|\phi\|^2 \\ &= (1 + \lambda^2 Q - 2\lambda\gamma)\|\phi\|^2, \end{aligned}$$

where $Q = (M^2 + N^2 + 2\mu\alpha)$, and $\gamma = (\alpha+\mu)$. Since $\gamma > 0$, we can choose λ such that

$$(1 + \lambda^2 Q - 2\lambda\gamma)^{1\backslash 2} < 1,$$

and hence, (8) follows.
Now we shall prove the existence part of the problem.
(i) Suppose that, $(A\phi, \varphi) = ((\phi, \varphi))$. Then the problem (4) means:

Find $\phi \in$ K such that

$$\begin{aligned} &((\phi, \varphi - \phi)) + \mu((\Upsilon\phi, \varphi - \phi)) \geq (f, \varphi - \phi) \\ &\qquad = ((\Upsilon f, \varphi - \phi)), \quad \forall \varphi \in K, \end{aligned} \tag{9}$$

that is

$$((\phi + \Upsilon\mu\phi - \Upsilon f, \varphi - \phi)) \geq 0.$$

If ϕ satisfies (9), then

$$\|\phi - \Upsilon(f - \mu\phi)\| \leq \|\varphi - \Upsilon(f - \mu\phi)\|,$$

and conversely. Since K is a closed convex set, therefore, there exists one and only one element of K which minimizes

$$\|\varphi - \Upsilon(f - \mu\phi)\|, \qquad \forall \quad \varphi \in K.$$

The solution of variational inequality (4) is thus given by

$$\phi = P_K\Upsilon(f - \mu\phi),$$

where P_K is the projection of V in K.

(ii) Consider the general case: Fix λ as in the above lemma A.

For $\phi \in$ V, define $\Phi(\phi) \in V'$ and setting

$$(\Phi(\phi), \varphi) = ((\phi, \varphi)) - \lambda[(A\phi, \varphi) + \mu(\phi, \varphi)] + \lambda(f, \varphi), \quad \forall \varphi \in V.$$

Then, for $\phi_1, \phi_2 \in V$, and by using the above lemma A, we have

$$\begin{aligned} &|(\Phi(\phi_1) - \Phi(\phi_2), \varphi)| = |((\phi_1 - \phi_2, \varphi)) - \lambda| \\ &\qquad [(A(\phi_1 - \phi_2), \varphi) + \mu(\phi_1 - \phi_2, \varphi)]| \end{aligned} \tag{10}$$

Then by (i), there exists a unique $\omega \in K$ such that

$$((\omega, \varphi - \omega)) + \mu((\omega, \varphi - \omega)) \geq (\Phi(\phi), \varphi - \omega), \qquad \varphi \in K,$$

and ω is given by

$$\omega = P_K\Upsilon\Phi(\phi) = T\phi, \tag{11}$$

the problem reduces to finding the fixed points of T. This defines a map $\phi \to T\,\phi$ from V into K. Moreover,

$$\|T\phi_1 - T\phi_2\| = \|P_K \Upsilon\Phi(\phi_1) - P_K \Upsilon\Phi(\phi_2)\| \leq \|\Upsilon\Phi(\phi_1) - \Upsilon\Phi(\phi_2)\| \leq \|\Phi(\phi_1) - \Phi(\phi_2)\| \leq \theta\|\phi_1 - \phi_2\|.$$

Since $\theta < 1$, $T\phi$ is a contraction and thus there exists one and only one solution ϕ such that

$$T\phi = \phi. \qquad (12)$$

Then ϕ belongs to K and satisfies

$$((\phi, \varphi - \phi)) \geq (\Phi(\phi), \varphi - \phi)) = ((\phi, \varphi - \phi)) - \lambda[(A\phi, \varphi - \phi) + \mu(\phi, \varphi - \phi)] - (f, \varphi - \phi), \qquad \forall\ \varphi \in K. \qquad (13)$$

It follows ($\lambda > 0$) that ϕ is a solution of (4). For more details, see Glowinski et al. and Lions et al. [8,4].

The Approximation Problem

Let us suppose we are given a parameter h converging to zero and a family $\{V_h\}_{0<h\leq 1}$ of closed subspaces of V (in practice, the V_h are finite dimen sional and the parameter h varies over a sequence). We are also given a family $\{K_h\}_h$ of closed convex nonempty subset of V. An approximation of (4) generally involves seeking a function ϕ_h in a set K_h which is a subset of a nite dimensional subspace V_h of V, h being an appropriate index. The approximation of (4) will then involve seeking $\phi_h \in K_h$ such that

$$(A\phi_h, \varphi_h - \phi_h) + \mu(\phi_h, \varphi_h - \phi_h) \geq (f, \varphi_h - \phi_h), \qquad \forall\ \varphi_h \in K_h. \qquad (14)$$

We are interested in determining sequences of solutions $\{\phi_h\}$ to (14), $\phi_h \in K_h \subset V_h$, and in investigating the behavior of the approximations as $h \to 0$. In particular, we want to determine conditions under which $\{\phi_h\}$ converges in some sense to a solution to (4) and estimating the error $\phi - \phi_h$. For a consistent approximation of the set K_h we use the following conditions: Let K be a subset of a space V. A sequence of subsets $\{K_h\}_h$ in V_h is said to converge to a set K if:

a) for every weakly convergent sequence $\{\phi_h\}_h$, $\phi_h \in K_h$, for all h, its weak limit ϕ belongs to K.

b) There exists $\zeta \subset V$, $\overline{\zeta} = K$ and $\lambda : \zeta \to K_h$ such that

$$\lim_{h\to 0} \lambda_h \phi = \phi, \text{ strongly in } V, \forall\, \phi \in \zeta.$$

Now we shall prove the theorem for seeking the solution of (14).

Theorem B: Prove that the problem (14) has a unique solution.

Proof: We can prove this theorem in the similar way as we proved the theorem A. In theorem A, taking V to be V_h and K to be K_h, we have the result. Now we prove the convergence results in the following way:

Theorem C: Let K is a closed convex nonempty subset of V, and $\{K_h\}_h$ be a family of closed convex nonempty subset of V. Then, we have

$$\lim_{h\to 0} \|\phi_h - \phi\|_V = 0, \qquad (15)$$

where ϕ_h is the solution of (14) and ϕ the solution of (4).

Proof: To prove this kind of convergence results, rst, we shall obtain a priori estimates for $\{\phi_h\}_h$ then the weak convergence of $\{\phi_h\}_h$ and finally, with the help of the weak convergence, we shall prove the strong convergence. For obtaining the estimates for ϕ_h we shall show that there exists two constants, called, β_1 and β_2 independent of h, such that

$$\|\phi_h\|^2 \leq \beta_1\|\phi_h\| + \beta_2, \qquad \forall\ h. \qquad (16)$$

Since we know that ϕ_h is the solution of (14), we have

$$(A\phi_h, \varphi_h - \phi_h) + \mu(\phi_h, \varphi_h - \phi_h) \geq (f, \varphi_h - \phi_h),$$

or,

$$(A\phi_h, \phi_h) + \mu(\phi_h, \phi_h) \leq (A\phi_h, \varphi_h) + \mu(\phi_h, \varphi_h) - (f, \varphi_h - \phi_h),$$

by using (2), we have

$$\alpha\|\phi_h\|^2 + \|\mu\|\|\phi_h\|^2 \leq \|A\|\|\phi_h\|\|\varphi_h\| + \|\mu\|\|\phi_h\|\|\varphi_h\| + \|f\|(\|\phi_h\| + \|\varphi\|), \qquad \forall\ \varphi_h \in K_h. \qquad (17)$$

Let $\varphi_0 \in \zeta$ and $\varphi_h = \lambda_h\varphi_0 \in K_h$, then by the condition on K_h (that is, (b)), we have, $\lambda_h\varphi_0 \to \varphi_0$ strongly in V and hence $\|\varphi_h\|$ is uniformly bounded by a constant m. Hence (17) can be written as

$$\|\phi_h\|^2 \leq 1/C\{(m\|A\| + m\|\| + \|f\|)\|\phi_h\| + m\|f\|\}, \quad \|\phi_h\|^2 \leq \beta_1|\phi_h\| + \beta_2, \qquad (18)$$

where $\beta_1 = 1/C\{m\|A\| + m\|\mu\| + \|f\|)$, $\beta_2 = m/C\|f\|$, and C (> 0) is constant. Then, (16) implies that

$$\|\phi_h\| \leq C, \qquad \forall\ h,$$

where here, and what follows, C denotes the various constant.

Now we shall discuss the weak convergence of $\{\phi_h\}_h$. Since the relation (16) implies that ϕ_h is uniformly bounded. Hence there exists a subsequence, say, $\{\phi_{hi}\}$ such that ϕ_{hi} converges to Φ^* weakly in V. Then, by the condition of $\{K_h\}_h$ (that is, (a)), we have $\Phi^* \in K$. Now we shall prove that Φ^* is a solution of (4). Since we know that

$$(A\phi_{h_i}, \phi_{h_i}) \leq (A\phi_{h_i}, \varphi_{h_i}) + \mu(\phi_{h_i}, \varphi_{h_i} - \phi_{h_i}) - (f, \varphi_{h_i} - \phi_{h_i}), \qquad (19)$$

by taking $\varphi_{h_i} = \lambda_{h_i}\varphi,\ and\ \varphi \in \zeta,$ then (19) becomes:

$$(A\phi_{h_i}, \phi_{h_i}) \leq (A\phi_{h_i}, \lambda_{h_i}\varphi) + \mu(\phi_{h_i}, \lambda_{h_i}\varphi - \phi_{h_i}) - (f, \lambda_{h_i}\varphi - \phi_{h_i}). \qquad (20)$$

Since $\lambda_{h_i}\varphi$ converges strongly to φ and ϕhi converges to Φ^* weakly as h_i tends to zero, so by taking the limit as h_i tends to zero in (19), we have

$$\liminf_{h_i\to 0}(A\phi_{h_i}, \phi_{h_i}) \leq (A\Phi^*, \varphi) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*), \ \forall\ \varphi \in \zeta. \qquad (21)$$

Since $\phi_{h_i} \to \Phi^*$ weakly, so by using the known

theorem [9] and from the continuity of "a", and the fact that

$$\liminf_{h_i\to 0}(A\phi_{h_i},\phi_{h_i}) \geq (A\Phi^*,\Phi^*), \tag{22}$$

so, from (21) and (22), we obtain

$$(A\Phi^*,\Phi^*) \leq \liminf_{h_i\to 0}(A\phi_{h_i},\phi_{h_i}) \leq (A\Phi^*,\varphi) + \mu(\Phi^*,\varphi-\Phi^*) - (f,\varphi-\Phi^*), \quad \forall\ \varphi\in\zeta.$$

Therefore, we have

$$(A\Phi^*,\Phi^*-\varphi) + \mu(\Phi^*,\varphi-\Phi^*) - (f,\varphi-\Phi^*) \geq 0, \quad \forall\ \varphi\in\zeta,\ \Phi^*\in K. \tag{23}$$

Since ζ is dense in K and A is continuous, so from (23), we get

$$(A\Phi^*,\Phi^*-\varphi) + \mu(\Phi^*,\varphi-\Phi^*) - (f,\varphi-\Phi^*) \geq 0, \quad \forall\ \varphi\in K,\ \Phi^*\in K. \tag{24}$$

Hence Φ* is a solution of (4). Since, by theorem A, the solution of (4) is unique and so $\Phi^* = \phi$ is the unique solution. Hence ϕ is the only limit point of $\{\phi_h\}_h$ in the weak topology of V. Hence the whole $\{\phi_h\}_h$ converges to ϕ weakly.

Now we shall discuss the strong convergence. By using (2), we have

$$0 \leq \alpha\|\phi_h-\phi\|^2 \leq (A\phi_h-\phi,\phi_h-\phi) = (A\phi_h,\phi_h)-(A\phi_h,\phi)-(A\phi,\phi_h)+(A\phi,\phi), \tag{25}$$

where ϕ_h is solution of (14) and ϕ is solution of (4). Since ϕ_h is solution of (14) and $\lambda_h\varphi\in K_h$ for any $\varphi\in\zeta$, so we get (14) of the form

$$(A\phi_h,\phi_h) \leq (A\phi_h,\lambda_h\varphi) + \mu(\phi_h,\lambda_h\varphi-\phi_h) - (f,\lambda_h\varphi-\phi_h), \quad \forall\varphi\in\zeta. \tag{26}$$

Since we know that

$$\lim_{h\to 0}\phi_h = \phi, \text{ weakly, in } V,$$

and

$$\lim_{h\to 0}\lambda_h\varphi = \varphi, \text{ strongly; in } V \text{ (by condition } (b));$$

we obtain (26) from (25), and after taking the limit, we have

$$0 \leq \alpha\liminf\|\phi_h-\phi\|^2 \leq \alpha\limsup\|\phi_h-\phi\|^2 \leq (A\phi,\varphi-\phi)+\mu(\phi,\varphi-\phi)-(f,\varphi-\phi), \ \forall\ \varphi\in\zeta. \tag{27}$$

By density and continuity, (27) also holds for all $\varphi \in K$; then taking $\varphi = \phi$ in (27), we get

$$\lim_{h\to 0}\|\phi_h-\phi\| = 0, \tag{28}$$

which proves the strong convergence.

We shall now determine the error estimate of the dierence of the solutions of related variational inequalities for the appropriate norm in the form of the following theorem.

Theorem D: Let us consider an operator $A : V = H_0^2(\Omega) \to V'$, which is linear, continuous and symmetric, and satises the coercivity condition (2), and let $\phi \in$ K and $\phi_h \in K_h$ be respective solutions of the variational inequalities:

$$\phi\in K\ \ (A\phi,\varphi-\phi)+\mu(\phi,\varphi-\phi)\geq(f,\varphi-\phi), \quad \forall\ \varphi\in K, \tag{29}$$

and

$$K_h\ \ (A\phi_h,\varphi_h-\phi_h)+\mu(\phi_h,\varphi_h-\phi_h)\geq(f,\varphi_h-\phi_h), \quad \forall\varphi_h\in K_h, \tag{30}$$

where K and K_h are closed convex subsets of $H_0^2(\Omega)$.

Let A be the operator such that

$$A\phi = -\sum_{i,j}\partial^2/\partial x_i^2(a_{ij}(x)\partial^2/\partial x_j^2\phi),$$

where

$$a_{ij}(x)\in L^\infty(\Omega),\ \ and\ \sum_{i,j}a_{ij}(x)\omega_i\omega_j \geq \alpha_0\sum_{i,h}\omega_i^2,\ \alpha>0,\ \ \forall\omega_i\in\Re^n.$$

If $(f-(A\phi+\mu\phi))\in L^2(\Omega)$, then

$$\|\phi-\phi_h\| \leq \{2/\alpha\|f-A\phi-\mu\phi\|\inf_{\varphi\in K}\|\phi_h-\varphi\| + \inf_{\varphi_h\in K_h}[2/\alpha\|f-A\phi-\mu\phi\|\|\phi-\varphi_h\|+C^2/\alpha^2\|\phi-\varphi_h\|^2]\}^{1/2}.$$

Proof: Adding the inequalities (29) and (30), we obtain

$$(A\phi,\phi-\varphi)+(A\phi_h,\phi_h-\varphi_h)+\mu(\phi,\phi-\varphi)+\mu(\phi_h,\phi_h-\varphi_h) -(f,\phi-\varphi)+(f,\phi_h-\varphi_h),$$

or

$$(A\phi,\phi)-(A\phi,\phi_h)-(A\phi_h,\phi)+(A\phi_h,\phi_h)+\mu(\phi,\phi) - \mu(\phi,\phi_h)-\mu(\phi_h,\phi) +\mu(\phi_h,\phi_h) \leq (A\phi,\varphi)+(A\phi_h,\varphi_h)-(A\phi_h,\phi)+ (A\phi,\phi_h)+\mu(\phi,\varphi) +\mu(\phi_h,\varphi_h)-\mu(\phi_h,\phi)-\mu(\phi,\phi_h)+(f,\phi-\varphi)+(f,\phi_h-\varphi_h)$$

After, regrouping, we obtain

$$(A(\phi-\phi_h),\phi-\phi_h)+\mu(\phi-\phi_h,\phi-\phi_h) \leq (f,\phi-\varphi_h)+(f,\phi_h-\varphi)$$

$$(A\phi,\varphi-\phi_h)+(A\phi_h,\varphi_h-\phi)+\mu(\phi,\varphi-\phi_h)+\mu(\phi_h,\varphi_h-\phi) = (f,\phi-\varphi_h)+(f,\phi_h-\varphi)+(A\phi,\varphi-\phi_h)+(A\phi_h,\varphi_h-\phi) +(A(\phi-\phi_h),\phi-\phi_h)+\mu(\phi-\phi_h,\phi-\phi_h) +\mu(\phi,\varphi-\phi_h)+\mu(\phi_h,\varphi_h-\phi).$$

Since

$$\mu(\phi-\phi_h,\phi-\phi_h)\geq 0,$$

so, by using the hypotheses of the theorem, we have

$$\alpha\|\phi-\phi_h\|^2 \leq \|f-A\phi-\mu\phi\|\|\phi-\varphi_h\|+\|f-A\phi-\mu\phi\|\|\phi_h-\varphi\| +C\|\phi_h-\phi\|\|\phi-\varphi_h\|+\|\mu\|\|\phi_h-\phi\|\|\phi-\varphi_h\|,$$

or

$$\alpha\|\phi-\phi_h\|^2 \leq \|f-A\phi-\mu\phi\|\|\phi-\varphi_h\|+\|f-A\phi-\mu\phi\|\|\phi_h-\varphi\| +C\|\phi_h-\phi\|\|\phi-\varphi_h\|,$$

where C denotes various constant. Since we know (Young's inequality) that

$$ab \leq a^2/2\epsilon + \epsilon b^2/2, \quad for\, all \;\; \epsilon > 0,$$

so we make the following estimate:

$$C\|\phi - \varphi_h\|\|\phi_h - \phi\| \leq C^2/2\alpha\|\phi - \varphi_h\|^2 + \alpha/2\|\phi - \phi_h\|^2,$$

so we have, for all $\varphi \in K \;\; and \; \varphi_h \in K_h$

$$\|\phi - \phi_h\| \leq \{2/\alpha(\|f - A\phi - \mu\phi\|\|\phi - \varphi_h\| + \|f - A\phi - \mu\phi\| \\ \|\phi_h - \varphi\|) \; + C^2/\alpha^2\|\phi - \varphi_h\|^2\}^{1/2},$$

which implies that

$$\|\phi - \phi_h\| \leq \{2/\alpha\|f - A\phi - \mu\phi\| \inf_{\varphi\in K} \|\phi_h - \varphi\| + \\ \inf_{\varphi_h \in K_h}[2/\alpha\|f - A\phi - \mu\phi\|\|\phi - \varphi_h\| + C^2/\alpha^2\|\phi - \varphi_h\|^2]\}^{1/2}. \quad (31)$$

Now we shall discuss about the computation work by taking a simple example of one-dimensional.

Computational Results

In order to solve the approximate variational inequality (4), we use the fact that the problem (4) is equivalent to the problem for finding $\phi \in K$ which minimizes

$$1/2(A\varphi, \varphi) - (\chi, \varphi), \quad \forall \; \varphi \in K, \quad (32)$$

where $\chi = f - \mu\phi$. It is very easy to prove this equivalence, since

$$I(\varphi) = 1/2(A\varphi, \varphi) - (\chi, \varphi), \quad \forall \; \varphi \in K,$$

is convex function of φ on K and every local minimum of $I(\varphi)$ in K is also a global minimum of $I(\varphi)$ in K. So we can have existence and uniqueness for the minimization problem. For more details see [8]. Now we shall consider a one-dimensional version of the problem of seepage ow through a homogeneous rectangular dam. We shall solve this problem numerically using the xed point method [8]. The example problem involves nding a solution $\phi \in K$ of a one-dimensional variational inequality

$$\int_0^1 \phi'(\varphi - \phi)'dx + \mu\int_0^1 (\varphi - \phi)dx \geq \int_0^1 f(\varphi - \phi)dx, \;\; \forall \; \varphi \in K, \quad (33)$$

where $\phi' = d\phi/dx$, and let

$$K = \{\varphi \in H^2(0,1) : \varphi(0) = 1/4, \varphi(1) = 0 \;\&\; \varphi(x) \geq 0 \;\; in\,(0,1)\}. \quad (34)$$

Let the domain Ω = (0; 1) be discrtized by a uniform mesh containing N nite elements. Within a nite element, every function $\varphi \in H^2(\Omega)$ is approximated by function of the form

$$\varphi = \varphi^1\Psi_1(\theta) + \varphi^2\Psi_2(\theta),$$

where φ^i is the value of the ith local nodal point, Ψ_i is the local interpolation function at ith nodal point, and is a local coordinate in the nite element. For a unit linear element

$$\Psi_1(\theta) = 1 - \theta, \quad and \quad \Psi_2(\theta) = \theta.$$

Upon assembling the elements, a global modal is obtained for which every function $\varphi \in H^2(\Omega)$ is approximated by

$$\varphi_h = \sum_{i=1}^{n} \varphi_i \psi^i(x), \quad (35)$$

where φ_i is the value of φ_h at ith (global) nodal point, and $\psi^i(x)$ is the global basis function corresponding to the ith (global) nodal point. Introducing the approximation (35) into (33) and (34), the variational inequality (33) on the admissible set (34) reduces to an optimization problem in $\Re^n$

$$\phi_j\psi'_j.(\varphi_i\psi'_i - \phi_i\psi'_i) + \mu\phi_i\psi_j(\varphi_i\psi_i - \phi_i\psi_i) \\ - f(\varphi_i\psi_i - \phi_i\psi_i) \geq 0,$$

or, for $\{\phi_i\} \in \Re_h$

$$(\varphi_i - \phi_i)(K^{ij}\phi_j + G^{ij}\phi_j - F^i) \geq 0, \;\; \forall \{\varphi_i\} \in \Re_h,$$

where repeated indices are summed and

$$\Re_h = \{\{\varphi_i\} \in \Re^n : \varphi_1 = 1/4, \varphi_N = 0 \;\&\; \varphi_i \geq 0, \\ i = 2, ..., N-1\}, \quad (37)$$

and

$$K^{ij} = \int_0^1 \psi'_i.\psi'_j dx, \quad G^{ij} = \mu\int_0^1 \psi_i\psi_j dx, \\ \text{and} \;\; F^i = -f\int_0^1 \psi_i dx, \quad (38)$$

and N is the total number of nodal points in the nite element model. We shall solve (36) and (37) using the xed point method. The iterative scheme we used, is dened in general as

$$\phi_i^{t+1} = max(0., \phi_i^t - \lambda_i(\sum_{j=1}^{i-1}(K^{ij} + G^{ij})\phi_j^{t+1} + \\ \sum_{j=i}^{n}(K^{ij} + G^{ij})\phi_j^{t+1} - F^i)), \quad (39)$$

where the projection P_k is dened pointwise by

$$P_k(\{.\}) = \{max(0., 0)\}, \quad with \quad 0 < \lambda < \frac{2\gamma}{Q}.$$

The terminal value $\phi_j^{t+1}, \, 1 \leq j \leq i$ are used in calculating the value ϕ_i^{t+1}. We solve the variational inequality (36) for $N = 20$, with $\mu = f = 1$, by using fixed point method. The exact solution and approximate nite element results on the variational inequality (36) are shown in the gure below. Also, in the below table, we showed the numerical results obtained by using the fixed point scheme and compared with the exact solution of (36). According to this, we found that the numerical results are same to the exact solution up to five signicant figures.

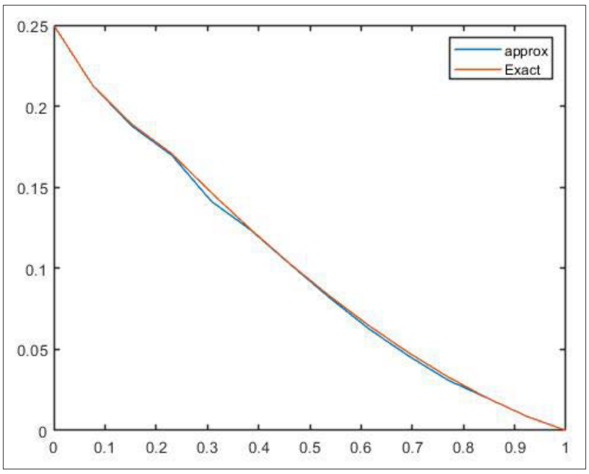

Table: Comparison of the Solutions with Iteration

Node:	Aproxi. Soln.	Exact Soln.	Error
1	0.2500000	0.2500000	0.0000000
2	0.2123777	0.2123787	0.0000010
3	0.1884904	0.1884915	0.0000011
4	0.1706367	0.1706378	0.0000011
5	0.1466371	0.1466384	0.0000013
6	0.1239702	0.1239721	0.0000019
7	0.1027054	0.1027072	0.0000018
8	0.0829138	0.0829156	0.0000018
9	0.0646738	0.0646728	0.0000010
10	0.0480571	0.0480583	0.0000012
11	0.0331544	0.0331555	0.0000011
12	0.0200515	0.0200521	0.0000006
13	0.0088398	0.0088405	0.0000007
14	0.0000000	0.0000000	0.0000000

factor = 0:05, at iteration 14.

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