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M-107

Part 2

Vectors and

Vector Valued Functions

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CHAPTER 10

Vectors

and

The Geometry of space

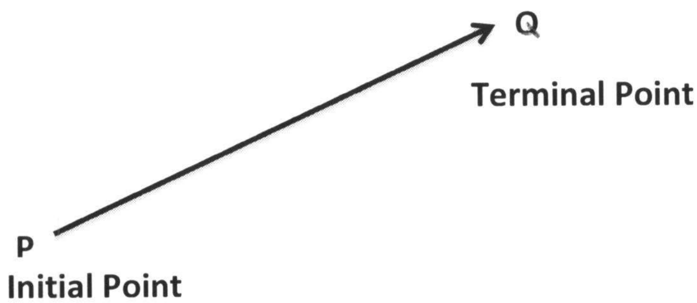
Chapter 10

Vectors and Surfaces

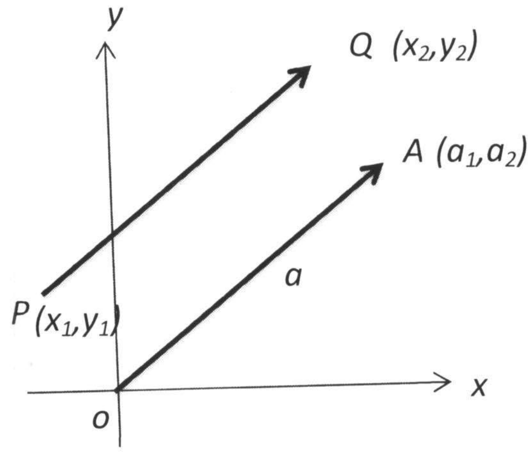
10.1 Vectors in Two dimensions

Vector is a quantity that is used by scientist to define a quantity that has **size(magnitude)** and **direction**. Example for vectors are displacement, velocity, acceleration, force, weight, momentum, and moment of inertia.

In geometry it is show by **directed line**. The directed line PQ has **initial point** P and **terminal point** Q.



Scalar is a quantity that has only magnitide but no direction. Examples for scalars are distance, speed, mass, temperature, length, area and volume. Scalars are denoted by real numbers with appropriate unit.



1. If P (x_1,y_1) and Q (x_2,y_2) are points in xy- plane, then

$$\overrightarrow{PQ} = < x_2 - x_1, y_2 - y_1 >$$

Vector $\overrightarrow{OA} = \overrightarrow{< a_1, a_2 >}$ has initial position at the origin O, is called **position vector.** a_1 and a_2 are called components of vector a.

2. Magnitude of a vector

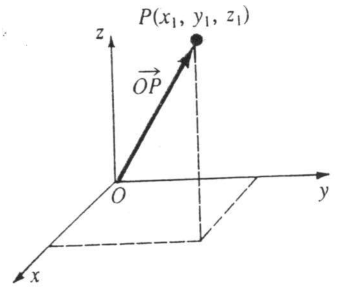
$$\|a\| = \sqrt{a_1^2 + a_2^2}$$

3. Let $a = < a_1, a_2 >$ and $b = < b_1, b_2 >$ be vectors in two dimensions
   - (i) Addition: $a + b = < a_1 + b_1, a_2 + b_2 >$
   - (ii) Scalar multiplication ka = k$< a_1, a_2 > = <$ k$a_1, ka_2 >$
   - (iii) Equality of Vectors $a = b$ if anly if $a_1 = b_1$ and $a_2 = b_2$
4. Unit Vector, u, A vector that has magnitude "one " is called unit vector

$$u = \frac{a}{\|a\|},$$

$$u = \frac{< a_1, a_2 >}{\sqrt{a_1^2 + a_2^2}}$$

10.2 Vectors in Three Dimensions



Figure

A **vector a in 3-space** is any ordered triple of real numbers

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

where a_1, a_2, and a_3 are the **components** of the vector. The **position vector** of a point $P(x_1, y_1, z_1)$ in space is the vector $\overrightarrow{OP} = \langle x_1, y_1, z_1 \rangle$ whose initial point is the origin O and whose terminal point is P. See Figure

The component definitions of addition, subtraction, scalar multiplication, and so on, are natural generalizations of those given for vectors in 2-space.

DEFINITION Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors in 3-space.

(*i*) Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

(*ii*) Scalar multiplication: $k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$

(*iii*) Equality: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$

(*iv*) Negative: $-\mathbf{b} = (-1)\mathbf{b} = \langle -b_1, -b_2, -b_3 \rangle$

(*v*) Subtraction: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

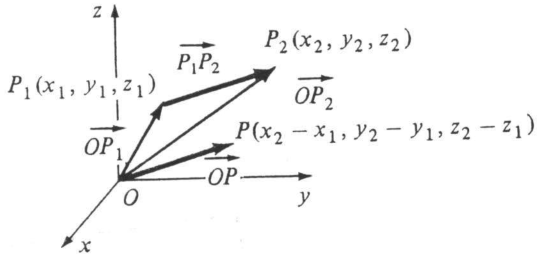
(*vi*) Zero vector: $\mathbf{0} = \langle 0, 0, 0 \rangle$

(*vii*) Magnitude: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

If $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are the position vectors of the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, then the vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

As in 2-space $\overrightarrow{P_1P_2}$ can be drawn either as a vector whose initial point is P_1 and whose terminal point is P_2 or as a position vector $\overrightarrow{OP}$ with terminal point $P(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. See Figure



Example. Given the points $P_1(5, 6, -2)$ and $P_2(-3, 8, 7)$, find the vector a in V_3 that corresponds to $\overrightarrow{P_1P_2}$

Solution

$$a = \overrightarrow{P_1P_2} = \langle -3-5, 8-6, 7+2 \rangle = \langle -8, 2, 9 \rangle.$$

i, j, k vectors

$i = \langle 1,0,0\rangle$, $j = \langle 0,1,0\rangle$, $k = \langle 0,0,1\rangle$

Any vector $a = \langle a_1, a_2, a_3\rangle$

$= \langle a_1, 0, 0\rangle + \langle 0, a_2, 0\rangle + \langle 0, 0, a_3\rangle$

$= a_1\langle 1,0,0\rangle + a_2\langle 0,1,0\rangle + a_3\langle 0,0,1\rangle$

$= a_1 i + a_2 j + a_3 k$

i, j and k are unit vectors along x-, y-, and z-axis respectively.

Ex If $a = \langle -2, 6, 1\rangle$ and $b = \langle 3, -3, -1\rangle$, find

(i) $a+b$ (ii) $a-b$ (iii) $5a-4b$ (iv) $\|a\|$ (v) $\|3a\|$

Solution

(i) $a+b = \langle -2,6,1\rangle + \langle 3,-3,-1\rangle$

$= \langle -2+3, 6-3, 1-1\rangle = \langle 1,3,0\rangle$

(ii) $a-b = \langle -2,6,1\rangle - \langle 3,-3,-1\rangle$

$= \langle -2-3, 6+3, 1+1\rangle = \langle -5, 9, 2\rangle$

(iii) $5a-4b = 5\langle -2,6,1\rangle - 4\langle 3,-3,-1\rangle$

$= \langle -10-12, 30+12, 5+4\rangle = \langle -22, 42, 9\rangle$

(iv) $\|a\| = \sqrt{4+36+1} = \sqrt{41}$

(v) $3a = 3\langle -2,6,1\rangle = \langle -6, 18, 3\rangle$

$\|3a\| = \sqrt{36+324+9} = \sqrt{369} = \sqrt{9\times 41} = 3\sqrt{41}$

Ex. If $a = 2\langle -2, +5, -1\rangle$, find the unit vector that has same direction as a.

Solution. $u = \dfrac{a}{\|a\|}$ or $a = \langle -4, 10, -2\rangle$

$\|a\| = \sqrt{16+100+4} = \sqrt{120} = 2\sqrt{30}$

$$\|a\| = \sqrt{4\cdot(4+25+1)} = 2\sqrt{30}$$

$$u = \frac{2}{2\sqrt{30}}\langle -2, 5, -1\rangle = \frac{1}{\sqrt{30}}\langle -2, 5, -1\rangle$$

Ex If $a = \langle -6, -3, 6\rangle$, find the vector that has

(i) the same direction as a and twice the magnitude of a,

(ii) the opposite direction of a and one-third magnitude of a,

(iii) the same direction of a and magnitude 2.

Soln.

(i) $b = 2a = 2\langle -6, -3, 6\rangle$

(ii) $b = -\frac{1}{3}a = -\frac{1}{3}\langle -6, -3, 6\rangle$

(iii) $u = \dfrac{a}{\|a\|} = \dfrac{1}{\sqrt{36+9+36}}\langle -6, -3, 6\rangle = \dfrac{1}{9}\langle -6, -3, -6\rangle$

$$b = 2u = \frac{2}{9}\langle -6, -3, -6\rangle$$

Note: 1 If $b = ka$, where k is scalar, then a and b are parallel.

Note: 2. Three points lie on same line, if two vectors from three points have

(i) Same initial point

(ii) and are parallel.

Exercise. Use vectors to determine whether the points lie on a straight line, points are $(1,-1,5)$, $(0,-1,6)$, $(3,-1,3)$

EXERCISES FOR VECTORS

In Exercise 1 – 4 , Find the vector $\overrightarrow{PQ}$,

1. P(3, 2, -1), Q(-1, 2, 3)
2. P(3, 3, -2), Q(2, 1, 0)
3. P(-2, 1, 0), Q(0, 2, 3)
4. P(0, 0, 0), Q(1, 2, 1)
5. Determine whether the points P (3, -4, 2) , Q(4, -2, 1) and R(6, 1, -1) lie on the same line.
6. Determine whether the points P (3, 2, -2) , Q(4, 4, -4) and R(2, 0, -1) lie on the same line.
7. Find terminal point of the vector if v = 5i –2j –k , initial point is (3,-2 1).
8. Find initial point of the vector v = < - 2, 0, 3>, if the terminal point is (5, 0, -2).
9. Find the vector **a** , given u = <2, -1, 0>, v = <-3, 2, 1> and w = <3, 0, 4 >

   i. a = u –v
   ii. a = u + v –w
   iii. a = 2u +3v –w
   iv. 3a – 2v + w = 0.

10. Determine which of the vectors are parallel to z = <3, -1, 2>

    (a) <-6, 2, -4>
    (b) $< 1, \frac{-1}{3}, \frac{2}{3} >$
    (c) $4i + \frac{4}{3}j + 6k$
    (d) $2i - 2j + 4k$

11. Find the magnitude of the vector **a**, hence find the unit vector

    i. a = $2i - 3j + k$
    ii. a = < 2, 4, 3 >
    iii. a = i – j
    iv. a = <-1 , 1, 1 >

12. Let u = < 1, -3 , 2 > , v = < 1, 0, 1 >, w = < 3, 1, 2 >, find
   a. 2u + v,
   b. 3u -5v
   c. $\|u\|$
   d. $\|-2u\|$
   e. $\|u+v\|$
   f. $\|u-v\|$
   g. $\left\|\frac{u}{\|u\|}\right\|$
   h. $\left\|\frac{u+v}{\|u+v\|}\right\|$
   i. $\|2u-v+3w\|$

13. Determine the value of k that satisfies $\|kv\| = 6$, where v = <1, 3, 5>.
14. Find the vector a with magnitude twice magnitude of vector u = 3i + 2j – k.

15. Find the vector which is two-third of vector from P(3, 2, 1) and Q(-2,1,0).

16. Find the sides of triangle in the form of vectors with vertices P(2, 1, 3), Q(-2, 0, 2) and R(3, 2, 1).

10.3 Dot Product / Scalar Product / Inner Product

Definition 1 The dot product $a \cdot b$ of $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ is

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example Find $a \cdot b$ if

(i) $a = \langle 2, 3, 4 \rangle$, $b = \langle 3, -2, 1 \rangle$

(ii) $a = i + 2j - 3k$, $b = 2i + j + k$

Solution (i) $a \cdot b = \langle 2, 3, 4 \rangle \cdot \langle 3, -2, 1 \rangle$

$$= (2)(3) + (3)(-2) + (4)(1)$$
$$= 6 - 6 + 4$$
$$= 4$$

(ii) $a \cdot b = (i + 2j - 3k) \cdot (2i + j + k)$

$$= 1.2 + 2.1 + (-3).1$$
$$= 2 + 2 - 3$$
$$= 4 - 3$$
$$= 1$$

Properties of dot product

(*i*) $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\theta = \frac{\pi}{2}$

(*ii*) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (Commutative law)

(*iii*) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (Distributive law)

(*iv*) $\mathbf{a} \cdot (k\mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$, k a scalar

(*v*) $\mathbf{a} \cdot \mathbf{a} \geq 0$

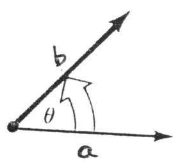
(*vi*) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

Definition 2 The dot Product

The **dot product** of two vectors **a** and **b** is the scalar

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

where θ is the angle between the vectors such that $0 \leq \theta \leq \pi$.



(a)

$$\cos\theta = \frac{a \cdot b}{\|a\|\,\|b\|} \Rightarrow \theta = \cos^{-1}\frac{a \cdot b}{\|a\|\,\|b\|}$$

Note. 1. If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ are parallel, then $b = ca$

Note. 2. If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ are parallel, then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Note. 3. Two non-zero vectors **a** and **b** are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b} = 0$. $\Rightarrow \cos\theta = 0$
$\Rightarrow \theta = \frac{\pi}{2}$

Note. 4. $i\cdot j = j\cdot i = 0, \quad j\cdot k = k\cdot j = 0, \quad k\cdot i = i\cdot k = 0$

<u>Example.</u> If $a = \langle -2, 3, 1 \rangle$, $b = \langle 7, 4, 5 \rangle$ and $c = \langle 1, -5, 2 \rangle$ find $(2a+b)\cdot 3c$

<u>Solution.</u>

$$2a = 2\langle -2, 3, 1\rangle = \langle -4, 6, 2\rangle$$

$$3c = 3\langle 1, -5, 2\rangle = \langle 3, -15, 6\rangle$$

$$(2a+b) = \langle -4, 6, 2\rangle + \langle 7, 4, 5\rangle = \langle 3, 10, 7\rangle$$

$$(2a+b)\cdot 3c = \langle 3, 10, 7\rangle \cdot \langle 3, -15, 6\rangle$$

$$= 9 - 150 + 42 = 51 - 150 = -99$$

<u>Ex.</u> Find the angle between $a = i - 7j + 4k$, $b = 5i - k$

<u>Solution</u>

$$a\cdot b = \|a\|\|b\|\cos\theta \Rightarrow \cos\theta = \frac{a\cdot b}{\|a\|\|b\|}$$

$$a\cdot b = (i - 7j + 4k)\cdot(5i - k) = 5 - 0 - 4 = 1$$

$$\|a\| = \sqrt{1+49+16} = \sqrt{66}$$

$$\|b\| = \sqrt{25+1} = \sqrt{26}$$

$$\cos\theta = \frac{1}{\sqrt{66}\sqrt{26}} \Rightarrow \theta = \cos^{-1}\left[\frac{1}{\sqrt{66}\sqrt{26}}\right]$$

Example. Show that $a = 3i - 2j + k$ and $b = 4i + 5j - 2k$ are orthogonal to each other.

Solution. a and b are orthogonal if $a \cdot b = 0$

$$a \cdot b = (3)(4) + (-2)(5) + (1)(-2) = 12 - 10 - 2 = 0$$

$\Rightarrow$ a and b are orthogonal to eacher other.

Example. Find the value of 'c' such that $a = 4i + 2j + ck$ and $b = i + 22j - 3ck$ are orthogonal.

Solution. If a and b are orthogonal, then $a \cdot b = 0$

$$a \cdot b = (4i + 2j + ck) \cdot (i + 22j - 3ck)$$

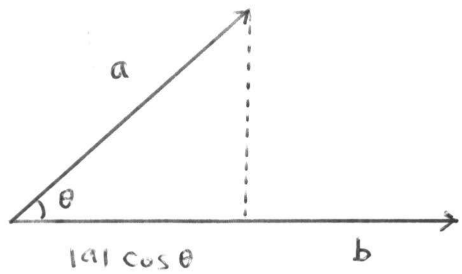
$$= 4 + 44 - 3c^2 = 48 - 3c^2,$$

$a \cdot b = 0 \Rightarrow 48 - 3c^2 = 0$ or $3c^2 = 48$, $c^2 = 16$

$c = +4$, and $c = -4$. $\quad c = \pm 4$

Component of a along b

Let a and b vectors in V_3 with $b \neq 0$.



The component of a along b

$$= \text{Comp}_b^a = \frac{a \cdot b}{\|b\|} = a \cdot \frac{b}{\|b\|}$$

Projection of a on b

Vector projection of a onto b

$$= \left(\text{Comp}_b^a\right) \frac{b}{\|b\|} = \left(\frac{a \cdot b}{\|b\|}\right)\left(\frac{b}{\|b\|}\right)$$

Ex. Let $a = 2i + 3j - 4k$ and $b = i + j + 2k$, Find

(a) $\text{Comp}_b\, a$ (b) $\text{comp}_a\, b$, (c) $\text{Proj}_b\, a$

Solution. (a) $\text{Comp}_b\, a = a \cdot \frac{b}{\|b\|}$

$$= (2i + 3j - 4k) \cdot \frac{(i + j + 2k)}{\sqrt{1+1+4}}$$

$$= \frac{2+3-8}{\sqrt{6}} = -\frac{3}{\sqrt{6}}$$

(c) $\text{Proj}_b\, a = \left(a \cdot \frac{b}{\|b\|}\right) \frac{b}{\|b\|}$

$$= -\frac{3}{\sqrt{6}} \cdot \frac{i+j+2k}{\sqrt{6}}$$

$$= -\frac{1}{2}(i+j+2k)$$

(b) $\text{Comp}_a\, b = b \cdot \frac{a}{\|a\|} = (i + j + 2k) \cdot \frac{(2i+3j-4k)}{\sqrt{4+9+16}}$

$$= \frac{2+3-8}{\sqrt{29}} = \frac{-3}{\sqrt{29}}$$

$\boxed{\text{Note}}$. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

Ex. Given points $P(3, -2, -1)$, $Q(1, 5, 4)$, $R(2, 0, -6)$ and $S(-4, 1, 5)$, find (i) $\overrightarrow{PQ} \cdot \overrightarrow{RS}$

(ii) The angle between $\overrightarrow{PQ}$ and $\overrightarrow{RS}$

(iii) The component of $\overrightarrow{PS}$ along $\overrightarrow{QR}$

Solution

$$\overrightarrow{PQ} = \langle 1-3,\ 5+2,\ 4+1 \rangle = \langle -2, 7, 5 \rangle$$

$$\overrightarrow{RS} = \langle -4-2,\ 1-0,\ 5+6 \rangle = \langle -6, 1, 11 \rangle$$

(i) $\overrightarrow{PQ} \cdot \overrightarrow{RS} = \langle -2, 7, 5 \rangle \cdot \langle -6, 1, 11 \rangle = 12 + 7 + 55 = 74$

(ii) $\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{RS}}{\|\overrightarrow{PQ}\| \|\overrightarrow{RS}\|} = \frac{74}{\sqrt{4+49+25}\sqrt{36+1+121}} = \frac{74}{\sqrt{78}\sqrt{158}}$

$$\theta = \cos^{-1}\left[\frac{74}{\sqrt{78}\sqrt{158}}\right] = \cos^{-1}\left(\frac{74}{110.6}\right) \approx 48^\circ$$

(iii) $\overrightarrow{PS} = \langle -4-3,\ 1+2,\ 5+1 \rangle = \langle -7, 3, 6 \rangle$

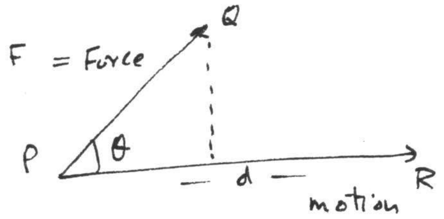
$\overrightarrow{QR} = \langle -1,\ 0-5,\ -6-4 \rangle = \langle 1, -5, -10 \rangle$

$$\text{Comp}_{\overrightarrow{QR}}\, \overrightarrow{PS} = \frac{\overrightarrow{PS} \cdot \overrightarrow{QR}}{\|\overrightarrow{QR}\|} = \frac{\langle -7, 3, 6 \rangle \cdot \langle 1, -5, -10 \rangle}{\sqrt{1+25+100}} = \frac{-7-15-60}{\sqrt{126}}$$

$$= \frac{-82}{\sqrt{126}}$$

WORK DONE

The work done by a force $\overrightarrow{PQ}$ as its point of application moves along the vector $\overrightarrow{PR}$ is $\overrightarrow{PQ} \cdot \overrightarrow{PR}$



$W = F \cdot d$

Ex. Find the work done by a constant force $F = 2i + 4j + k$ if its point of application moves from $P(1,1,3)$ to $Q(4,6,2)$

Solution

$$d = \overrightarrow{PQ} = \langle 4-1, 6-1, 2-3 \rangle = \langle 3, 5, -1 \rangle$$

$$W = F \cdot d = \langle 2, 4, 1 \rangle \cdot \langle 3, 5, -1 \rangle$$
$$= 6 + 20 - 1$$
$$= 25$$

Ex. A constant force of magnitude 4 pounds has the same direction as the vector $a = i + j + k$. If distance is measured in feet, find the work done if the point of application moves along the y-axis from $(0,2,0)$ to $(0,-1,0)$.

Solution

$$\text{Force} = 4\frac{(i+j+k)}{\sqrt{1+1+1}} = \frac{4}{\sqrt{3}}(i+j+k)$$

$$d = \langle 0-0, -1-2, 0-0 \rangle = \langle 0, -3, 0 \rangle$$

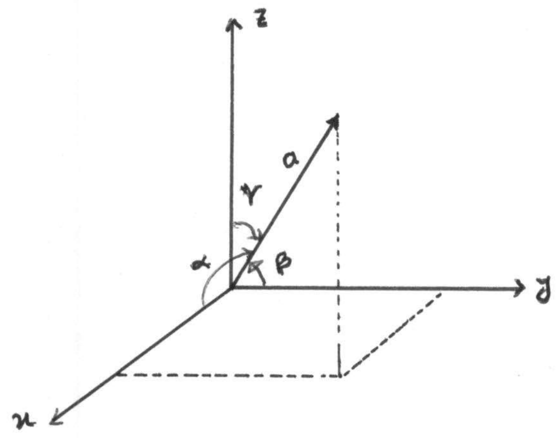
$$W = F \cdot d = \frac{4}{\sqrt{3}} \langle 1, 1, 1 \rangle \cdot \langle 0, -3, 0 \rangle$$

$$= \frac{+4}{\sqrt{3}} \cdot (-3) = \frac{-12}{\sqrt{3}} = -4\sqrt{3}.$$

work done $= |-4\sqrt{3}| = 4\sqrt{3}$ foot poundals

DIRECTION ANGLES / DIRECTION COSINES

The direction angles of a non- zero vector $a = \langle a_1, a_2, a_3 \rangle$ are the angles α, β and γ with the base vector i, j and k respectively.



The cosine of these angles $\cos\alpha, \cos\beta$ and $\cos\gamma$, are called direction cosine of vector a and are defined as

$$\cos\alpha = \frac{a.i}{|a||i|} = \frac{a_1}{|a|},$$

$$\cos\beta = \frac{a.j}{|a||j|} = \frac{a_2}{|a|}, \text{ and}$$

$$\cos\gamma = \frac{a.k}{|a||k|} = \frac{a_3}{|a|}.$$

Because $\frac{a}{|a|}$ is a unit vector, it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Example: 1.

Find the direction cosines and direction angles of the vector a = 2i + 3j + 4k and also show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Solution:

$$|a| = \sqrt{29}$$

$$\cos\alpha = \frac{a_1}{|a|} = \frac{2}{\sqrt{29}} \quad \Rightarrow \quad \alpha \approx 68.2^0$$

$$\cos\beta = \frac{a_2}{|a|} = \frac{3}{\sqrt{29}} \quad \Rightarrow \quad \beta \approx 56.1^0$$

$$\cos\gamma = \frac{a_3}{|a|} = \frac{4}{\sqrt{29}} \quad \Rightarrow \quad \gamma \approx 42.0^0$$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{4}{29} + \frac{9}{29} + \frac{16}{29} = \frac{29}{29} = 1$.

EXERCISES FOR SCALAR PRODUCT

SECTION: SCALAR PRODUCT

1. Given vector $u = \langle 3, -2, 2 \rangle$, $v = \langle 4, 5, 6 \rangle$ and $w = \langle -1, 2, -3 \rangle$, find
   i. u.v, v.w
   ii. u.(v + w),
   iii. $comp_{v}^{u}$,
   iv. $comp_{w}^{v}$,
   v. $comp_{v}^{u+w}$,
   vi. $comp_{w}^{w}$,
   vii. Find angle between u and v,
   viii. Find angle between u and w,
   ix. Find angle between v and w.
2. Determine whether u and v are perpendicular to each other,
   i. $u = j$, $v = k$,
   ii. $u = 2i - 3j - 5k$, $v = 3i + 2j$,
   iii. $u = 3i - 5j + 7k$, $v = 2i + 4j + 2k$.

3. Which of the vectors u , v and w are mutually perpendicular

   $u = 4i + j - k$, $v = 3i + 7j + 13k$ and $w = 20i - 29j + 11k$

4. Find value of m such that u and v are orthogonal ,

   i. $u = \langle 2, m, 2 \rangle$, $v = \langle 2, 3, 2 \rangle$.
   ii. $u = i + j + k$, $v = 3i - 2j + m k$.
5. Find the scalar m so that the vector $\langle 2, 1, m \rangle$ is perpendicular to the sum of the vectors $\langle 1, -1, 2 \rangle$ and $\langle 3, 2, 1 \rangle$.

6. Show that $\langle 3, -2, 1 \rangle$, $\langle 1, -3, 5 \rangle$ and $\langle 2, 1, -4 \rangle$ form a right angle triangle.

7. Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ where P(2, 3, -1) , Q(2, 1, 3), R(1, 2, 1) and S(2, 1, 1).

8. A constant force F moves an object along a straight line form th point P to the point Q . Find the work done when F, P and Q are :

   i. $F = 3i + 2j$, P(1, 5) , Q(-1, 4),
   ii. $F = i + j - 2k$, P(0, 1, 1), Q(0,1,-2),
   iii. $F = -5i - 3j + k$, P(3,-4,5), Q(1,-3,6).
9. The constant forces $2i + j - 3k$, $-i + 2j - k$ and $2i + 5k$ act on a particle which is displaced from point (4,-3,-2) to (6,7,-3). Find the total work done.

10. Forces of magnitudes 5 and 3 units are acting on the direction $a = 6i + 2j + 3k$ and $b = -2i + 3j + 6k$ respectively act on a body which is displaced from point P(2, 2,1) to Q (3,4,2). Find the total work done by the forces.

11. Find the three angles of triangle with vertices P(2, 1, 3), Q(-2, 0, 2) and R(3, 2, 1).

12. Determine whether points P(1,-2,3), Q(2,1,0) and R(4,7,-6) lie on the same line.

13. Find a vector $a\,i + b\,j + c\,k$ which is orthogonal to both $2\,i + j - k$ and $3\,i + j - k$ and satisfies $(a\,i + b\,j + c\,k).(i + j + k) = 1$.

14. Find direction cosines and direction angles of vector a

(i) $a = 2i + 3j + 6k$

(ii) $a = \langle 3, -1, 5\rangle$

(iii) $a = \langle 2, 1, -2\rangle$

Also show that squares of direction cosines is 1.

Review

Since knowledge of determinants of order 2 and order 3 is important to the discussion that follows, we recall the following facts:

(a) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

(b) $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

This is called **expanding the determinant by cofactors** of the first row.

(c) When two rows of a determinant are interchanged, the resulting determinant is the negative of the original.

See Appendix I for a review of the properties of determinants.

10.4 Vector Product / Cross Product

Definition 1 The vector product (or cross product) a x b of

$a = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$ is

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k .$$

Definition 2

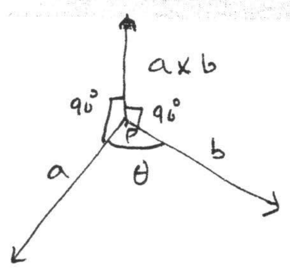
Cross Product of Two Vectors*

The **cross product** of two vectors **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin\theta)\mathbf{n}$$

where θ is the angle between the vectors such that $0 \le \theta \le \pi$ and **n** is a unit vector perpendicular to the plane of **a** and **b** with direction given by the right-hand rule.

Note: a x b is orthogonal to both a and b.



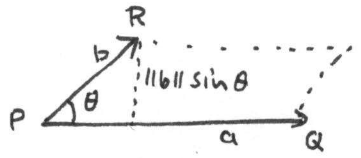
$(a \times b) \cdot a = 0$

$(a \times b) \cdot b = 0$

Note. 2. a and b are parallel if $a \times b = 0$

Note. 3. a and b are orthogonal if $a \cdot b = 0$

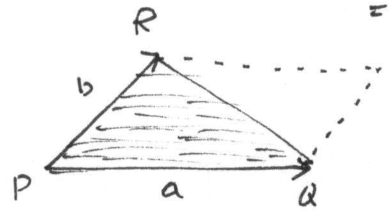
Note. 4. The magnitude of $a \times b$ equals the area of the parallelogram determined by a and b



parallelogram

$$\text{Area of parallelogram} = (\text{base})(\text{height})$$
$$= \|a\| \|b\| \sin\theta$$
$$= \|a \times b\|$$

Note. 5.



$$\text{Area of triangle } PQR = \frac{\|a \times b\|}{2}$$

Ex. Let $a = \langle -6, -10, 4 \rangle$, $b = \langle 3, 5, -2 \rangle$, use vector product to show that a and b are parallel.

Solution

$$a \times b = \begin{vmatrix} i & j & k \\ -6 & -10 & 4 \\ 3 & 5 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -10 & 4 \\ 5 & -2 \end{vmatrix} i - \begin{vmatrix} -6 & 4 \\ 3 & -2 \end{vmatrix} j + \begin{vmatrix} -6 & -10 \\ 3 & 5 \end{vmatrix} k = 0$$

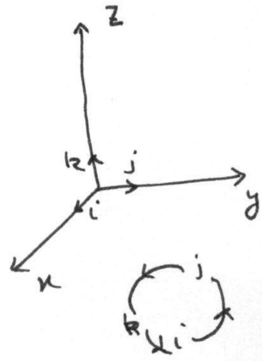
$\Rightarrow$ a and b are parallel.

Note

$i \times j = k$ $\qquad$ $j \times i = -k$

$j \times k = i$ $\qquad$ $k \times j = -i$

$k \times i = j$ $\qquad$ $i \times k = -j$



$i \times i = 0$

$j \times j = 0$

$k \times k = 0$

Ex Let $P(-3,0,5)$, $Q(2,-1,-3)$, $R(4,1,-1)$ be the point in a plane,

(i) Find a vector perpendicular to the plane determined by P, Q and R

(ii) Find the area of parallelogram made by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$,

(iii) Find the area of triangle PQR

(iv) Find a unit vector perpendicular to the plane determined by P, Q and R.

Solution

$$\overrightarrow{PQ} = \langle 2+3, -1+0, -3-5\rangle = \langle 5,-1,-8\rangle$$

$$\overrightarrow{PR} = \langle 4+3, 1+0, -1-5\rangle = \langle 7,1,-6\rangle$$

(i) $$\vec{A} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 5 & -1 & -8 \\ 7 & 1 & -6 \end{vmatrix} = \begin{vmatrix} -1 & -8 \\ 1 & -6 \end{vmatrix} i - \begin{vmatrix} 5 & -8 \\ 7 & -6 \end{vmatrix} j + \begin{vmatrix} 5 & -1 \\ 7 & 1 \end{vmatrix} k$$

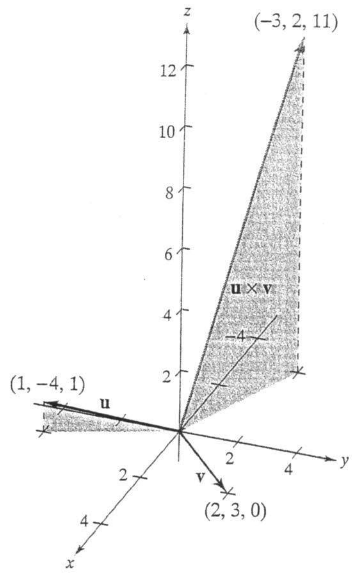
$$= 14i - 26j + 12k$$

(ii) Area of $\|^{\text{gram}}$ is $\|\overrightarrow{PQ}\times\overrightarrow{PR}\|$

$$\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \sqrt{(14)^2+(-26)^2+(12)^2} = \sqrt{196+676+144}$$
$$= \sqrt{1016}$$
$$= 31.87 \text{ unit}^2$$

(iii) Area of triangle PQR $= \frac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \frac{1}{2}(31.87) = 15.94 \text{ unit}^2$

(iv) $$u = \frac{\vec{A}}{\|\vec{A}\|} = \frac{1}{31.87}\langle 14,-26,12\rangle$$



The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
Figure 10.37

EXAMPLE 2 Using the Cross Product

Find a unit vector that is orthogonal to both

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{v} = 2\mathbf{i} + 3\mathbf{j}.$$

Solution The cross product $\mathbf{u} \times \mathbf{v}$, as shown in Figure 10.37, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix}$$
$$= -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$$

Because $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 11^2} = \sqrt{134}$, a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

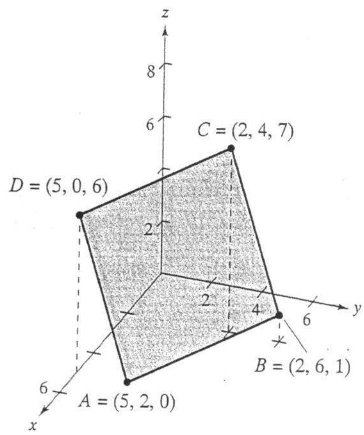
$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{3}{\sqrt{134}}\mathbf{i} + \frac{2}{\sqrt{134}}\mathbf{j} + \frac{11}{\sqrt{134}}\mathbf{k}.$$

NOTE In Example 2, note that you could have used the cross product $\mathbf{v} \times \mathbf{u}$ to form a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. With that choice, you would have obtained the negative of the unit vector found in the example.

EXAMPLE 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram, and find its area.

$$A = (5, 2, 0) \qquad B = (2, 6, 1)$$
$$C = (2, 4, 7) \qquad D = (5, 0, 6)$$



The area of the parallelogram is approximately 32.19.
Figure 10.38

Solution From Figure 10.38 you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \qquad \overrightarrow{CD} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB}$$
$$\overrightarrow{AD} = 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} \qquad \overrightarrow{CB} = 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}$$

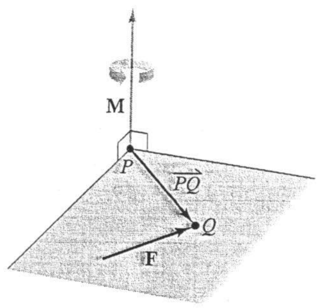
Thus, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$, and you can conclude that the quadrilateral is a parallelogram with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as adjacent sides. Moreover, because

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix}$$
$$= 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

the area of the parallelogram is

$$\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{1036} \approx 32.19.$$

Is the parallelogram a rectangle? You can tell whether it is by finding the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.



The moment of F about P
Figure 10.39

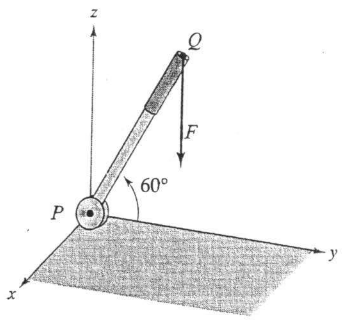
In physics, the cross product can be used to measure **torque**—the **moment M of a force F about a point** P, as shown in Figure 10.39. If the point of application of the force is Q, the moment of $\mathbf{F}$ about P is given by

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F}. \qquad \text{Moment of } \mathbf{F} \text{ about } P.$$

The magnitude of the moment $\mathbf{M}$ measures the tendency of the vector $\overrightarrow{PQ}$ to rotate counterclockwise (using the right-hand rule) about an axis directed along the vector $\mathbf{M}$.

EXAMPLE 4 An Application of the Cross Product

A vertical force of 50 pounds is applied to the end of a 1-foot lever that is attached to an axle at point P, as shown in Figure 10.40. Find the moment of this force about the point P when $\theta = 60°$.



A vertical force of 50 pounds is applied at point Q.
Figure 10.40

Solution If you represent the 50-pound force as $\mathbf{F} = -50\mathbf{k}$ and the lever as

$$\overrightarrow{PQ} = \cos(60°)\mathbf{j} + \sin(60°)\mathbf{k} = \frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$$

the moment of $\mathbf{F}$ about P is given by

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -50 \end{vmatrix} = -25\mathbf{i}.$$

The magnitude of this moment is 25 foot-pounds.

NOTE In Example 4, note that the moment (the tendency of the lever to rotate about its axle) is dependent on the angle θ. When $\theta = \pi/2$, the moment is $\mathbf{0}$. The moment is greatest when $\theta = 0$.

The Triple Scalar Product

For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in space, the dot product of $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the **triple scalar product**. The proof of this theorem is left as an exercise (see Exercise 53).

NOTE The value of a determinant is multiplied by -1 if two rows are interchanged. After two such interchanges, the value of the determinant will be unchanged. Thus, the following triple scalar products are equivalent.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) =$$
$$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) =$$
$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

THEOREM 10.9 The Triple Scalar Product

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

1. Scalar Triple Product

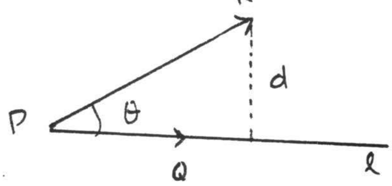
$$(a \times b) \cdot c = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(a \times b) \cdot c = (a \cdot b) \times c = c \cdot (a \times b) = b \cdot (c \times a)$$

Note: vectors a, b and c will be coplanar if $a \cdot (b \times c) = 0$.

2. Distance of a point R to line ℓ

$$d = \|\vec{PR}\| \sin\theta = \frac{\|\vec{PQ}\| \|\vec{PR}\| \sin\theta}{\|\vec{PQ}\|}$$



$$d = \frac{\|\vec{PQ} \times \vec{PR}\|}{\|\vec{PQ}\|}$$

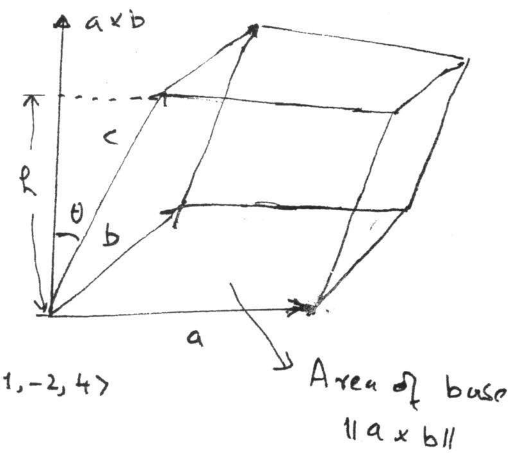
3. Volume of a box

Volume of a parallelpipe with edges a, b and c or Volume of a box

$V = $ (area of base)(height)

$= (\|a \times b\|)(\|c\| \cos\theta)$

$= |(a \times b) \cdot c|$



Ex. Let $a = \langle 2, 0, -1 \rangle$, $b = \langle -3, 1, 0 \rangle$ and $c = \langle 1, -2, 4 \rangle$

Find $a \times (b \times c)$

Soln. $b \times c = \begin{vmatrix} i & j & k \\ -3 & 1 & 0 \\ 1 & -2 & 4 \end{vmatrix} = 4i + 12j + 5k$ $\quad a \times (b \times c) = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 4 & 12 & 5 \end{vmatrix} = 12i - 14j + 24k$

Note. Properties

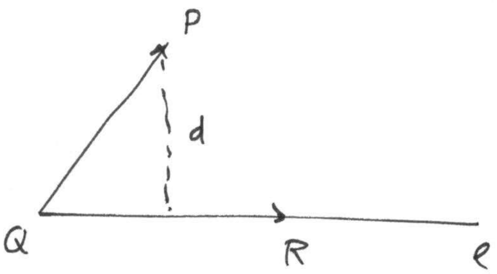
1. $b \times a = -(a \times b)$
2. $a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$
3. $a \times (b + c) = a \times b + a \times c$
4. $(a \times b) \cdot c = a \cdot (b \times c)$
5. $i \times j = k \quad j \times k = i \quad k \times i = j.$

   $j \times i = -k \quad k \times j = -i \quad i \times k = -j$

   $i \times i = j \times j = k \times k = 0.$

Ex. Find the distance from $P(3,1,-2)$ to the line through $Q(2,5,1)$ and $R(-1,4,2)$

Solution



$$d = \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|}$$

$$\overrightarrow{QP} = \langle 3-2, 1-5, -2-1 \rangle = \langle 1, -4, -3 \rangle$$

$$\overrightarrow{QR} = \langle -1-2, 4-5, 2-1 \rangle = \langle -3, -1, 1 \rangle$$

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 1 & -4 & -3 \\ -3 & -1 & 1 \end{vmatrix} = -7i + 8j - 13k$$

$$\|\overrightarrow{QR} \times \overrightarrow{QP}\| = \sqrt{49+64+169} = \sqrt{282} = 16.79$$

$$\|\overrightarrow{QR}\| = \sqrt{9+1+1} = \sqrt{11} = 3.3$$

$$d = \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|} = \frac{16.79}{3.3} = 5.09 \text{ unit}$$

Ex Find the volume of box having adjacent sides AB, AC and AD where $A(2,1,-1)$, $B(3,0,2)$, $C(4,-2,1)$, $D(5,-3,0)$

Solution
$$a = \overrightarrow{AB} = \langle 1, -1, 3 \rangle$$
$$b = \overrightarrow{AC} = \langle 2, -3, 2 \rangle$$
$$c = \overrightarrow{AD} = \langle 3, -4, 1 \rangle$$

$$V = |(a \times b) \cdot c| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \\ 3 & -4 & 1 \end{vmatrix} = 5-4+3$$
$$= 4 \text{ unit}^3$$

Note: If volume of box is zero, vectors lie in same plane. (co-planner)

EXERCISES FOR VECTOR PRODUCT

1. Find Vector product /Cross product, u x v of vectors u and v

   i. u = < 2, 1, 3 > v = < 2, 2, 1 >
   ii. u = <-2, 0 2 > v = < 5, 4, -6 >
   iii. u = 4i v = 3j
   iv. u = i + j – k, v = j + k
   v. u = 6i + 3j +4k v = 2i –j –k
   vi. u = i + k v = j +2k

2. Find a unit vector orthogonal to

   i. u = < 3, 2, -1 >, v = < 2, 1, 3 >
   ii. u = i – j + k , v = 3i + j – k

3. Given points P, Q and R, find

   (a) A unit vector perpendicular to the plane determined by P, Q and R,
   (b) The area of the triangle PQR,
   (c) The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$
   (d) The component of $\overrightarrow{PQ}$ along $\overrightarrow{PR}$ and
   (e) The distance from R to the line through P and Q

   i. P(1, -1 ,0), Q(2, 1, 1) and R(-1, 1, 2)
   ii. P(5, 3, 3), Q(6, 0, 1) and R(7, 4, 1)
   iii. P(2, -1 ,1), Q(-3, 2,0) and R(4, -5,3)

4. Find a vector orthogonal to the plane determined by the points P, Q and R and also find the area of triangle PQR where P,Q and R are as follow:

   i. P(-1, 2 ,0), Q(0, 2, -3) and R(4,1 ,-1)
   ii. P(1, -2 ,-2), Q(1, 2, -2) and R(3,-2, 1)
   iii. P(0, 0 ,0), Q(2, 1, 3) and R(3,-1,-3)
   iv. P(4, 0 ,4), Q(0, 3, 2) and R(3,1,0)

5. Find the volume of the parallelepiped having adjacent sides AB, AC and AD, where

   i. A (0, 0 ,0), B(2, -2, 3), C(1, 1, -1) and D (4, -1, -1)
   ii. A (1,-1 ,1), B(3, 4,-5), C(-2, 1, 1) and D (1, 2, 4)
   iii. A (1, 0 ,-1), B(2, 3, 5), C(0, -1, -2) and D (4, 1, -1)
   iv. A (1, 1 , 1), B(2, 0, 3), C(4, 1, 7) and D (3, -1, -2)

6. Find a.(b x c), where

i. a = < 1, 2, 0 >, b = <2, 0, 1 > and c = <1, 2, 1 >
ii. a = i + j + k, b = 2i + 3j – k and c = i – j + k

7. Determine the volume of the parallelepiped having adjacent sides formed by the vectors

i. a = 2i – 3j , b = i + 3j –k and c = 3i + 2j +k
ii. a = < 2, -1, 5 >, b = < 4, 6, 0 > and c = < 0, 0, 3 >
iii. a = 3j , b = -2i +j and c = -i + j + k
iv. a = 3i – 4j +k , b = 4i +7j +2k and c = i –2k

8. Show that the quadrilateral with vertices at points P, Q, R and S is a parallelogram and find its area:

i. P(4, 1, -1), Q(1, 5, 0), R(1, 3, 6) and S(4, -1, 5)
ii. P(3, 0, 2), Q(6, 2, 5), R(1, 2, 2) and S(4, 4, 5)

9. Use cross product to find the distance between point P(2, 1, 3) to the line through points R(3, 2, 1) and S(1, 2, 3).

10. Use dot product to find the distance between point P(2, 1, 3) to the line through points R(3, 2, 1) and S(1, 2, 3).

11. Given three points A(1, 1, 1) , B(0, 3, 0) and C(2, 3, 1) , find
    (a) A vector orthogonal to $\overrightarrow{AB}$,
    (b) The projection of $\overrightarrow{AB}$ in the direction of $\overrightarrow{Ac}$,
    (c) The angle between $\overrightarrow{AB}$ and $\overrightarrow{Ac}$,
    (d) A unit vector orthogonal to both $\overrightarrow{AB}$ and $\overrightarrow{Ac}$,
    (e) The area of the parallelogram formed by $\overrightarrow{AB}$ and $\overrightarrow{Ac}$ and
    (f) The area of the triangle ABC.

12. A force F = 2i + 3j – k is applied at the point A (2, 1, -1). Find moment of the force about the point B (3, 1, -2). Note: moment of the force M = AB x F

13. Find the vector X and scalar λ which satisfy the equations
    $A \times X = B + \lambda A$ and $A \,.\, X = 4$
    where $A = 2i + j - k$, $B = i - 2j + 3k$

14. Prove that the vectors are coplaner a = < 2, 3, 1 > , b = < 1, 0, 2 > and c = < 0, 3, -3 >.

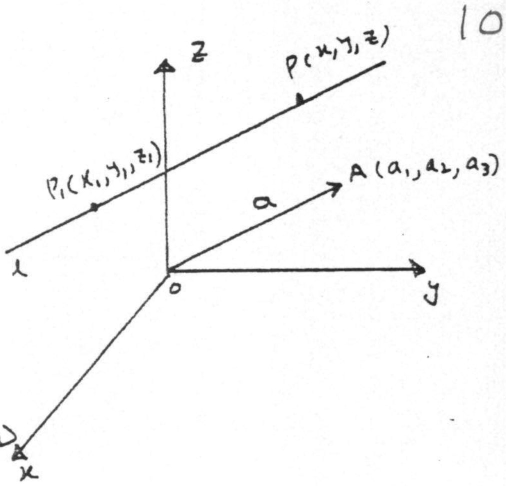
15. If a = < -1, 1, 1 > , b = < 2, 3, -1 > and c = < 1, -1, +2 >, find the following
    (i) (2a)xb (ii) (a+b) x c (iii) (a x b) x c (iv) a . (b x c) (v) (3a + b - 2c) x c

16. Determine if the three vectors
    $a =< 1,4,-7 >,\ b =< 2,-1,4 >\ and\ c =< 0,-9,18 >$ lie in the same plane .

10.5 Line and Plane

Equation of a line

The line l through $P_1(x_1, y_1, z_1)$ is parallel to vector a



$\Rightarrow$ $\overrightarrow{P_1P}$ is parallel to $\overrightarrow{OA}$

$$\overrightarrow{P_1P} = t\,\overrightarrow{OA}$$

$$\langle x - x_1 , y - y_1 , z - z_1 \rangle = t \langle a_1, a_2, a_3 \rangle$$

Equations of the Line through the point $P_1(x_1,y_1,z_1)$ and parallel to vector $a = \langle a_1,a_2,a_3 \rangle$ is

Parametric Form

$x = x_1 + a_1t$
$y = y_1 + a_2t$
$z = z_1 + a_3t$, $t \in R$.

Symmetric Form

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3} = t \quad , t \in R.$$

Ex. Find equation of the line passing Point $P(4,3,2)$ and parallel to vector $a = i + 2j + 3k$.

Solution. Parametric Form
$x = 4 + t$
$y = 3 + 2t$
$z = 2 + 3t$

Symmetric Form $\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3}$

Ex. Find equation of line passing through $P_1(-3,1,-1)$ and $P_2(7,11,-8)$

<u>Solution</u>

$$\overrightarrow{P_1P_2} = \langle 7+3\,,\,11-1\,,\,-8+1\rangle$$
$$= \langle 10, 10, -7\rangle$$

Eq. of line passing through point $P_1(-3,1,-1)$ or $P_2(7,11,-8)$ and parallel to vector $\overrightarrow{P_1P_2} = \langle 10, 10, -7\rangle$ is

Parametric form

$$x = -3 + 10t \qquad \text{or} \qquad x = 7 + 10t$$
$$y = 1 + 10t \qquad\qquad\quad y = 11 + 10t$$
$$z = -1 - 7t \qquad\qquad\quad z = -8 - 7t$$

Symmetric form
$$\frac{x+3}{10} = \frac{y-1}{10} = \frac{z+1}{-7}$$

or
$$\frac{x-7}{10} = \frac{y-11}{10} = \frac{z+8}{10}$$

Ex. If l has parametric equations
$$x = 5 - 3t$$
$$y = -2 + t$$
$$z = 1 + 9t,$$

find parametric equations for the line through $P(-6,4,-3)$ that is parallel to l.

Solution. Vector parallel to line l: $a = \langle -3, 1, 9\rangle$

Parametric equations passing through point $P(-6,4,-3)$ and parallel to vector $a = \langle -3, 1, 9\rangle$ are

$$x = -6 - 3t$$
$$y = 4 + t$$
$$z = -3 + 9t$$

Example. (a) Find parametric equations for the line, ℓ, passin through the points $A(-3, 2, 1)$, $B(1, 3, 2)$.

(b) At what point does 'ℓ' intersect xy-plane?

Solution

.a. Direction of line 'ℓ' is same as the vector

$$\overrightarrow{AB} = \langle 1+3,\ 3-2,\ 2-1 \rangle = \langle 4, 1, 1 \rangle$$

Equation of line passing through point $A(-3, 2, 1)$ and parallel to vector $\overrightarrow{AB} = \langle 4, 1, 1 \rangle$ is

$$x = -3 + 4t, \quad y = 2 + t, \quad z = 1 + t, \quad t \in R.$$

.b. At the point where the line intersects the xy-plane, is when $z = 0$

From equation of line $1 + t = 0 \Rightarrow t = -1$.

Substituting this value into the parametric equation, we obtained the required point

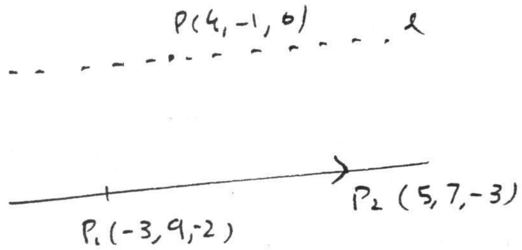
$$x = -3 - 4 = -7$$

$$y = 2 - 1 = 1$$

$$z = 1 - 1 = 0$$

Point is $(-7, 1, 0)$.

Ex. Find parametric equations for the line through the point $P(4,-1,0)$ that is parallel to the line through the points $P_1(-3, 9, -2)$ and $P_2(5, 7, -3)$



Solution

$$\overrightarrow{P_1P_2} = \langle 5+3, 7-9, -3+2 \rangle$$
$$= \langle 8, -2, -1 \rangle$$

Eq. of line passing through $P(4,-1,0)$ and parallel to vector $\overrightarrow{P_1P_2} = \langle 8, -2, -1 \rangle$ is

$$x = 4 + 8t$$
$$y = -1 - 2t$$
$$z = 0 - t$$

Orthogonal and Parallel lines

Let a and b be direction vectors for lines l_1 and l_2,

(i) l_1 and l_2 are orthogonal if $a \cdot b = 0$, and

(ii) l_1 and l_2 are parallel if $a = kb$, where k is scalar.
or $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = k$.

Ex. Determine whether the lines

l_1 : $x = 4 - 2t$, $y = 1 + 4t$, $z = 3 + 10t$

l_2 : $x = v$, $y = 6 - 2v$, $z = \frac{1}{2} - 5v$ are parallel.

Solution.

$$a = \langle -2, 4, 10 \rangle$$
$$b = \langle 1, -2, -5 \rangle$$

Since $\frac{-2}{1} = \frac{4}{-2} = \frac{10}{-5}$ $\Rightarrow$ lines are parallel

or

$a = -2b$ $\Rightarrow$ lines are parallel.

Ex. Determine whether the lines

$l_1: \quad x = -6 - t, \quad y = 10 + 3t, \quad z = 3 + 2t$

$l_2: \quad x = 3 + 2v, \quad y = -5 - 4v, \quad z = -1 + 7v$

are orthogonal.

Solution. vectors are

$$a = \langle -1, 3, 2 \rangle$$
$$b = \langle 2, -4, 7 \rangle$$

$$a \cdot b = -2 - 12 + 14 = 0$$

$\Rightarrow$ l_1 and l_2 are orthogonal.

Ex. Determine whether the lines

$l_1: \quad x = 1 - 6t, \quad y = 3 + 2t, \quad z = 1 - 2t$

$l_2: \quad x = 2 + 2v, \quad y = 6 + v, \quad z = 2 + v$

intersect, and if so, find the point of intersection.

Solution. Let the point of intersection be $P_0(x_0, y_0, z_0)$ is common to both lines, we must have

$$\begin{aligned} 1 - 6t_0 &= 2 + 2v_0 \\ 3 + 2t_0 &= 6 + v_0 \\ 1 - 2t_0 &= 2 + v_0 \end{aligned} \Rightarrow \begin{aligned} 6t_0 + 2v_0 &= -1 \quad \rightarrow 1 \\ 2t_0 - v_0 &= 3 \quad \rightarrow 2 \\ 2t_0 + v_0 &= -1 \quad \rightarrow 3 \end{aligned}$$

We solve any two the equations simultaneously and use the remaining equation as check. taking Eq 2. and Eq. 3.

$E_2 + E_3 \Rightarrow 4t_0 = 2$ or $t_0 = \frac{1}{2}$

$E_2 - E_3 \Rightarrow -2v_0 = 4$ or $v_0 = -2$

substituting in Eq. 1 $\quad 6(\frac{1}{2}) + 2(-2) = 3 - 4 = -1$

$\Rightarrow$ l_1 and l_2 intersects.

To find point of intersection, we substitute $v_0 = -2$ in l_2

$$\Rightarrow \begin{aligned} x_0 &= -2 \\ y_0 &= 4 \\ z_0 &= 0 \end{aligned} \Rightarrow \text{Point of intersection is } P_0(-2, 4, 0)$$

///

EXAMPLE 7 Finding the Distance Between a Point and a Line

Find the distance between the point $Q(3, -1, 4)$ and the line given by

$$x = -2 + 3t, \quad y = -2t, \quad \text{and} \quad z = 1 + 4t.$$

Solution Using the direction numbers 3, -2, and 4, you know that the direction vector for the line is

$$\mathbf{u} = \langle 3, -2, 4 \rangle. \qquad \text{Direction vector for line}$$

To find a point on the line, let $t = 0$ and obtain

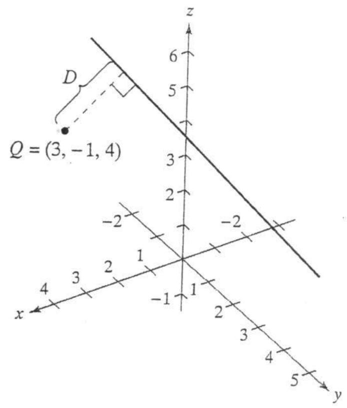
$$P = (-2, 0, 1). \qquad \text{Point on the line}$$

Thus,

$$\overrightarrow{PQ} = \langle 3 - (-2), -1 - 0, 4 - 1 \rangle = \langle 5, -1, 3 \rangle$$

and you can form the cross product

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = 2\mathbf{i} - 11\mathbf{j} - 7\mathbf{k} = \langle 2, -11, -7 \rangle.$$



The distance between the point Q and the line is $\sqrt{6} \approx 2.45$.
Figure 10.55

Finally, using Theorem 10.14, you can find the distance to be

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{174}}{\sqrt{29}} = \sqrt{6} \approx 2.45. \qquad \text{(See Figure 10.55.)}$$

Ex. Find angle between l_1 and l_2

l_1 : $x = 5 + 3t$, $y = 4 - t$, $z = 3 + 2t$

l_2 : $x = -t$, $y = 1 - 2t$, $z = 3 + t$

Solution. Angle between lines is angle between their parallel vectors.

$a = \langle 3, -1, 2 \rangle$ $\qquad \|a\| = \sqrt{9 + 1 + 4} = \sqrt{14}$

$b = \langle -1, -2, 1 \rangle$ $\qquad \|b\| = \sqrt{1 + 4 + 1} = \sqrt{6}$

$$\theta = \cos^{-1} \frac{a \cdot b}{\|a\|\|b\|} = \cos^{-1} \frac{-3+2+2}{\sqrt{14}\sqrt{6}} = \cos^{-1} \frac{1}{\sqrt{14}\sqrt{6}} \approx 84^0$$

EXERCISES 3
FOR LINE

1. Find equation of the line through the point P(3,-1,4) and parallel to vector a = < 2, 1, 7>
2. Find equation of the line through the point P(2,1,5) and parallel to vector a = 3i +3j – 5k.

3. Find equation of the line through the point P(0,1,2) and parallel to vector a = < 2, 0, 3>.

4. Find equation of the line passing through the points P(2,1,-3) and Q(5,-1,4) .

5. Find distance from the point P to the line l
   i. P(3,-1,2) line: x =-2+3t, y = 2, z = -1 +2t.
   ii. P(2,1,-1) line: x = 3t, y = 1+2t, z = -5 – t.
   iii. P(1,2,3) line: $\frac{x+1}{2} = \frac{y-3}{-1} = \frac{z-4}{3}$.

6. Find the shortest distance between lines;

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z}{2} \quad \text{and} \quad \frac{x}{1} = \frac{y-1}{2} = \frac{z}{2}$$

7. Find the distance of the point (1,2,0) from the line joining the points (0,1,-1) and (2,0,-2).

8. Find the parametric equations of the line l_1 through the point (5,3,5) an intersecting at right angle the line l_2 whose parametric equations are x = 4+3t, y = 1 + t and z = -3t.

9. Find the perpendicular distance between the skew lines :
   L_1: through points A(-2,1,-1) and B(0,4,-2)
   L_2: through points C(1,-1,2) and D(0,1,6).

10. Let L_1 be the line through points A(2,1,5) and B(1,2,3) and let L_2 be the line through C(2,0,1) and D(4,-3,0), find the shortest distance between the two skew lines L_1 and L_2.
11. Determine whether the following line are parallel or they intersect. If they intersect then find the point of intersection:
   (i) Line1: x = 3 + t, y = 5 - t, z = -2 +2t.
   Line2: x = 2 + v, y = 3 – 2v, z = -1 + 3v.

(ii) Line1: $x = 4 + t$, $y = 5 + t$, $z = -1 + 2t$.
Line2: $x = 5 + 2v$, $y = 11 + 4v$, $z = 3 + v$.

(iii) Line1: $x = 5 + 2t$, $y = 3 - t$, $z = 1 - 2t$.
Line2: $x = 11 + 7v$, $y = 8v$, $z = 8 - 2v$.

(iv) Line1: $x = t + 2$, $y = 2t - 1$, $z = 3t + 3$
Line2: $x = 3s + 5$, $y = 2s + 1$, $z = s + 4$

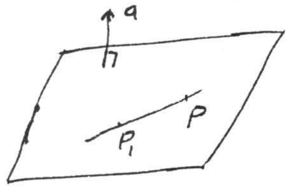
11. Find the distance from the point (1, 0, 2) to the line through the point (1, -1 ,1) in the direction of the vector $i - 2j - 2k$.

Equation of the Plane

Let $P_1(x_1, y_1, z_1)$ lies on the plane and a non-zero vector $a = \langle a_1, a_2, a_3 \rangle$ is normal to the plane

Let $P(x, y, z)$ be any point in the plane, then

$$\overrightarrow{P_1P} = \langle x - x_1, y - y_1, z - z_1 \rangle$$



Now $a \cdot \overrightarrow{P_1P} = 0$

$$\Rightarrow \langle a_1, a_2, a_3 \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$$

is equation of the plane.

Ex. Find an equation of the plane through the point $P(-11, 4, -2)$ with normal vector $a = 6i - 5j - k$

Soln.

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$$

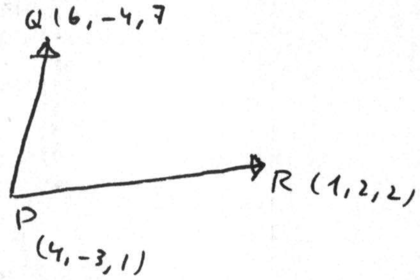
$$6(x + 11) - 5(y - 4) - (z + 2) = 0$$

$$6x + 66 - 5y + 20 - z - 2 = 0$$

$$6x - 5y - z + 84 = 0$$

Ex. Find an equation of the plane determined by the Points $P(4,-3,1)$, $Q(6,-4,7)$, and $R(1,2,2)$

Soln.



$\overrightarrow{PQ} = \langle 2,-1,6 \rangle$

$\overrightarrow{PR} = \langle -3,5,1 \rangle$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is normal to the plane determined by P, Q and R

$$n = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix} = -31i - 20j + 7k$$

Eq. of plane with $P(4,-3,1)$ and normal vector $\hat{n}$

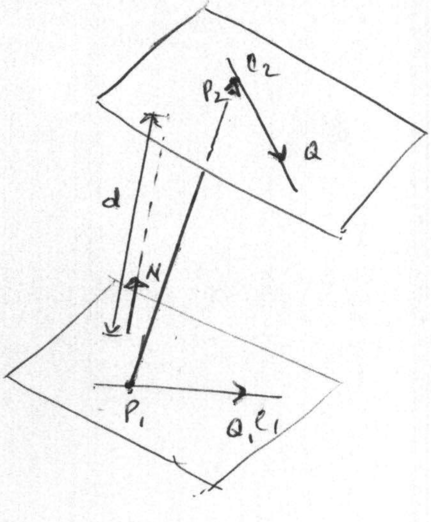
$$-31(x-4) - 20(y+3) + 7(z-1) = 0$$

$$-31x - 20y + 7z + 57 = 0$$

$\boxed{\text{Note 1}}$ Distance from a point $P(x_0, y_0, z_0)$ to the plane $ax+by+cz+d=0$

$$h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

$\boxed{\text{Note.2}}$ Shortest distance d between skew lines l_1 and l_2



$$d = \frac{1}{\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\|} |(\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}) \cdot \overrightarrow{P_1P_2}|$$

Note.3 Planes P_1 and P_2 are orthogonal if $n_1 \cdot n_2 = 0$

Note.4 Planes P_1 and P_2 are parallel if $n_1 = k n_2$

Ex Show that planes $4x-2y+6z=3$ and $-6x+3y-9z=4$ are paralled and find the distance between the planes.

Solution. Two planes are parallel, when vectors normal to the planes are parallel.

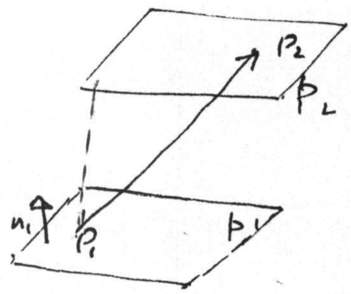
vector normal to plane P_1 is $n_1 = \langle 4,-2,6\rangle$

vector normal to plane P_2 is $n_2 = \langle -6,3,-9\rangle$

n_1 and n_2 are parallel because

$$\frac{4}{-6} = -\frac{2}{3} = \frac{6}{-9} \quad \text{or} \quad n_2 = -\frac{3}{2}n_1$$

distance between planes P_1 and P_2 will be $\text{Comp}_{n_1}\overrightarrow{P_1P_2}$



To find P_1, Let $x=1,\ y=1 \Rightarrow z=\frac{1}{6}$

To find P_2 Let $x=1,\ y=1 \Rightarrow z=-\frac{7}{9}$

2nd method

$P_2(0,0,-\frac{4}{9})$

$$d = \frac{|6\cdot(-\frac{4}{9})-3|}{\sqrt{16+4+36}} = \frac{17}{3\sqrt{56}} = \frac{17}{6\sqrt{14}}$$

$P_1(1,1,\frac{1}{6})$ on P_1

$P_2(1,1,-\frac{7}{9})$ on P_2

$\overrightarrow{P_1P_2} = \langle 0,0,\frac{17}{18}\rangle$

$\hat{n} = \langle 4,-2,6\rangle$

$$h = \text{Comp}_n\overrightarrow{P_1P_2} = \frac{|\overrightarrow{P_1P_2}\cdot n|}{|n|} = \frac{17/3}{\sqrt{16+4+36}} = \frac{17}{6\sqrt{14}}$$

Ex. Find the distance from $P(1,-1,2)$ to the plane $3x-7y+z-5=0$

Solution

$$h = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$$

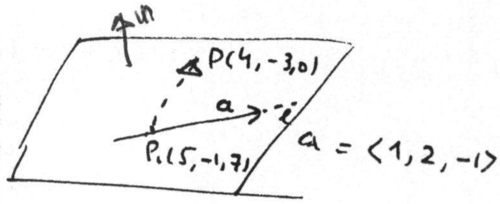
$$= \frac{|3(1)-7(-1)+1(2)-5|}{\sqrt{9+49+1}} = \frac{7}{\sqrt{59}}$$

Ex Find the equation of the plane that contains the point $P(4,-3,0)$ and the line $x = t+5,\ y = 2t-1,\ z = -t+7$

Solution.

vector $a = \langle 1, 2, -4 \rangle$ is parallel to line l

and $P_1\ (5,-1,7)$ is point on the line.



$$\overrightarrow{P_1P} = \langle 5-4, -1+3, 7-0 \rangle = \langle 1, 2, 7 \rangle$$

A vector normal to the plane is $n = a \times \overrightarrow{P_1P}$

$$n = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & 2 & 7 \end{vmatrix} = 16i - 8j + 0k$$

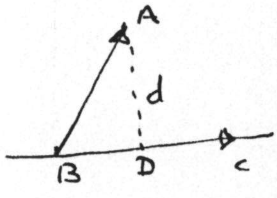
Eq. of the plane containing point $P(4,-3,0)$ with normal vector $n = \langle 16, -8, 0 \rangle$ is

$$16(x-4) - 8(y-3) + 0(z-0) = 0$$

$$16x - 8y - 88 = 0$$

Ex. Use dot product to find the distance from $A(2,-6,1)$ to the line through $B(3,4,-2)$ and $C(7,-1,5)$

Solution. $\overrightarrow{BA} = \langle -1, -10, 3 \rangle,\quad \overrightarrow{BC} = \langle 4, -5, 7 \rangle$



$$|BD| = \mathrm{Comp}_{\overrightarrow{BC}} \overrightarrow{BA} = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BC}\|} = \frac{67}{\sqrt{90}}$$

$$\|\overrightarrow{BA}\| = \sqrt{110}$$

$$AD = \sqrt{\|\overrightarrow{BA}\|^2 - \|\overrightarrow{BD}\|^2} = \sqrt{110 - \frac{(67)^2}{90}} = \sqrt{\frac{5411}{90}} \doteq 7.75$$

Note. The graph of every linear equation $ax + by + cz + d = 0$ is a plane with normal vector $\langle a, b, c \rangle$

Ex. Find parametric equations for the line of intersection of the planes

$$x + 2y - 9z = 7 \quad \rightarrow P_1$$
$$2x - 3y + 17z = 0 \quad \rightarrow P_2$$

Solution. Vector n_1 normal to P_1 is $\langle 1, 2, -9 \rangle$

vector n_2 normal to P_2 is $\langle 2, -3, 17 \rangle$

vector parallel to line of intersection of planes is $n_1 \times n_2$

$$a = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -9 \\ 2 & -3 & 17 \end{vmatrix} = 7i - 35j - 7k$$

To find a point on the line of intersection of P_1 and P_2, let $z = 0$

$$x + 2y = 7 \quad \rightarrow E_1$$
$$2x - 3y = 0 \quad \rightarrow E_2$$

$-2E_1 + E_2$
$$-2x - 4y = -14$$
$$2x - 3y = 0 \quad \Rightarrow \quad -7y = -14 \Rightarrow y = 2$$
$$x = 7 - 2y$$
$$x = 7 - 4 = 3$$

Point is $(3, 2, 0)$.

Equation of line through point $(3, 2, 0)$ parallel to vector $a = \langle 7, -35, -7 \rangle$ is
$$x = 3 + 7t$$
$$y = 2 - 35t$$
$$z = -7t$$

<u>METHOD 2.</u> by Linear Algebra

$$\begin{bmatrix} 1 & 2 & -9 & 7 \\ 2 & -3 & 17 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -9 & 7 \\ 0 & -7 & 35 & -14 \end{bmatrix}_{-2R_1+R_2}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -5 & 2 \end{bmatrix} \begin{matrix} -2R_2' + R_1 \\ -\frac{1}{7}R_2 \end{matrix}$$

$$x + z = 3 \qquad x = 3 - z$$
$$y - 5z = 2 \qquad y = 2 - 5z$$

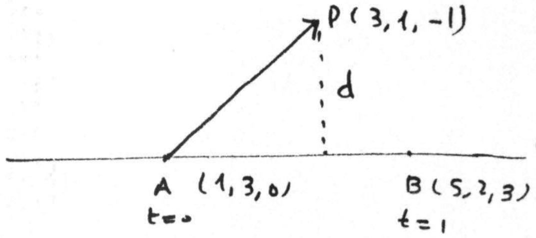
Let $z = t$

$$x = 3 - t$$
$$y = 2 - 5t$$
$$z = t$$

are parametric equations of the line of intersection.

Ex. Find the distance from the point $P(3,1,-1)$ to the line: $x = 1+4t,\ y = 3-t,\ z = 3t$.

Solution.



$$d = \frac{\|\overrightarrow{AB} \times \overrightarrow{AP}\|}{\|\overrightarrow{AB}\|}$$

$\overrightarrow{AB} = \langle 4, -1, 3\rangle$

$\overrightarrow{AP} = \langle 2, -2, -1\rangle$

$$\overrightarrow{AB} \times \overrightarrow{AP} = \begin{vmatrix} i & j & k \\ 4 & -1 & 3 \\ 2 & -2 & -1 \end{vmatrix} = 7i + 10j - 6k$$

$\|\overrightarrow{AB} \times \overrightarrow{AP}\| = \sqrt{49+100+36} = \sqrt{185}$

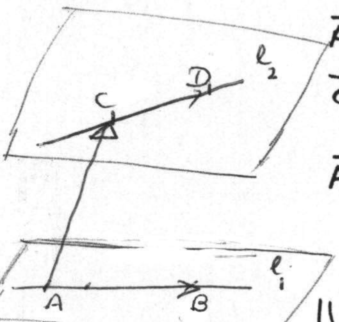
$\|\overrightarrow{AB}\| = \sqrt{16+1+9} = \sqrt{26}$

$$d = \sqrt{\frac{185}{26}} = 2.67$$

Ex. Find the shortest distance between the lines l_1 through the points $A(1,-2,3)$, $B(2,0,5)$ and line l_2 through the points $C(4,1,-1)$, $D(-2,3,4)$. l_1 and l_2 are skew lines

Soln. Two lines are skew if they are not paralled and do not intersect.

$$h = \frac{|(\overrightarrow{AB} \times \overrightarrow{CD}) \cdot \overrightarrow{AC}|}{\|\overrightarrow{AB} \times \overrightarrow{CD}\|}$$



$\overrightarrow{AB} = \langle 1, 2, 2\rangle$

$\overrightarrow{CD} = \langle -6, 2, 5\rangle$

$\overrightarrow{AC} = \langle 3, 3, -4\rangle$

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ -6 & 2 & 5 \end{vmatrix} = 6i - 17j + 14k$$

$\|\overrightarrow{AB} \times \overrightarrow{CD}\| = \sqrt{36+289+196} = \sqrt{521} = 22.8$

$(\overrightarrow{AB} \times \overrightarrow{CD}) \cdot \overrightarrow{AC} = \langle 6, -17, 14\rangle \cdot \langle 3, 3, -4\rangle = 18 - 51 - 56 = -89$

$$h = \frac{|-89|}{22.8} = \frac{89}{22.8} = 3.9$$

EXERCISES FOR PLANE

1. Find the equation of the plane containing point P(3,6,1) and a vector perpendicular to plane is $a = 2i + 8j - 3k$.

2. Find the equation of the plane containing point P(2,0,3) and a vector perpendicular to plane is $a = \langle 2, 5, -1 \rangle$.

3. Find the equation of the plane containing point P(2,0,-5) and perpendicular to the line $x = 2 + 4t$, $y = 3$, $z = -1+t$.

4. Find the equation of the plane containing point P(1,-1,2) and

   the line $x = -2 + t$, $y = -1+t$, $z = -5+2t$.

5. Find the equation of the plane containing line $x = 4t + 1$, $y = -6t+3$, $z = 4$ and is perpendicular to the plane $2x + 3y - 4z = 5$.

6. Find the equation of the plane containing point P(2,5,0) and perpendicular to the line $\frac{x+2}{3} = \frac{y-4}{-1} = \frac{z-7}{4}$.

7. Given points P,Q and R,

   (a) Find the equation of the plane containing these points,
   (b) Find the equation of the orthogonal line to the plane and
   (c) If T(2,3,1) is a point not on the plane then find its distance from the plane:

   i. P(2,2,1), Q(3,1,5) and R(3,3,4).
   ii. P(2,1,1), Q(-3,1,5) and R(3,2,0).

8. Given point P(2,2,1), and line $x = 2 + t$, $y = 3 - 2t$, $z = 4$,

   (a) Find the equation of the plane containing the point and the line,
   (b) Find the equation of the line orthogonal to the plane and
   (d) If T(4,3,-1) is a point not on the plane then find its distance from the plane.

In Exercise 9 to 14, find the equation of the plane :

9. containing lines $\frac{x-1}{2} = \frac{y}{1} = \frac{z+1}{2}$ and $\frac{x+1}{2} = \frac{y-2}{2} = \frac{z-1}{3}$.

10. passing through the points (1,0,2), (-1,3,4) and (3,5,7).

11. passing through the points (-1,1,2), (1,-2,1) and (2,2,4).

12. Passing through the point P(1,4,2) and is parallel to the plane z=3.

13. Passing through the points (2,1,2) and (1,-1,0) and is perpendicular to the plane $x-2y+z=4$.

14. Passing through the points (2,3,5) and (0,0,1) and is parallel to y – axis.

15. Find the equation of the plane passing through the point (1,2,3) and parallel to the plane $x+y+z=1$. Also find the distance between these two parallel planes.

16. Given the planes p_1: $3x-6y+2z=7$
p_2: $2x+y-2z=5$
 (a) Find a vector parallel to the line of intersection of the planes,
 (b) Find the angle between of the planes, and
 (c) Find the equation of the plane passing through the line of intersection of the above planes and through the point (0,0,1)

17. Find the distance of the point P(2,3,-1) from the plane $3x+y+5z=12$.

18. Find the distance of the point P(2,-1,3) from the plane $3x-4y+5z=-6$.

19. Consider the line L with the parametric equations $x=0$, $y=t$, $z=t$,

 i. Show that L lies in the plane $6x+4y-4z=0$,
 ii. Show that L is parallel to the plane $5x-3y+3z=1$, and also find whether L lies above or below the plane.

20. Find the distance between the planes:
p_1: $2x-4y+8z=7$
p_2: $x-2y+4z=0$.

21. Find the angle between the planes:
p_1: $2x-5y-3z=8$
p_2: $3x+2y-4z=7$.

22. Find the point intersection and angle of intersection of the plane $x+2y+z=2$ and line $\frac{x-2}{3}=\frac{y+2}{2}=\frac{z-1}{2}$.

23. Find the parametric equations of the line of intersection of the planes: $x-2y+4z=2$, $x+y-2z=5$.

24. Find the parametric equations of the line of intersection of the planes: $2x+3y-4z-6=0$, $3x-y+2z+4=0$.

25. Show that the line $x=1+2t$, $y=-1+2t$, $z=2+4t$ is parallel to the plane $x-2y+z=6$.

26. Determine whether the line $x = 3 + 8t,\ y = 4 + 5t,\ z = -3-t$ and the plane $x - 3y + 5z = 12$ are parallel, if the above line and the plane are not parallel find their point of intersection.

27. Let the equation of the plane P_1 be $3x - y - z = 2$
   i)Find the equation of the line orthogonal to the plane P_1 and passing through the point (2,-3,6),
   ii)Find the equation of the plane P_2 passing through the point (2,-1,2) and parallel to plane P_1.
   iii)Find the distance between the planes P_1 and P_2.

28. Find the equation of the plane passing through the point (1, 2, 3) and parallel to plane $x + y + z = 1$. Also find the distance between these planes.

29. Show that the line $x = 1 + 2t,\ y = -1 + 6t,\ z = 3 - 8t$ is parallel to the plane $x + 3y - 4z = 5$.

30. Find the equation of the line through the origin and perpendicular to the plane $x + y + z = 1$.

31. Find the point, if any, at which the line
$$x=2+3t, \quad y=-4t, \quad z=5+t, \quad t \in R$$
intersects the plane $4x+5y-2z=18$.

32. Find the point, if any, at which the line
$$\frac{x+1}{3}=\frac{y-2}{4}=\frac{z-1}{1}$$
intersects the plane $x-y+2z=18$.

33. If the line $\frac{x}{3}=\frac{y}{5}=\frac{z}{3}$ is perpendicular to a plane which contains the line $x=1-4t,\ y=3t,\ z=2-t$, find the equation of that plane.

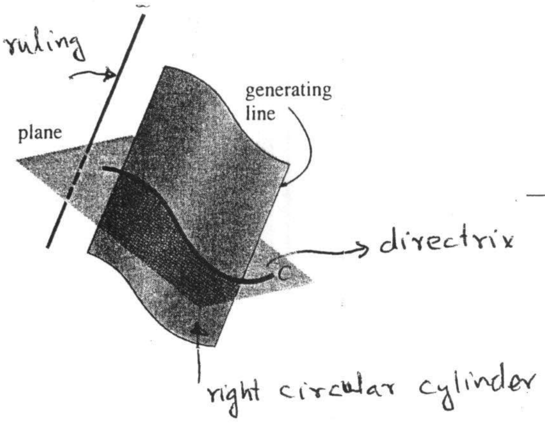
34. Find the equation of the line that passes through the point (1,2,-3) and is perpendicular to the plane $x+3y-2z=4$.

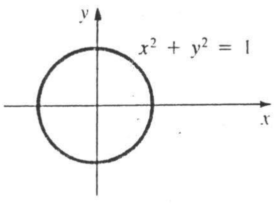
10.6 SURFACES

Cylinder

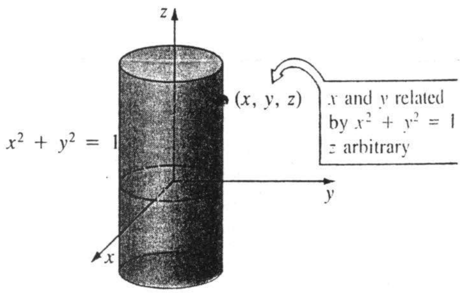
Def.

Let C be a curve in a plane, and let l be a line that is not in a parallel plane. The set of points on all lines that are parallel to l and intersect C is a cylinder.



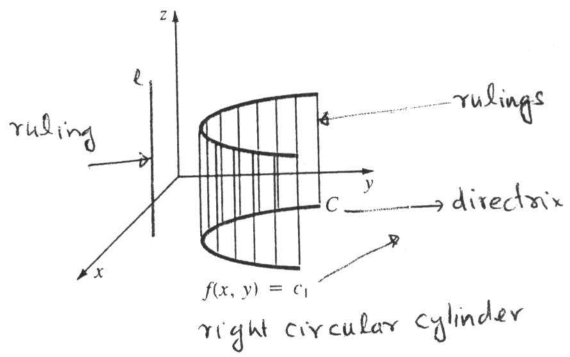


(a) circle

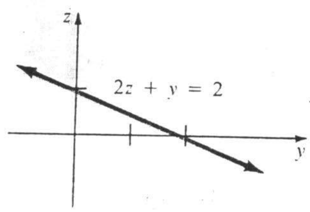


(b) circular cylinder

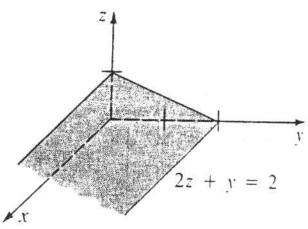
Figure 14.63



Example. 1

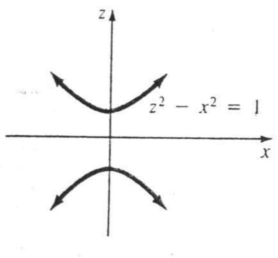


(a) line

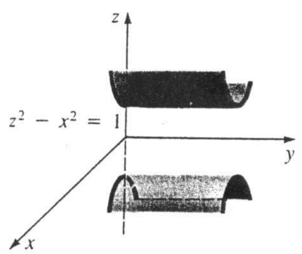


(b) plane

Example. 2

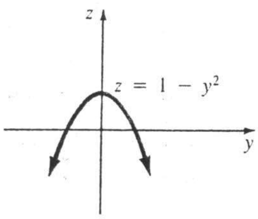


(a) hyperbola

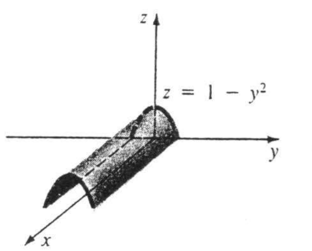


(b) hyperbolic cylinder

Example. 3



(a) parabola



(b) parabolic cylinder

Sketching Planes in Space

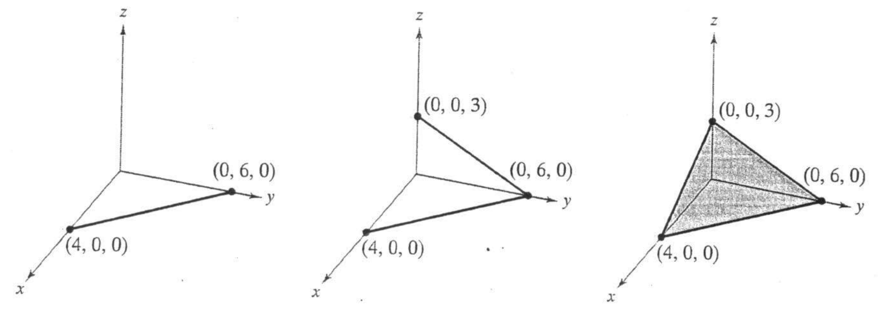
If a plane in space intersects one of the coordinate planes, we call the line of intersection the **trace** of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane given by

$$3x + 2y + 4z = 12. \qquad \text{Equation of plane}$$

We find the xy-trace by letting $z = 0$ and sketching the line

$$3x + 2y = 12 \qquad xy\text{-trace}$$

in the xy-plane. This line intersects the x-axis at $(4, 0, 0)$ and the y-axis at $(0, 6, 0)$. In Figure 10.49, we continue this process by finding the yz-trace and the xz-trace, and then shading in the triangular region lying in the first octant.



xy-trace ($z = 0$): $3x + 2y = 12$

yz-trace ($x = 0$): $2y + 4z = 12$

xz-trace ($y = 0$): $3x + 4z = 12$

Traces of the plane $3x + 2y + 4z = 12$
Figure 10.49

If the equation of a plane has a missing variable such as $2x + z = 1$, the plane must be *parallel to the axis* represented by the missing variable, as shown in Figure 10.50. If two variables are missing from the equation of a plane, it is *parallel to the coordinate plane* represented by the missing variables, as shown in Figure 10.51.

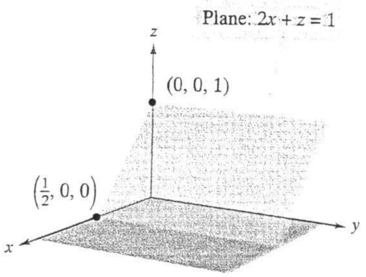
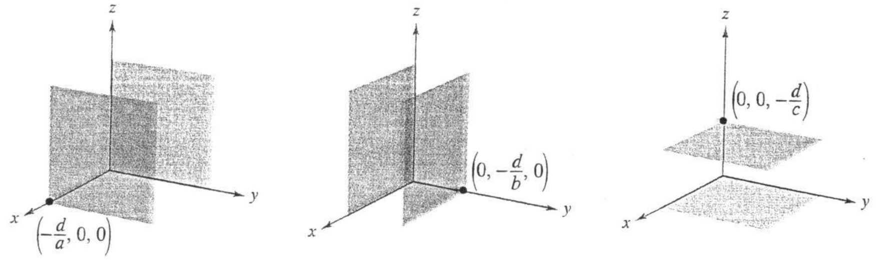


Figure 10.50



Plane $ax + d = 0$ is parallel to yz-plane.

Plane $by + d = 0$ is parallel to xz-plane.

Plane $cz + d = 0$ is parallel to xy-plane.

Figure 10.51

Quadric Surface

The graph of a second - degree equation in x, y and z,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fy + Gx + Hy + Iz + J = 0$$

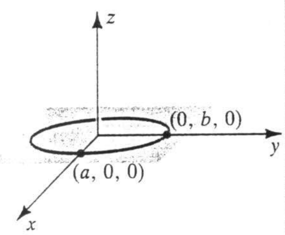
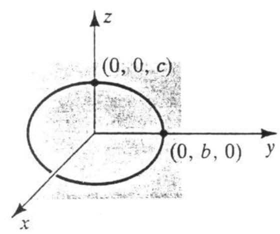
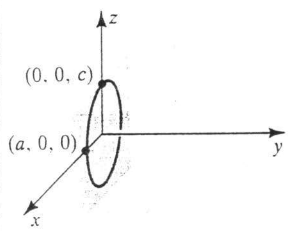
is a quadric surface.

There are three types of quadric surfaces:

- ellipsoids
- hyperboloids and
- paraboloids.

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Ellipse	
yz-trace	$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipse	
xz-trace	$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$	Ellipse	

Ex. Let a surface be $36x^2 + 16y^2 + 9z^2 = 144$

(i) write the name of the surface,

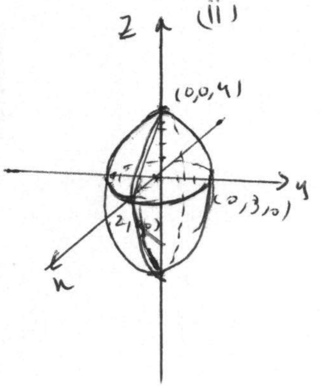
(ii) write the names and the equation of the traces of the surface on the coordinate planes and

(iii) Sketch the surface.

Solution. Dividing both sides of the equation by 144, we obtained

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

(i) This is an ellipsoid, axis is Z-axis.



(ii)

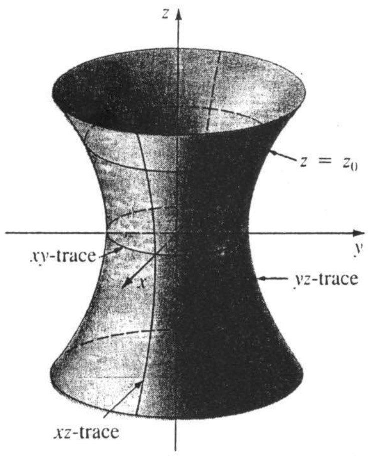
Trace	Equation of Trace	Discription
xy-plane ($z=0$)	$\frac{x^2}{4} + \frac{y^2}{9} = 1$	Ellipse
yz-plane ($x=0$)	$\frac{y^2}{9} + \frac{z^2}{16} = 1$	Ellipse
xz-plane ($y=0$)	$\frac{x^2}{4} + \frac{z^2}{16} = 1$	Ellipse

Hyperbolid of One Sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Coordinate plane	*Trace*
xy $(z=0)$	ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
xz $(y=0)$	hyperbola: $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$
yz $(x=0)$	hyperbola: $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

(a)



(b)

EXAMPLE ▪ Sketch the graph of $16x^2-9y^2+36z^2=144$, and identify the surface.

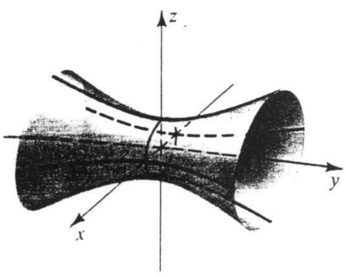
SOLUTION Dividing both sides of the equation by 144 leads to

$$\frac{x^2}{9}-\frac{y^2}{16}+\frac{z^2}{4}=1,$$

where the negative y-term indicates that the axis of this hyperboloid of one sheet is the y-axis. Traces in coordinate planes are as follows.

Trace	Equation of trace	Description of trace
xy-plane	$\frac{x^2}{9}-\frac{y^2}{16}=1$	Hyperbola
yz-plane	$\frac{z^2}{4}-\frac{y^2}{16}=1$	Hyperbola
xz-plane	$\frac{x^2}{9}+\frac{z^2}{4}=1$	Ellipse

$16x^2-9y^2+36z^2=144$



The graph of the equation is sketched in Figure 10.70. Traces in planes parallel to the xz-plane are ellipses, and traces in planes parallel to the xy- or yz-planes are hyperbolas.

Hyperbolid of Two Sheets

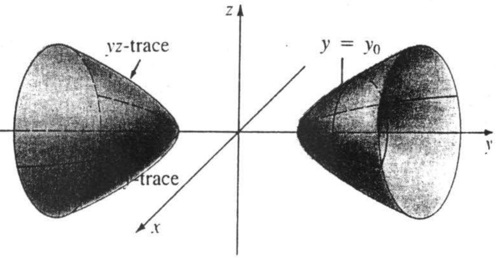
, a graph of

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad a>0,\ b>0,\ c>0$$

is appropriately called a **hyperboloid of two sheets**.

Coordinate plane	*Trace*
$xy\ (z=0)$	hyperbola: $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
$xz\ (y=0)$	no locus
$yz\ (x=0)$	hyperbola: $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

(a)



(b)

For $|y_0|>b$ the equation $\frac{x^2}{a^2}+\frac{z^2}{c^2}=\frac{y_0^2}{b^2}-1$ describes the elliptical curve of intersection of the surface with the plane $y=y_0$.

Ex. Sketch the graph of $3x^2-4y^2-z^2=12$, name the surface and describe the traces.

Solution. Dividing both sides of the equation by 12, we obtained

$$\frac{x^2}{4}-\frac{y^2}{3}-\frac{z^2}{12}=1$$

(i) This hyperboloid of two sheets. positive x-term indicates that axis of this hyperboloid of Two sheets is the x-axis

(ii)

Trace	Eq.	Discription
$xy\ (z=0)$	$\frac{x^2}{4}-\frac{y^2}{3}=1$	Hyperobola
$xz\ (y=0)$	$\frac{x^2}{4}-\frac{z^2}{12}=1$	Hyperbola
$yz\ (x=0$	$-\frac{y^2}{3}-\frac{z^2}{12}=1$	No

When $|x|>2$, equation $\frac{y^2}{3}+\frac{z^2}{12}=\frac{x^2}{4}-1$ describes the elliptical curves.

Cone

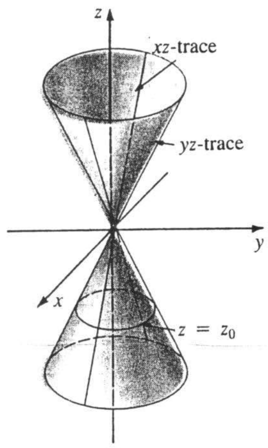
The graph of an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, \qquad a > 0,\ b > 0,\ c > 0 \qquad (14.58)$$

is called an **elliptical** (or circular if $a = b$) **cone**.

Coordinate plane	*Trace*
xy $(z = 0)$	point: $(0, 0)$
xz $(y = 0)$	lines: $z = \pm \frac{c}{a} x$
yz $(x = 0)$	lines: $z = \pm \frac{c}{b} y$

(a)



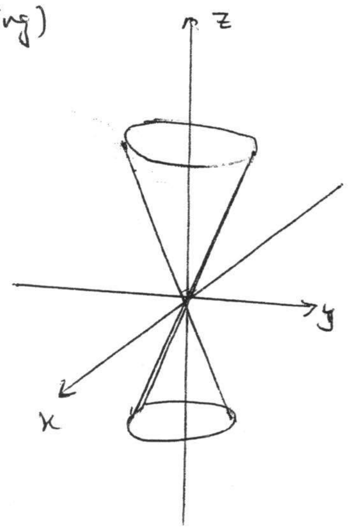
(b)

Ex. Name the surface $\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{4} = 0$, discribe the traces, sketch the graph.

Solution (i) This is a cone with the axis along the z-axis

Plane	Eq.	Trace
xy $(z=0)$	$(0,0)$	point
xz $(y=0)$	$z = \pm \frac{2}{3} x$	lines (intersecting)
yz $(x=0)$	$z = \pm y$	lines (intersecting)

$z = z_0$ are the ellipses.



Paraboloid

The graph of an equation of the form

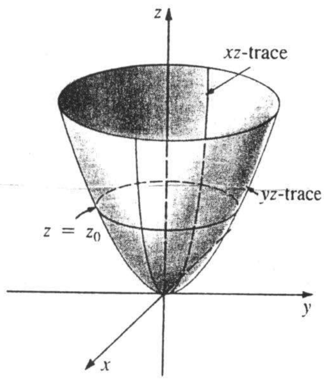
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$$

is called a **paraboloid**. In Figure 14.77(b) we see that for $c > 0$, plane $z = z_0 > 0$, parallel to the xy-plane, slice the surface in ellipses whose equation are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz_0$$

Coordinate plane	*Trace*
xy ($z = 0$)	point: $(0, 0)$
xz ($y = 0$)	parabola: $\frac{x^2}{a^2} = cz$
yz ($x = 0$)	parabola: $\frac{y^2}{b^2} = cz$

(a)



(b)

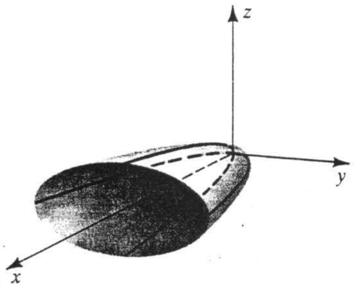
EXAMPLE ▪ 5 Sketch the graph of $y^2 + 4z^2 = x$, and identify the surface.

SOLUTION Traces are as follows.

Trace	Equation of trace	Description of trace
xy-plane	$y^2 = x$	Parabola
yz-plane	$y^2 + 4z^2 = 0$	Origin
xz-plane	$4z^2 = x$	Parabola

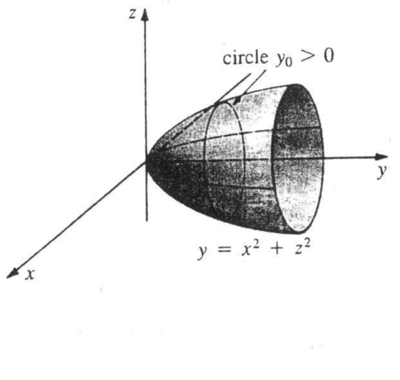
The trace in a plane $x = k$ parallel to the yz-plane has an equation of the form $y^2 + 4z^2 = k$, which is an ellipse if $k > 0$. Traces in planes parallel to the xz- or xy-planes are parabolas. The surface, a paraboloid having the x-axis as its axis, is sketched in Figure 10.71.

Figure 10.71
$y^2 + 4z^2 = x$



Ex. Identify $y = x^2 + z^2$ and sketch.

Soln. It is paraboloid with axis along y-axis



EXAMPLE 2 Sketching a Quadric Surface

Classify and sketch the surface given by $4x^2 - 3y^2 + 12z^2 + 12 = 0$.

Solution Begin by writing the equation in standard form.

$$4x^2 - 3y^2 + 12z^2 + 12 = 0 \qquad \text{Original equation}$$

$$\frac{x^2}{-3} + \frac{y^2}{4} - z^2 - 1 = 0 \qquad \text{Divide by } -12.$$

$$\frac{y^2}{4} - \frac{x^2}{3} - \frac{z^2}{1} = 1 \qquad \text{Standard form}$$

From the table on pages 750 and 751, you can conclude that the surface is a hyperboloid of two sheets with the y-axis as its axis. To sketch the graph of this surface, it helps to find the traces in the coordinate planes.

xy-trace ($z = 0$):	$\frac{y^2}{4} - \frac{x^2}{3} = 1$	Hyperbola
xz-trace ($y = 0$):	$\frac{x^2}{3} + \frac{z^2}{1} = -1$	No trace
yz-trace ($x = 0$):	$\frac{y^2}{4} - \frac{z^2}{1} = 1$	Hyperbola

The graph is shown in Figure 10.59.

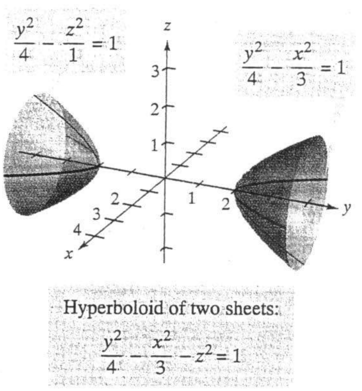


Figure 10.59

EXAMPLE 3 Sketching a Quadric Surface

Classify and sketch the surface given by $x - y^2 - 4z^2 = 0$.

Solution Because x is raised only to the first power, the surface is a paraboloid. The axis of the paraboloid is the x-axis. In the standard form, the equation is

$$x = y^2 + 4z^2. \qquad \text{Standard form}$$

Some convenient traces are as follows.

xy-trace ($z = 0$):	$x = y^2$	Parabola
xz-trace ($y = 0$):	$x = 4z^2$	Parabola
parallel to yz-plane ($x = 4$):	$\frac{y^2}{4} + \frac{z^2}{1} = 1$	Ellipse

The surface is an *elliptic* paraboloid, as shown in Figure 10.60.

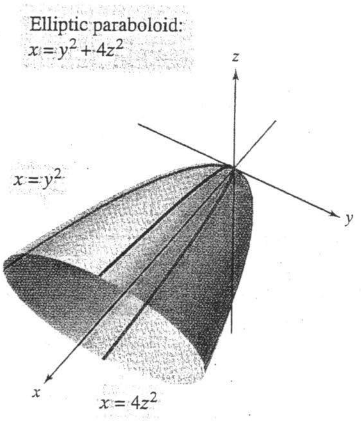


Figure 10.60

Some second-degree equations in x, y, and z do not represent one of the basic types of quadric surfaces. Here are two examples.

$$x^2 + y^2 + z^2 = 0 \qquad \text{Single point}$$

$$x^2 + y^2 = 1 \qquad \text{Right circular cylinder}$$

Hyperbolic Paraboloid

The last quadric surface we shall consider, known as a **hyperbolic paraboloid**, is the graph of any equation of the form

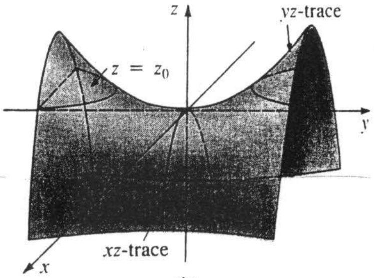
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = cz, \qquad a > 0,\ b > 0 \tag{14.59}$$

Note that for $c > 0$, planes $z = z_0$, parallel to the xy-plane, cut the surface in hyperbolas whose equations are

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = cz_0$$

Coordinate plane	*Trace*
xy $(z = 0)$	lines: $y = \pm\frac{a}{b}x$
xz $(y = 0)$	parabola: $-\frac{x^2}{b^2} = cz$
yz $(x = 0)$	parabola: $\frac{y^2}{a^2} = cz$

(a)



(b)

<u>Example</u> Identify the surface $y = x^2 - z^2$

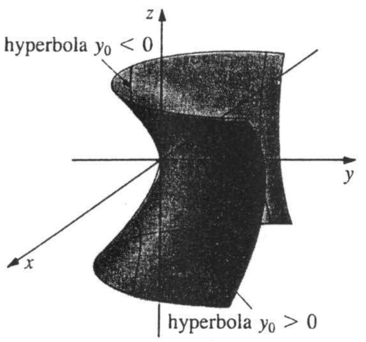
Solution. The graph of $x^2 - z^2 = y$ is hyperbolic paraboloid

$z = 0$ xy-plane $x^2 = y$ are parabolas

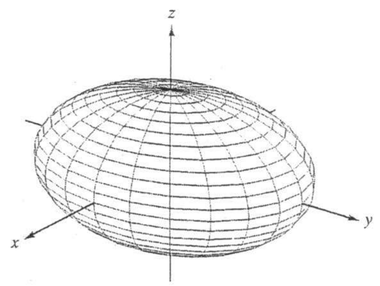
$y = 0$ xz-plane $x = \pm z$ are lines

$y = y_0$ planes parallel to xz-planes are hyperbolas

$x = 0$ yz-plane $-z^2 = y$ are parabolas



$y = x^2 - z^2$

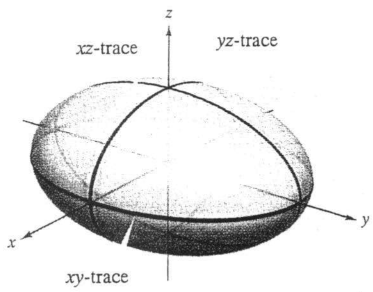


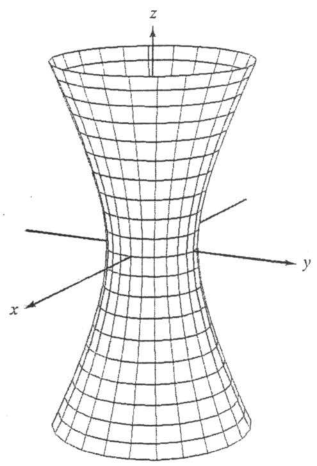
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a = b = c \neq 0$.



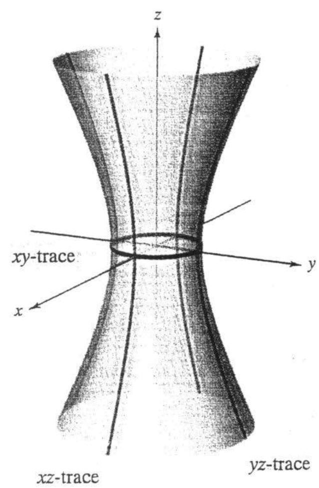


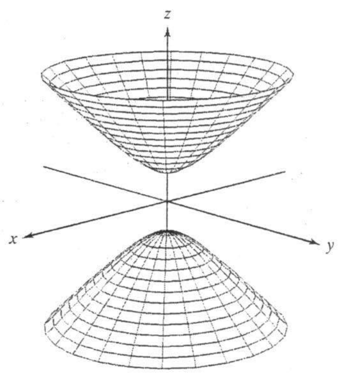
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



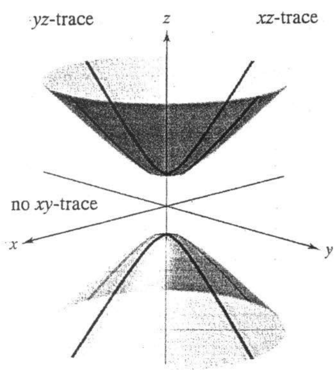


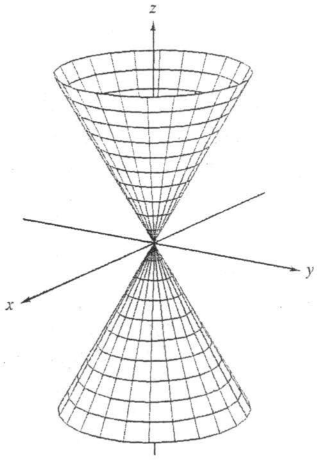
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



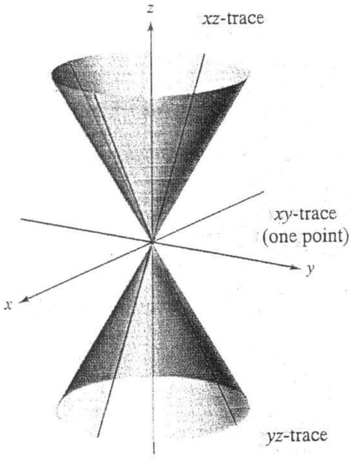


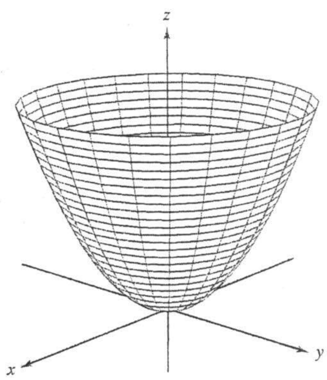
Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	*Plane*
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



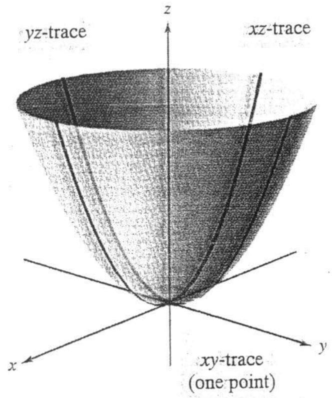


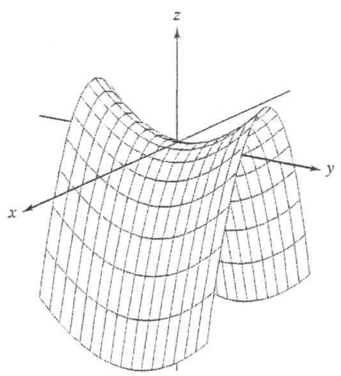
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



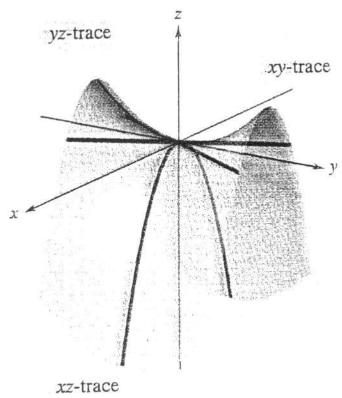


Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



EXERCISES FOR SURFACES IN SPACE

Exercise 1 – 7 , Sketch the graph of cylinder in xyz- coordinate system:

1. $x^2 + y^2 = 1$
2. $z = 4x^2$
3. $z = y^2$
4. $y^2 + z^2 = 9$
5. $y = \sin x \qquad 0 \le x \le 2\pi$
6. $x = 3$
7. $x + y = 2$

Exercise 8 – 18 , identify and sketch the graph. Find its traces in the coordinate planes:

8. $z^2 + y^2 - x^2 + 1 = 0.$
9. $20x^2 - 8y^2 - z^2 + 1 = 0.$
10. $4x^2 + 2y^2 - z^2 = 16.$
11. $36z = y^2 + 9x^2.$
12. $4z^2 + y^2 + 4x^2 = 4.$
13. $36y = 4z^2 + 9x^2.$
14. $9x^2 + 25y^2 = 144z^2.$
15. $x^2 = 9y^2 + z^2.$
16. $4x^2 - 4y + z^2 = 0.$
17. $x^2 + 9y^2 + 4z^2 - 2x + 18y + 16z = 10.$
18. $16x^2 - 4y^2 - (z-2)^2 = 100.$

ALGEBRA

Arithmetic

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\tfrac{a}{b}\right)\left(\tfrac{d}{c}\right) = \frac{ad}{bc}$$

Factoring

$$x^2-y^2 = (x-y)(x+y)$$

$$x^3-y^3 = (x-y)(x^2+xy+y^2)$$

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$

$$x^4-y^4 = (x-y)(x+y)(x^2+y^2)$$

Binomial

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

Exponents

$$x^n x^m = x^{n+m}$$

$$\frac{x^n}{x^m} = x^{n-m}$$

$$(x^n)^m = x^{nm}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{n/m} = \sqrt[m]{x^n}$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Lines

Slope m of line through (x_0, y_0) and (x_1, y_1)

$$m = \frac{y_1-y_0}{x_1-x_0}$$

Through (x_0, y_0), slope m

$$y - y_0 = m(x-x_0)$$

Slope m, y-intercept b

$$y = mx + b$$

Quadratic Formula

If $ax^2+bx+c=0$

then $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

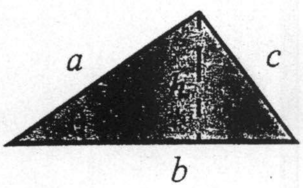
Distance

Distance d between (x_1, y_1) and (x_2, y_2)

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

GEOMETRY

Triangle



Area $= \frac{1}{2}bh$

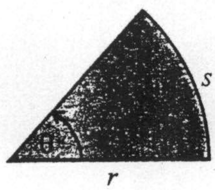
$c^2 = a^2+b^2-2ab\cos\theta$

Circle

Area $= \pi r^2$

$C = 2\pi r$

Sector of a Circle

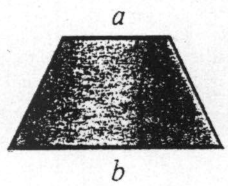


Area $= \frac{1}{2}r^2\theta$

$s = r\theta$

(for θ in radians only)

Trapezoid



Area $= \frac{1}{2}(a+b)h$

Sphere

Volume $= \frac{4}{3}\pi r^3$

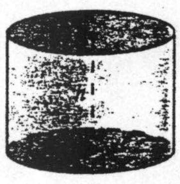
Surface Area $= 4\pi r^2$

Cone

Volume $= \frac{1}{3}\pi r^2 h$

Surface Area $= \pi r\sqrt{r^2+h^2}$

Cylinder



Volume $= \pi r^2 h$

Surface Area $= 2\pi r h$

DERIVATIVE FORMULAS

General Rules

$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

$\frac{d}{dx}[cf(x)] = c\,f'(x)$

$\frac{d}{dx}[f(g(x)] = f'(g(x))\,g'(x)$

$\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x)+f(x)\,g'(x)$

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x)-f(x)g'(x)}{[g(x)]^2}$

Power Rules

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(cx) = c$

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

Exponential

$\frac{d}{dx}[e^x] = e^x$

$\frac{d}{dx}[a^x] = a^x \ln a$

$\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)}\,u'(x)$

$\frac{d}{dx}[e^{rx}] = r\,e^{rx}$

Trigonometric

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x\,\tan x$

$\frac{d}{dx}(\csc x) = -\csc x\,\cot x$

Inverse Trigonometric

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\,\sqrt{x^2-1}}$

$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\,\sqrt{x^2-1}}$

Hyperbolic

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x$

$\frac{d}{dx}(\coth x) = -\mathrm{csch}^2 x$

$\frac{d}{dx}(\mathrm{sech}\, x) = -\mathrm{sech}\, x\,\tanh x$

$\frac{d}{dx}(\mathrm{csch}\, x) = -\mathrm{csch}\, x\,\coth x$

Inverse Hyperbolic

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

$\frac{d}{dx}(\mathrm{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$

$\frac{d}{dx}(\mathrm{csch}^{-1} x) = -\frac{1}{|x|\,\sqrt{x^2+1}}$

other $\frac{d}{dx}(\ln u) = \frac{1}{u}$, $\sinh u = \frac{e^u + e^{-u}}{2}$, $\cosh u = \frac{e^u - e^{-u}}{2}$

INTEGRAL FORMULAS

General Rules

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad \int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int [cf(x)]\,dx = c\int f(x)\,dx \qquad \int [f(x)\,g'(x)]\,dx = f(x)\,g(x) - \int f'(x)g(x)\,dx$$

Power Rules

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1) \qquad \int a\,dx = a\,x + c$$

$$\int \frac{1}{x}\,dx = \ln|x| + c \qquad \int \sqrt{x}\,dx = \tfrac{2}{3}x^{3/2} + c$$

Exponential

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{1}{\ln a}\,a^x + c$$

Trigonometric

$$\int \sin x\,dx = -\cos x + c \qquad \int \cos x\,dx = \sin x + c \qquad \int \sec^2 x\,dx = \tan x + c$$

$$\int \csc^2 x\,dx = -\cot x + c \qquad \int \sec x\,\tan x\,dx = \sec x + c \qquad \int \csc x\,\cot x\,dx = -\csc x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c \qquad \int \cot x\,dx = \ln|\sin x| + c \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + c$$

Inverse Trigonometric

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c \qquad \int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$$

Hyperbolic

$$\int \sinh x\,dx = \cosh x + c \qquad \int \cosh x\,dx = \sinh x + c \qquad \int \operatorname{sech}^2 x\,dx = \tanh x + c$$

Inverse Hyperbolic

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + c \qquad \int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + c \qquad \int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + c$$