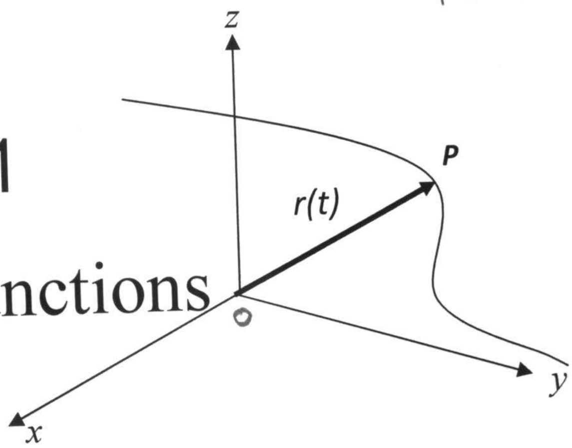
CHAPTER 11

Vector Valued Functions



Chapter 11

Vector Valued Functions

11.1 Vector valued functions

A vector valued function is a function whose domain is set of real numbers and whose range is a set of vectors.

For every value of 't' there is a unique vector r , denoted by *r(t)* a three dimensional vector in R^3 space.

If *f(t), g(t)* and *h(t)* are real valued functions , called components of vector *r(t)* and is denoted by

$$r(t) = f(t)i + g(t)j + h(t)k$$

't' is independent variable and denotes time is most of the applications.

Note: Domain of *r(t)* is common domain of its components.

Example.1. Domain of r(t)

$r(t) = (3 + 2t)i + \sqrt{1-t}j + t^2k$ be a vector valued function , the components of the function are $f(t) = 3 + 2t,\ g(t) = \sqrt{1-t},\ and\ h(t) = t^2$
The domain of *r(t)* consists of all values of ' t 'for which r(t) is defined.

$$D_r = D_f \cap D_g \cap D_h$$

$$D_r = (-\infty, \infty) \cap (-\infty, 1] \cap (-\infty, \infty) = (-\infty, 1]$$

□

Example.2. Domain of r(t)

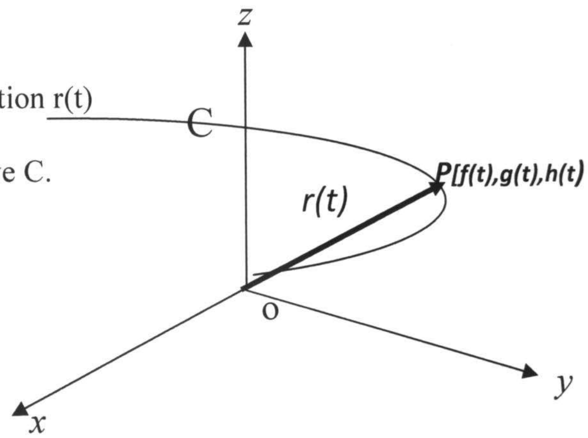
$r(t) = (3 + 2t)i + (2 + t)j + k$ be a vector valued function , the components of the function are $f(t) = 3 + 2t,\ g(t) = (2 + t),\ and\ h(t) = 1$
The domain of *r(t)* consists of all values of ' t 'for which r(t) is defined.

$$D_r = D_f \cap D_g \cap D_h$$

$$D_r = (-\infty, \infty) \cap (-\infty, \infty) \cap (-\infty, \infty) = (-\infty, \infty)$$

□

11.2 Graph



C is the space curve describing the vector valued function r(t)

$\overrightarrow{OP}$ = r(t) is the position vector of point P on the curve C.

$$r(t) = f(t)i + g(t)j + h(t)k$$

Where *f(t), g(t)* and *h(t)* are real valued functions ,

called components of vector *r(t).*

Example.3. Describe the curve defined by the vector valued function

$$r(t) =< 3 + 2t, 1 - t, -2 + 4t >$$

Solution: The corresponding parametric equations are:

$$x = 3 + 2t,$$
$$y = 1 - t$$
$$z = -2 + 4t$$

These are parametric equation of a line passing through a point (3, 1, -2) and parallel to vector < 2, -1, 4 >. □

Example.4. Describe the curve defined by the vector valued function

$$r(t) =< 2, 4\cos t, 9\sin t >, \quad t \geq 0$$

Solution: The corresponding parametric equations are:

$$x = 2,$$
$$y = 4\cos t$$
$$z = 9\sin t$$

$$\frac{y}{4} = \cos t, \quad \frac{z}{9} = \sin t,$$
$$sin^2 t + cos^2 t = 1,$$

$\frac{z^2}{81} + \frac{y^2}{16} = 1$ is equation of ellipse in yz – plane , x = 2. □

Example.5. Let $r(t) = ti + (9 - t^2)j \ for - 3 \leq t \leq 3$.

a) Sketch the curve C determined by r(t),
b) Sketch r(t) for t = -3, -2, 0, 2, 3

Solution: (a) The curve has parametric equations

$$x = t, \quad y = 9 - t^2; \ -3 \leq t \leq 3$$

Eliminating ' t ' $y = 9 - x^2$, which is parabola opening downwards.

(b) at t = -3 $r(-3) = -3i$

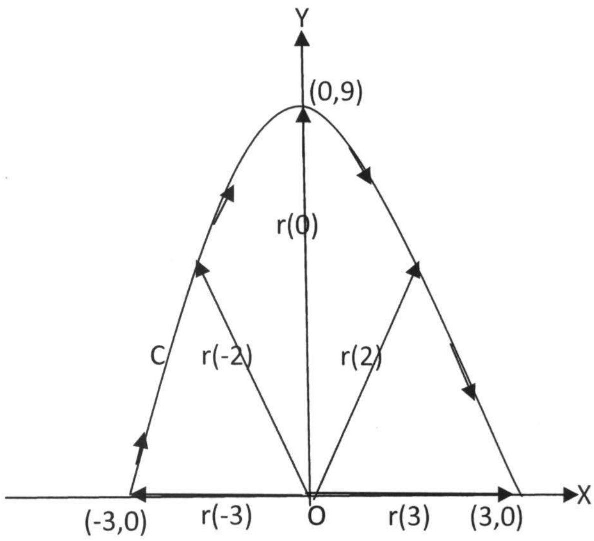
at t = -2 $r(-2) = -2i + 5j$

at t = 0 $r(0) = 9j$

at t = 2 $r(2) = 2i + 5j$

at t = 3 $r(3) = 3i$

These vectors are sketched on the graph.



□

Example.6. Let $r(t) = 3t\,i + (1 - 9t^2)j$ for t∈ R.

a) Sketch $r(0)$ *and* $r(1)$
b) Give the graphic description of the curve $r(t)$

Solution: (a) $r(t) = < 3t, 1 - 9t^2 >$

$$r(0) = < 0, \ 1 > = j$$

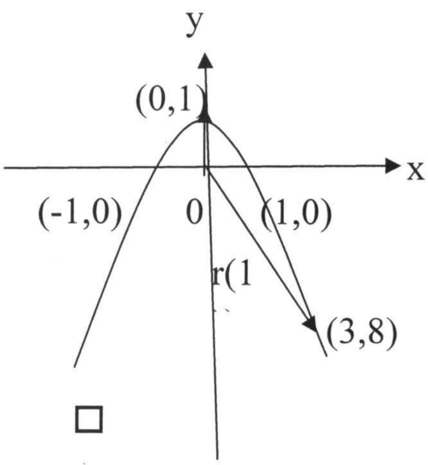
$$r(1) = < 3, -8 > = 3i - 8j$$

(b) Graphic description

$$x = 3t, \quad y = 1 - 9t^2$$

$$y = 1 - x^2$$

It is a parabola opening downwards.



□

Note . 1: $r(t) = at^2 i + 2atj$
represents parabola $y^2 = 4ax$ *in the* $xy-$*pl*ane because its parametric equations are $x = at^2$, *and* $y = 2at$.

Note . 2: $r(t) = a\ cost\ i + bsint\ j$
represents ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in xy – plane because its parametric equations are $x = \mathrm{a}\cos t$, *and* $y = b\sin t$.

EXERCISES FOR VECTOR VALUED FUNCTIONS

In Exercise 1 – 4 , find the domain of the vector valued function:

1. $r(t) = 3ti + \frac{1}{t^2} j$
2. $r(t) = ti + t^2 j + tk$
3. $r(t) = \sin ti + 4\cos tj + tk$
4. $r(t) = ti + \sqrt{t^2 - 1} j + t^2 k$

5. Represent the curve by a vector valued function;

   i. $y = x^2 + 4$
   ii. $x^2 + y^2 = 16$

In Exercise 6 – 9 , evaluate the vector valued function for the given values of t,

6. $r(t) = \frac{1}{2} t^2 i + (t+1) j$
   (a) t = 0, (b) t =1, (c) t =s+1.

7. $r(t) = ti - 4\cos tj + 4\sin tk$
   (a) t = 0, (b) $t = \frac{\pi}{2}$, (c) $t = \pi$.

8. $r(t) = 9\sin ti + 3\cos tj + tk$
   (a) t = 0, (b) $t = \frac{\pi}{4}$.

9. $r(t) = ti + t^2 j + t^3 k$
   (a) t = 0, (b) t =-1, (c) t =2.

In Exercise 10 – 13 , Sketch the curve C determined by *r(t)*

10. $r(t) = (1 - 4t^2)i + 2tj \qquad t \in R$
11. $r(t) = (2+t)i + (3-2t)j + (1+3t)k \qquad t \in R$
12. $r(t) = 2\cos ti + 2\sin tj + tk$
13. $r(t) = ti + t^2 j + \frac{1}{2} tk$

11.2 Limits and Derivatives

LIMIT

Let $r(t) = f(t)i + g(t)j + h(t)k$, then

$$\lim_{t\to a} r(t) = [\lim_{t\to a} f(t)]i + [\lim_{t\to a} g(t)]j + [\lim_{t\to a} h(t)]k$$

Provide the limit of the components functions exist.

Example. 7. Find $\lim_{t\to 0} r(t)$, where $r(t) = (1-t)i + 4e^t j + \frac{\sin 2t}{t}k$.

Solution: $\lim_{t\to 0} r(t) = \lim_{t\to 0}(1-t)i + \lim_{t\to 0} 4e^t j + \lim_{t\to 0}\frac{\sin 2t}{t}k$

$= i + 4j + 2k.$ □

CONTINUITY

A vector valued function r(t) is continuous at t = a if

$$\lim_{t\to a} r(t) = r(a).$$

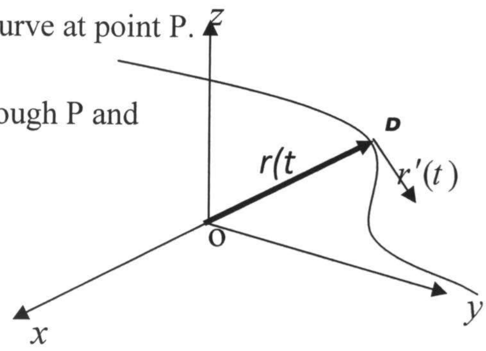
Derivative

If $r(t) = f(t)i + g(t)j + h(t)k$ and components $f, g, and\ h$ are differentiable, then $\frac{d}{dt}r(t) = \frac{d}{dt}f(t)i + \frac{d}{dt}g(t)j + \frac{d}{dt}h(t)k$

$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Note.1: The vector $r'(t)$ is called **tangent vector** to the curve at point P.

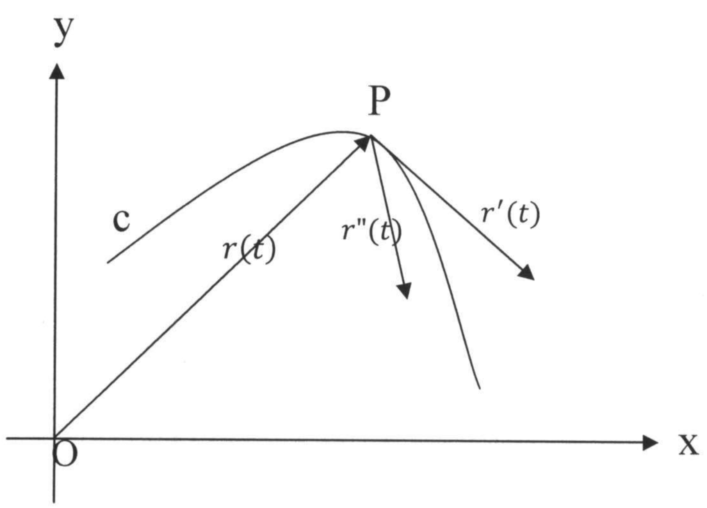
Note.2: The **tangent line** to C at P is defined to be line through P and Parallel to vector $r'(t)$.



Note.3: The unit tangent vector is

$$T(t)=\frac{r'(t)}{|r'(t)|}$$

Note:4. Geometrical Interpretation of $r'(t)$ *and* $r''(t)$



Note: $\lim_{t\to o} r(t)$, $where\ r(t)=(1-t)i+4e^{t}j+\frac{1}{t}k$ does not exits because $\lim_{t\to 0}\frac{1}{t}$ does not exists.

Example. 8: For the vector valued function $r(t) = ti + t^2j + t^3k, \quad t \geq 0$. Find

$$a)r'(t), \quad b)r''(t), \quad c)r'(t).r''(t), \quad d)r'(t) \text{ x } r''(t)$$

e) Find the parametric equations of the tangent line when t = 2.

Solution:

$$a) \quad r'(t) = i + 2tj + 3t^2k$$

$$b) \quad r''(t) = \quad 2j + 6tk$$

$$c) \quad r'(t).r''(t) = 4t + 18t^3$$

$$d) \quad r'(t) \text{ x } r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = (12t^2 - 6t^2)i - 6tj + 2k$$

$$= 6t^2i - 6tj + 2k$$

$e)$ Tangent vector to C at t = 2 is $r'(2) = i + 4j + 12k$

Point P corrsponding to t =2 at curve C is (2,4,8)

The parametric equations of the tangent line are

x = 2 + t, y = 4+ 4t, z= 8 + 12t, t in R. □

Example. 9: Find equation of the tangent line to the curve C

$$r(t) = t\, sint\; i + t\cos t\; j + tk, \quad at \;\; t = \frac{\pi}{2}.$$

Solution:

$$r(t) = < t\sin t, t\cos t, t >$$

$$r'(t) = < \sin t + t\cos t, \cos t - t\sin t, 1 >$$

$$r'(\frac{\pi}{2}) = < 1, -\frac{\pi}{2}, 1 > \qquad \sin\frac{\pi}{2} = 1, \quad \cos\frac{\pi}{2} = 0.$$

The point corresponding to $t = \frac{\pi}{2}$ is $(\frac{\pi}{2}, 0, \frac{\pi}{2})$

Parametric equations of the tangent line are

$x = \frac{\pi}{2} + t$, $y = -\frac{\pi}{2}t$, $z = \frac{\pi}{2} + t$ □

Ex. Find parametric equations for the tangent line to C, which given parametrically by

$x = 2t^3 - 1,\ y = -5t^2 + 3,\ z = 8t + 2$ at point $P(1,-2,10)$

Solution

$r(t) = \langle 2t^3 - 1, -5t^2 + 3, 8t + 2 \rangle$

a tangent vector to C at the point correspond to t is

$r'(t) = \langle 6t^2, -10t, 8 \rangle$

point $P(1,-2,10) \implies$ $8t + 2 = 10$

$8t = 10 - 2$

$8t = 8$

$t = 1$

$r'(1) = \langle 6, -10, 8 \rangle$

Equation of the tangent line from point $(1,-2,10)$ and parallel to vector $r'(t)$ is

$x = 1 + 6t$

$y = -2 - 10t$

$z = 10 + 8t$

Ex. Find the equation of the tangent line to the curve

$x = t \sin t, \quad y = t \cos t, \quad z = t$ at $t = \frac{\pi}{2}$

Solution.

$r(t) = \langle t \sin t, t \cos t, t \rangle$

$r'(t) = \langle t \cos t + \sin t, -t \sin t + \cos t, 1 \rangle$ $\qquad \cos\frac{\pi}{2} = 0$

$r'(\frac{\pi}{2}) = \langle 1, -\frac{\pi}{2}, 1 \rangle$ $\qquad \sin\frac{\pi}{2} = 1$

The point on the curve corresponding to $t = \frac{\pi}{2}$ is

$(\frac{\pi}{2}, 0, \frac{\pi}{2})$

∴ Equation of tangent line are

$x = \frac{\pi}{2} + t$

$y = -\frac{\pi}{2} t$

$z = \frac{\pi}{2} + t$

DIFFERENTIATION RULES

THEOREM:

Suppose u and v are differentiable vector valued functions, c is scalar and f(t) is differentiable real valued function, then

1. $\dfrac{d}{dt}[u(t)+v(t)] = u'(t)+v'(t)$
2. $\dfrac{d}{dt}[cu(t)] = cu'(t)$
3. $\dfrac{d}{dt}[f(t)u(t)] = f'(t)u(t)+f(t)u'(t)$
4. $\dfrac{d}{dt}[u(t).v(t)] = u'(t).v(t)+u(t).v'(t)$
5. $\dfrac{d}{dt}[u(t)\,\mathrm{x}\,v(t)] = u'(t)\,\mathrm{x}\,v(t)+u(t)\,\mathrm{x}\,v'(t)$
6. $\dfrac{d}{dt}[u(f(t))] = f'(t)u'(f(t)),$ Chain Rule

Example. : For vector valued function

$$r(t) = ti + 2j + t^2k, \quad and \quad u(t) = i - t^2j + t^3k$$

Find $(a)\dfrac{d}{dt}[r(t).u(t)]$, (b)$\dfrac{d}{dt}[u(t).u'(t)]$

Solution:

$$(a)\frac{d}{dt}[r(t).u(t)] = r'(t).u(t)+r(t).u'(t)$$

$$= (i+2tk).(i - t^2 j + t^3 k) + (ti+2j+t^2k)(-2tj+3t^2kj)$$

$$= (1+2t^4)+(-4t+3t^{2})$$

$$= 3t^5+2t^4-4t+1$$

(b) $\frac{d}{dt}[u(t)u'(t)] = u(t)u''(t) + u'(t)u'(t)$

$$= (i - t^2 j + t^3 k).(2k) + (-2tj + 3t^2k).(-2tj + 3t^2k)$$

$$= 2t^3 + 4t^2 + 9t^4$$

11.3 Velocity and Acceleration (Motion in Space)

If the scalar variable t denotes the time and $\vec{r}$ is the position vector of the moving particle P. If $\vec{v}$ represents the velocity vector of the particle P, then

$$\vec{v}(t) = \frac{dr}{dt}$$

If $\vec{a}$ represents the acceleration of the particle P, the n

$$\vec{a}(t) = \frac{d^2r}{dt^2}$$

Speed $\quad \|\vec{v}(t)\| = \|r'(t)\| = \sqrt{(f')^2 + (g')^2 + (h')^2}$

Example. 12: **Find velocity and Acceleration**

Find velocity, Acceleration and speed of the vector valued function

$$r(t) =< t, t^3, 2t^2 > \text{ at } t = 1.$$

Solution:

$$r(t) =< t, t^3, 2t^2 >,$$

$$v(t) = r'(t) =< 1, 3t^2, 4t > \quad \text{Velocity Vector}$$

$$a(t) = r''(t) =< 0, 6t, 4 > \quad \text{Acceleration Vector}$$

At t = 1, the velocity and acceleration vectors are given by

$$v(1) = r'(1) =< 1, 3, 4 >$$

$$a(1) = r''(1) =< 0, 6, 4 >$$

Speed $\|v(1)\| = \|r'(1)\| = \sqrt{1+9+16} = \sqrt{26}$ $\square$

Example. 12: **Find velocity , speed and Acceleration**

Find velocity, Acceleration and speed of the vector valued function of the particle moving along the curve $r(t) = t\cos t\, i + t\, sin\, t\, j + t^2 k$, $t = \dfrac{\pi}{2}$.

Solution: $r(t) = t\cos t\, i + t\, sin\, t\, j + t^2 k$,

$$Velocity: \quad r'(t) = (\cos t - t\sin t)i + (\sin t + t\cos t)j + 2tk$$

$$Accelration: r''(t) = (-2\sin t - t\cos t)i + (2\cos t - t\sin t)j + 2k$$

At $t = \dfrac{\pi}{2}$

$$Velocity: \quad \mathrm{v}\left(\frac{\pi}{2}\right) = r'\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}i + j + \lambda k$$

$$Accelration: a\left(\frac{\pi}{2}\right) = r''\left(\frac{\pi}{2}\right) = -2i - \frac{\pi}{2}j + 2k$$

$$Speed: \quad \left\|\mathrm{v}\left(\frac{\pi}{2}\right)\right\| = \sqrt{\frac{\pi^2}{4} + 1 + \pi^2} = \sqrt{\frac{5\pi^2}{4} + 1}$$

Example:13 . A particle moves along the curve at anytime t is given by

$$x = t^2, y = t - 4 \text{ and } z = t^3 - 3$$

Find the components of velocity and acceleration at t = 1 in the direction b = 2 i – 3j + k.

Solution: If r is apposition vector of any point (x, y, z) on the given curve, then

$$\text{Postion Vector}: \quad r(t) = t^2 i + (t-4)\,j + (t^3-3)\,k$$

$$\text{Velocity}: \quad v(t) = r'(t) = 2ti + j + 3t^2 k$$

$$\text{Acclecration}: \quad a(t) = r''(t) = 2i + 6tk$$

$$\text{at } t = 1$$

$$v(1) = r'(1) = 2i + j + 3k$$

$$a(t) = r''(1) = 2i + 6k$$

$$comp_b^v = \frac{v.b}{\|b\|} = \frac{4-3+3}{\sqrt{14}} = \frac{4}{\sqrt{14}}$$

$$comp_c^a = \frac{a.b}{\|b\|} = \frac{4+6}{\sqrt{14}} = \frac{10}{\sqrt{14}} \qquad \square$$

11.4 Integration of Vector Valued Function

The indefinite integral of a continuous vector valued function

$r(t) = f(t)i + g(t)j + h(t)k$

$$\int r(t)dt = \left[\int f(t)dt\right]i + \left[\int g(t)dt\right]j + \left[\int h(t)dt\right]k$$

The definite integral of a continuous vector valued function

$r(t) = f(t)i + g(t)j + h(t)k$ on interval [a, b] is

$$\int_a^b r(t)dt = \left[\int_a^b f(t)dt\right]i + \left[\int_a^b g(t)dt\right]j + \left[\int_a^b h(t)dt\right]k$$

Example.14 : Evaluate $\int[(t^2+2)i+sin2tj+2te^{t^2}k]dt$

Solution.

$$\int[(t^2+2)i+sin2tj+2te^{t^2}k]dt=\int[(t^2+2)dti+\int sin2t\,dtj+\int 2te^{t^2}\,dt\,k$$

$$=\left(\frac{t^3}{3}+2t+c_1\right)i+\left(-\frac{\cos 2t}{2}+c_2\right)j+\left(e^{t^2}+c_3\right)k$$

$$=\left(\frac{t^3}{3}+2t\right)i-\frac{\cos 2t}{2}j+e^{t^2}k+c_1i+c_2j+c_3k$$

$$=\left(\frac{t^3}{3}+2t\right)i-\frac{\cos 2t}{2}j+e^{t^2}k+c$$

Integration of Vector Valued Function

Example.15 : Find the path of the curve when acceleration of the particle moving along this curve is $a(t)=-2\cos t\,i-2\sin t\,j+2k$, initial velocity of the particle is v(0) = 2 jAnd it starts from the point (2, 0, 0).

Solution. We are required to find r (t)

Acceleration is $a(t)=-2\cos t\,i-2\sin t\,j+2k$

Initial conditions are $v(0)=2j\ \ and\ \ r(0)=2i+0j+0k$

$$v(t)=\int(-2\cos t\,i-2\sin t\,j+2k)dt+c_1$$

$$v(t)=-2\sin t\,i+2\cos tj+2t\,k+c_1$$

To find c_1 , use $v(0)=2j$, $2j=-2(sin0)i+2(cos0)j+(0)k+c_1$

$$c_1=0$$

$$v(t)=-2\sin t\,i+2\cos tj+2t\,k$$

$$r(t) = \int(-2\sin t\, i + 2\cos t j + 2t\, k)dt$$

$$r(t)=2\cos t\, i+2\sin t\, j+t^2\, k+ C_2$$

To find C_2, we use r(0) =2 i

$$r(0) = 2i = 2(\cos 0\,)i + 2(\sin 0)\, j + (0)k + C_2$$

$$C_2 = 0$$

$$r(t) = 2\cos t\; i + 2\sin t\; j + t^2 k$$

is the required equation of the curve □

EXAMPLE 4 Finding a Position Function by Integration

An object starts from rest at the point $P(1, 2, 0)$ and moves with an acceleration of

$$\mathbf{a}(t) = \mathbf{j} + 2\mathbf{k} \qquad \text{Acceleration vector}$$

where $\|\mathbf{a}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ seconds.

Solution From the description of the object's motion, you can deduce the following *initial conditions*. Because the object starts from rest, you have

$$\mathbf{v}(0) = \mathbf{0}.$$

Moreover, because the object starts at the point $(x, y, z) = (1, 2, 0)$, you have

$$\begin{aligned}\mathbf{r}(0) &= x(0)\mathbf{i} + y(0)\mathbf{j} + z(0)\mathbf{k}\\ &= 1\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}\\ &= \mathbf{i} + 2\mathbf{j}.\end{aligned}$$

To find the position function, you should integrate twice, each time using one of the initial conditions to solve for the constant of integration. The velocity vector is

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int (\mathbf{j} + 2\mathbf{k})\, dt = t\mathbf{j} + 2t\mathbf{k} + \mathbf{C}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$. Letting $t = 0$ and applying the initial condition $\mathbf{v}(0) = \mathbf{0}$, you obtain

$$\mathbf{v}(0) = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k} = \mathbf{0} \quad \Longrightarrow \quad C_1 = C_2 = C_3 = 0.$$

Thus, the *velocity* at any time t is

$$\mathbf{v}(t) = t\mathbf{j} + 2t\mathbf{k}. \qquad \text{Velocity vector}$$

Integrating once more produces

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int (t\mathbf{j} + 2t\mathbf{k})\, dt = \frac{t^2}{2}\mathbf{j} + t^2\mathbf{k} + \mathbf{C}$$

where $\mathbf{C} = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k}$. Letting $t = 0$ and applying the initial condition $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$, you have

$$\mathbf{r}(0) = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k} = \mathbf{i} + 2\mathbf{j} \quad \Longrightarrow \quad C_4 = 1, C_5 = 2, C_6 = 0.$$

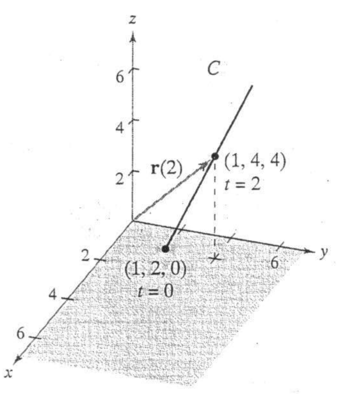
Thus, the *position* vector is

$$\mathbf{r}(t) = \mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + t^2\mathbf{k}. \qquad \text{Position vector}$$

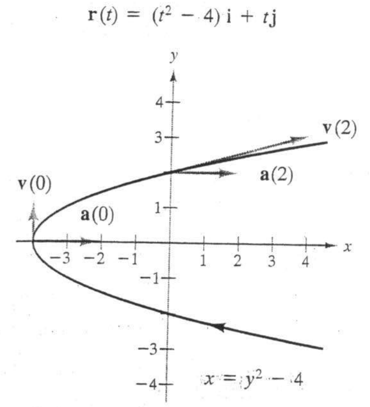
The location of the object after 2 seconds is given by $\mathbf{r}(2) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, as shown in Figure 11.15.

Curve:

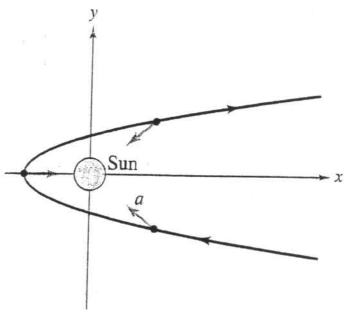
$\mathbf{r}(t) = \mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + t^2\mathbf{k}$



The object takes 2 seconds to move from point (1, 2, 0) to point (1, 4, 4) along C.
Figure 11.15



At each point on the curve, the acceleration vector points to the right.
Figure 11.12



At each point in the comet's orbit, the acceleration vector points toward the sun.
Figure 11.13

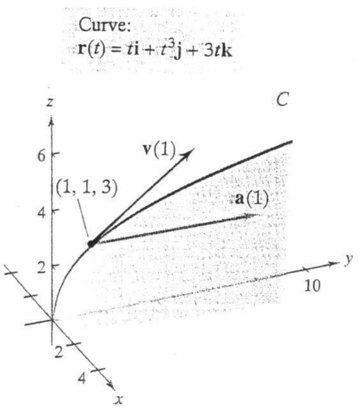


Figure 11.14

EXAMPLE 2 Sketching Velocity and Acceleration Vectors in the Plane

Sketch the path of an object moving along the plane curve given by

$$\mathbf{r}(t) = (t^2 - 4)\mathbf{i} + t\mathbf{j} \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 0$ and $t = 2$.

Solution Using the parametric equations $x = t^2 - 4$ and $y = t$, you can determine that the curve is a parabola given by $x = y^2 - 4$, as shown in Figure 11.12. The velocity vector (at any time) is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j} \qquad \text{Velocity vector}$$

and the acceleration vector (at any time) is

$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}. \qquad \text{Acceleration vector}$$

When $t = 0$, the velocity and acceleration vectors are given by

$$\mathbf{v}(0) = 2(0)\mathbf{i} + \mathbf{j} = \mathbf{j} \quad \text{and} \quad \mathbf{a}(0) = 2\mathbf{i}.$$

When $t = 2$, the velocity and acceleration vectors are given by

$$\mathbf{v}(2) = 2(2)\mathbf{i} + \mathbf{j} = 4\mathbf{i} + \mathbf{j} \quad \text{and} \quad \mathbf{a}(2) = 2\mathbf{i}.$$

For the object moving along the path shown in Figure 11.12, note that the acceleration vector is constant (it has a magnitude of 2 and points to the right). This implies that the speed of the object is decreasing as the object moves toward the vertex of the parabola, and the speed is increasing as the object moves away from the vertex of the parabola.

This type of motion is *not* characteristic of comets that travel on parabolic paths through our solar system. For such comets, the acceleration vector always points to the origin (the sun), which implies that the comet's speed increases as it approaches the vertex of the path and decreases as it moves away from the vertex. (See Figure 11.13.)

EXAMPLE 3 Sketching Velocity and Acceleration Vectors in Space

Sketch the path of an object moving along the space curve C given by

$$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}, \quad t \geq 0 \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 1$.

Solution Using the parametric equations $x = t$ and $y = t^3$, you can determine that the path of the object lies on the cubic cylinder given by $y = x^3$. Moreover, because $z = 3t$, the object starts as $(0, 0, 0)$ and moves upward as t increases, as shown in Figure 11.14. Because $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}$, you have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j} + 3\mathbf{k} \qquad \text{Velocity vector}$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\mathbf{j}. \qquad \text{Acceleration vector}$$

When $t = 1$, the velocity and acceleration vectors are given by

$$\mathbf{v}(1) = \mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{a}(1) = \mathbf{r}''(1) = 6\mathbf{j}.$$

EXERCISES FOR LIMITS AND DERIVATIVES

In Exercises 1 – 4 , evaluate the limit,

1. $\lim_{t\to 3}\left(ti + \frac{1}{t} j + \frac{t^2-9}{t-3} k\right)$

2. $\lim_{t\to 0}\left(e^t i + \frac{1-\cos t}{t} j + e^{-t} k\right)$

3. $\lim_{t\to 1}\left(\sqrt{t} i + \frac{t+1}{t} j + 2t^2 k\right)$

4. $\lim_{t\to \infty}\left(\frac{2t^2+2t}{3t^2+1} i + \frac{1}{t} j + \frac{2t}{t+1} k\right)$

In Exercises 4 – 10 , find $r'(t)$ *and* $r''(t)$

5. $r(t) = t^2 i + tj$
6. $r(t) = \frac{1}{t} i + \cos 2tj$
7. $r(t) = t^2 i + t\tan tj + 2k$
8. $r(t) = (t^2 + 4t)i + \sec tj + tk$
9. $r(t) = a\cos^3 ti + a\sin^3 tj + k$
10. $r(t) =< \sin t + t\cos t, \cos t + t\cos t, t^3 >$

In Exercises 11 – 12 , find $\frac{d}{dt}[u(t).v(t)]$ and $\frac{d}{dt}[u(t) \text{ x } v(t)]$

11. $u(t) = ti + 2tj + 3tk$
    $v(t) = ti + t^2 j + t^3 k$
12. $u(t) = ti + t^2 j + t^3 k$
    $v(t) = \sin ti + \cos tj + tk$
13. For the vector valued function given $r(t) = ti + \cos tj + \sin tk$
    Find (a) $r'(t)$, (b) $r''(t)$, (c) $r'(t).r''(t)$, and (d) $r'(t)$ x $r''(t)$
14. Show that if

    $\vec{r}(t) = \vec{a}\sin\omega t + \vec{b}\cos\omega t$, where $\vec{a}, \vec{b}, \omega$ are constants, then

    $$\frac{d^2\vec{r}}{dt^2} = -\omega\vec{r} \quad \text{and} \quad \vec{r} \text{ x } \frac{d\vec{r}}{dt} = -\omega\, \vec{a} \text{ x } \vec{b}$$
15. A particle moves along the curve $r(t) = (t^3 + 1)\, i + 2t\, j + t^2 k$, where t is time.
    Find the component of velocity and acceleration at t = 1 in direction of vector

$b = i + 2j + k$

In Exercise 14 – 16 , Evaluate the indefinite integral

14. $\int (3t^2 i + 2tj + k)dt$
15. $\int (\frac{1}{t} i + j - t^{\frac{1}{2}} k)dt$
16. $\int (e^t i + \sin tj + \cos tk)dt$

In Exercise 17 – 19 , find r(t) for the given conditions

17. $r'(t) = i + 2tj + 3t^2 k, \quad r(0) = i + j + k$
18. $r''(t) = i + j + k, \quad r'(0) = 0, \quad r(0) = 0$
19. $r''(t) = -\cos ti - \sin tj, \quad r'(0) = j + k, \quad r(0) = i.$

In Exercise 20 -21, find the position vector r(t) if the given acceleration and initial conditions are given:

20. $a(t) = 2\,i + 3j + k, \quad r(0) = 2k, \ v(0) = i + j.$

21. $a(t) = i + 2tj + 3t^2 k, \quad r(0) = i + j + k, \ v(0) = 0.$

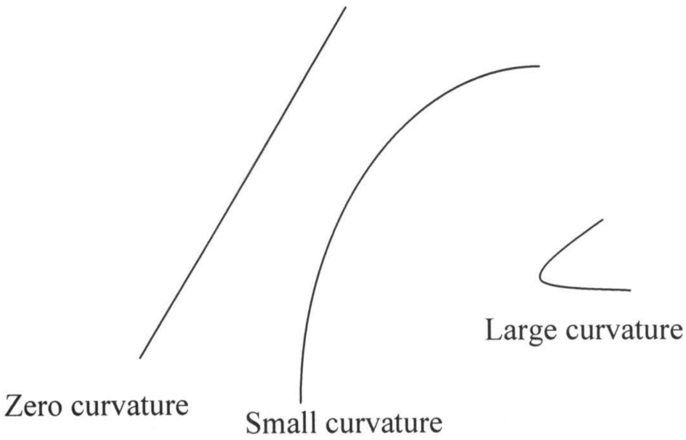
In Exercise 21 – 24, find the position vector r (t) if the given velocity and initial condition for the position vector are given

21. $v(t) = t\,i + j + t^2 k, \qquad r(0) = i + j + 2k.$
22. $v(t) = 2t\,i + t^3 j + t^2 k, \quad r(0) = i + j.$
23. $v(t) = t\,i + e^{-t} j + t^2 k, \quad r(0) = 2k.$
24. $v(t) = \frac{t}{1+t^2}\,i + \frac{2}{\sqrt{1-t^2}}\,j + t^2 k, \quad r(0) = 2k.$

11.5 Curvature

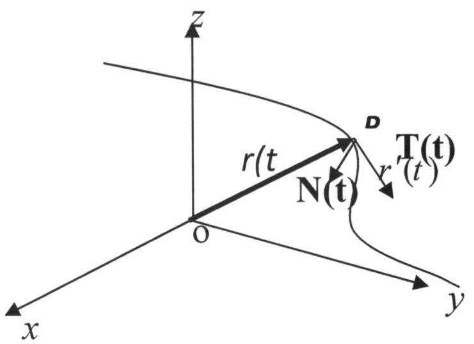
Curvature is the measure how sharp a curve bends.

A line has zero curvature, sharply bending curve has a large curvature.



1. Unit Tangent Vector

$$T(t) = \frac{r'(t)}{|r'(t)|}$$



2. Principal Normal Vector

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

3. Curvature of the curve C when $r(t) = f(t)i + g(t)j + h(t)k$ is

$$\kappa = \frac{|T'(t)|}{|r'(t)|}$$

4. <u>Curvature of the curve C when, $x = f(t),\ y = g(t)$</u>

$$\kappa = \frac{\left|f'(t)g''(t) - g(t)'f''(t)\right|}{\left|(f'(t))^2 + (g'(t))^2\right|^{\frac{3}{2}}}$$

5. <u>Curvature of the curve C when $y = f(x)$</u> is $\kappa = \dfrac{|y''|}{\left|1+(y')^2\right|^{\frac{3}{2}}}$

Example: 16. Find unit tangent vector and principal normal vector of the curve

$$r(t) = \cos t\, i + \sin t\, j + t\, k$$

Solution:

$$r(t) = \cos ti + \sin tj + tk,$$

$$r'(t) = -\sin ti + \cos tj + k, |r'(t)| = \sqrt{2}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}(-\sin ti + \cos tj + k)$$

$$T'(t) = \frac{1}{\sqrt{2}}(-\cos ti - \sin tj), |T'(t)| = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\frac{1}{\sqrt{2}}(-\cos ti - \sin tj)}{\frac{1}{\sqrt{2}}} = (-\cos ti - \sin tj)$$

EXAMPLE 3 Finding the Principal Unit Normal Vector

Find $\mathbf{N}(t)$ and $\mathbf{N}(1)$ for the curve represented by

$$\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}.$$

Solution By differentiating, you obtain

$$\mathbf{r}'(t) = 3\mathbf{i} + 4t\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{9 + 16t^2}$$

which implies that the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{9 + 16t^2}}(3\mathbf{i} + 4t\mathbf{j}). \qquad \text{Unit tangent vector}$$

Using Theorem 11.2, differentiate $\mathbf{T}(t)$ with respect to t to obtain

$$\mathbf{T}'(t) = \frac{1}{\sqrt{9 + 16t^2}}(4\mathbf{j}) - \frac{16t}{(9 + 16t^2)^{3/2}}(3\mathbf{i} + 4t\mathbf{j}) = \frac{12}{(9 + 16t^2)^{3/2}}(-4t\mathbf{i} + 3\mathbf{j})$$

$$\|\mathbf{T}'(t)\| = 12\sqrt{\frac{9 + 16t^2}{(9 + 16t^2)^3}} = \frac{12}{9 + 16t^2}.$$

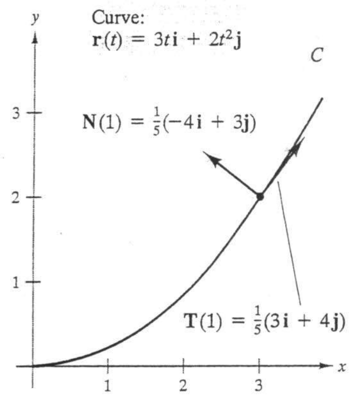
Therefore, the principal unit normal vector is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{9 + 16t^2}}(-4t\mathbf{i} + 3\mathbf{j}). \qquad \text{Principal unit normal vector}$$

When $t = 1$, the principal unit normal vector is

$$\mathbf{N}(1) = \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j})$$

as shown in Figure 11.21.



The principal unit normal vector points toward the concave side of the curve.
Figure 11.21

Example: Find the unit tangent vector of the curve

$$r(t) = 3\cos t\, i + 3\sin t\, j + t^2 k$$

Solution. unit tangent vector

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$r'(t) = -3\sin t\, i + 3\cos t\, j + 2t\, k$$

$$\|r'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2} = \sqrt{9 + 4t^2}$$

$$T(t) = \frac{1}{\sqrt{9+4t^2}}(-3\sin t\, i + 3\cos t\, j + 2t\, k)$$

EXAMPLE 5 Finding the Curvature of a Space Curve

Find the curvature of the curve given by $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{1}{3}t^3\mathbf{k}$.

Solution It is not apparent whether or not this parameter is arc length, so you should use the formula $K = \|\mathbf{T}'(t)\|/\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2 \qquad \text{Length of } \mathbf{r}'(t)$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k}}{t^2 + 2}$$

$$\mathbf{T}'(t) = \frac{(t^2 + 2)(2\mathbf{j} - 2t\mathbf{k}) - (2t)(2\mathbf{i} - 2t\mathbf{j} - t^2\mathbf{k})}{(t^2 + 2)^2}$$

$$= \frac{-4t\mathbf{i} + (4 - 2t^2)\mathbf{j} - 4t\mathbf{k}}{(t^2 + 2)^2}$$

$$\|\mathbf{T}'(t)\| = \frac{\sqrt{16t^2 + 16 - 16t^2 + 4t^4 + 16t^2}}{(t^2 + 2)^2}$$

$$= \frac{2(t^2 + 2)}{(t^2 + 2)^2}$$

$$= \frac{2}{t^2 + 2} \qquad \text{Length of } \mathbf{T}'(t)$$

Therefore,

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2}{(t^2 + 2)^2}. \qquad \text{Curvature}$$

Example: 17. Show that curvature of circle at any point is $\kappa = \frac{1}{a}$.

Solution: Equation of circle is $x = a\cos t,\ y = a\sin t$

$r(t) = a\cos t\, i + a\sin t\, j$

$$r'(t) = -a\sin t i + a\cos t j$$

$$|r'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{-a\sin t i + a\cos t j}{a} = -\sin t i + \cos t j$$

$$\kappa = \frac{|T'(t)|}{|r'(t)|} = \frac{|-\cos t i + \sin t j|}{a} = \frac{1}{a}$$

□

Example: 18. Find the curvature of curve $y = x^4$ at point P(1,1).

Solution:

$$y = x^4, y' = 4x^3, y'' = 12x^2$$

at $x = 1$,

$$y' = 4, y'' = 12$$

Curvature at P(1,1) is $\kappa = \frac{|y''|}{|1+(y')^2|}$, $\qquad \kappa = \frac{12}{|1+16|^{\frac{3}{2}}} = \frac{12}{(17)^{\frac{3}{2}}}$

□

Example: 19. Find the curvature of curve $x = cos^3 t$, $y = sin^3 t$ at point $p\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$.

Solution:

$$f'(t) = -3\cos^2 t \sin t,\ \ g'(t) = 3\sin^2 t \cos t$$

$$f''(t) = 6\cos t \sin^2 t - 3\cos^3 t,\ \ g''(t) = 6\sin t \cos^2 t - 3\sin^3 t$$

$$\cos^3 t = \frac{\sqrt{2}}{4} = \left(\frac{1}{\sqrt{2}}\right)^3 \Rightarrow \cos t = \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4},$$

$$NOTE:\ \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ \ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$at\ t = \frac{\pi}{4},\ f'(\frac{\pi}{4}) = -3\cos^2\frac{\pi}{4}\sin\frac{\pi}{4} = -3\frac{1}{2\sqrt{2}},\ \ g'(\frac{\pi}{4}) = 3\sin^2\frac{\pi}{4}\cos\frac{\pi}{4} = 3\frac{1}{2\sqrt{2}}$$

$$f''(\frac{\pi}{4}) = 6\cos\frac{\pi}{4}\sin^2\frac{\pi}{4} - 3\cos^3\frac{\pi}{4} = 6\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 3\frac{1}{2\sqrt{2}}$$

$$g''(\frac{\pi}{4}) = 6\sin\frac{\pi}{4}\cos^2\frac{\pi}{4} - 3\sin^3\frac{\pi}{4} = 6\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 3\frac{1}{2\sqrt{2}}$$

Substituting values in the formula

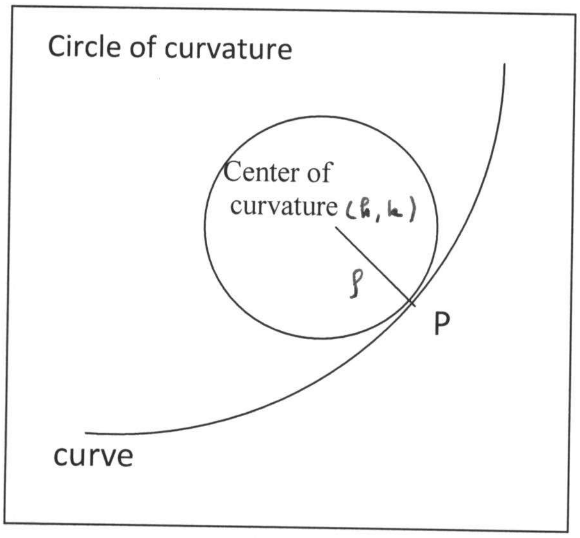
$$\kappa = \frac{\left|f'(t)g''(t) - g(t)'f''(t)\right|}{\left|(f'(t))^2 + (g'(t))^2\right|^{\frac{3}{2}}}$$

$$at\ \ t = \frac{\pi}{4}$$

$$\kappa = \frac{\left|-\left(3\frac{1}{2\sqrt{2}}\right)\left(3\frac{1}{2\sqrt{2}}\right)t) - \left(3\frac{1}{2\sqrt{2}}\right)\left(3\frac{1}{2\sqrt{2}}\right)\right|}{\left|\left(\left(3\frac{1}{2\sqrt{2}}\right)\right)^2 + \left(\left(3\frac{1}{2\sqrt{2}}\right)\right)^2\right|^{\frac{3}{2}}} = \frac{\frac{9}{4}}{\left(\frac{9}{4}\right)^{\frac{3}{2}}} = \frac{2}{3}$$

$\square$

11.9 Radius of Curvature ρ



Radius of Curvature

$$\rho = \frac{1}{\kappa}$$

Center of Curvature (h, k)

$$h = x - \frac{y'\left(1+(y')^2\right)}{y''} \quad , \quad k = y + \frac{\left(1+(y')^2\right)}{y''}$$

Example: 17. Find the Radius and center of curvature of the curve $y = x^4$ at point P(1,1).

Solution:

$$y = x^4, y' = 4x^3, y'' = 12x^2$$

at $x = 1,$

$$y' = 4, y'' = 12$$

Curvature at P(1,1) is $\kappa = \dfrac{|y''|}{\left|1+(y')^2\right|}$, $\kappa = \dfrac{12}{|1+16|^{\frac{3}{2}}} = \dfrac{12}{(17)^{\frac{3}{2}}}$

Radius of Curvature

$$\rho = \frac{1}{\kappa} = \frac{17^{\frac{3}{2}}}{12}$$

Center of Curvature (h, k)

$$h = x - \frac{y'\left(1+(y')^2\right)}{y''} = 1 - \frac{4(1+16)}{12} = 1 - \frac{68}{12} = \frac{-56}{12} = \frac{-14}{3}$$

$$k = y + \frac{\left(1+(y')^2\right)}{y''} = 1 + \frac{1+16}{12} = 1 + \frac{17}{12} = \frac{29}{12}$$

Center of curvature is $\left(\dfrac{-14}{3}, \dfrac{29}{12}\right)$ $\square$

Ex. Find the curvature, radius of curvature and the center of curvature of the curve $y = 2 - x^3$ at the point $(1,1)$

Solution. ① <u>curvature</u> $\quad K = \dfrac{|y''|}{(1+y'^2)^{3/2}}$

$$y = 2 - x^3, \qquad y' = -3x^2, \qquad y'' = -6x$$

$$y'\Big|_{(1,1)} = -3, \qquad y''\Big|_{(1,1)} = -6$$

$$K = \frac{|-6|}{(1+9)^{3/2}} = \frac{6}{10^{3/2}}$$

② <u>Radius of curvature</u> $\quad \rho = \dfrac{1}{K} = \dfrac{10^{\frac{3}{2}}}{6}$

③ <u>center of curvature</u> (h, k)

$$h = x - \frac{y'(1+y'^2)}{y''} = 1 - \frac{(-3)(1+9)}{-6}$$

$$= 1 - \left(\frac{-30}{-6}\right) = 1 - 5 = -4$$

$$k = y + \frac{(1+y'^2)}{y''} = 1 + \frac{10}{-6} = 1 - \frac{5}{2} = -\frac{2}{3}$$

center of curvature is $\left(-4, -\frac{2}{3}\right)$

Ex. Find curvature, radius of curvature and center of curvature of the curve $y = x^4$ at $P(1,1)$.

<u>Solution</u>

$$y = x^4, \qquad y' = 4x^3, \qquad y'' = 12x^2$$

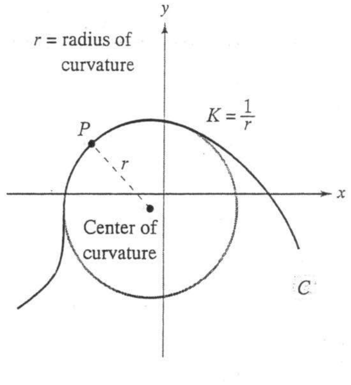
$$y'\Big|_{(1,1)} = 4 \qquad y''\Big|_{(1,1)} = 12$$

① $K = \dfrac{|12|}{(1+16)^{3/2}} = \dfrac{12}{17^{3/2}}$ $\qquad$ ② $\rho = \dfrac{1}{K} = \dfrac{17^{\frac{3}{2}}}{12}$

③ $h = 1 - \dfrac{4(1+16)}{12} = 1 - \dfrac{68}{12} = \dfrac{-56}{12} = -\dfrac{14}{3}$

$k = 1 + \dfrac{1+16}{12} = 1 + \dfrac{17}{12} = \dfrac{29}{12}$

center of curvature is $\left(-\frac{14}{3}, \frac{29}{12}\right)$



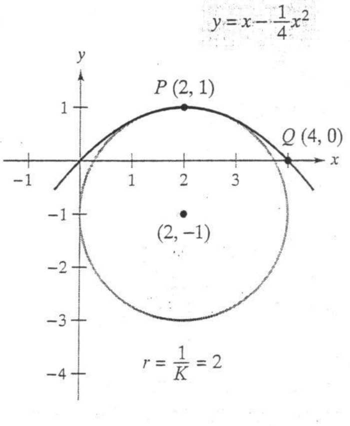
The circle of curvature
Figure 11.35

Let C be a curve with curvature K at point P. The circle passing through point P with radius $r = 1/K$ is called the **circle of curvature** if the circle lies on the concave side of the curve and shares a common tangent line with the curve at point P. The radius is called the **radius of curvature** at P, and the center of the circle is called the **center of curvature.**

The circle of curvature gives us a nice way to estimate graphically the curvature K at a point P on a curve. Using a compass, you can sketch a circle that snuggles up against the concave side of the curve at point P, as shown in Figure 11.35. If the circle has a radius of r, you can estimate the curvature to be $K = 1/r$.

EXAMPLE 6 Finding Curvature in Rectangular Coordinates

Find the curvature of the parabola given by $y = x - \frac{1}{4}x^2$ at $x = 2$. Sketch the circle of curvature at $(2, 1)$.



The circle of curvature
Figure 11.36

Solution The curvature at $x = 2$ is as follows.

$$y' = 1 - \frac{x}{2} \qquad\qquad y' = 0$$

$$y'' = -\frac{1}{2} \qquad\qquad y'' = -\frac{1}{2}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} \qquad K = \frac{1}{2}$$

Because the curvature at $P(2, 1)$ is $\frac{1}{2}$, it follows that the radius of the circle of curvature at that point is 2. Thus, the center of curvature is $(2, -1)$, as shown in Figure 11.36. [In the figure, note that the curve has the greatest curvature at P. Try showing that the curvature at $Q(4, 0)$ is $1/2^{5/2} \approx 0.177$.]

EXERCISES For CURVATURE

In Exercises 1- 4 , find the curvature of the curve described by a position vector of a particle moving along curve at given time :

1. $r(t) = ti + 2\cos tj,$ at $t = \frac{\pi}{2}$
2. $x = 3\cos t, y = 3\sin t,$ at $t = \frac{\pi}{2}$
3. $x = t - \sin t, y = 1 - \cos t,$ at $t = \frac{\pi}{2}$
4. $x = t^2, y = t^3,$ at $t = 1.$
5. $x = t^2 - 1, y = t$ at p(3,2).
6. $x = t + t^2, y = t^3$ at p(2,1).
7. $x = t - \sin t, y = 1 - \cos t,$ at p(0,0).
8. $x = \sin^2 t, y = \cos^2 t,$ at $p(\frac{1}{2}, \frac{1}{2})$.

In Exercise 9 - 19 , find the curvature κ, radius of curvature ρ and center of curvature *(h,k)* fir the curve C at the given point P:

9. $y = \sin x + \cos x,$ $P(\frac{\pi}{2},1)$.
10. $y = x^3 + 2,$ P(1,3).
11. $y = \sqrt{x},$ P(1,1).
12. $y = 2x^2 - x + 2,$ P(1,3).
13. $y^2 = x,$ P(0,0).
14. $y = \sin 4x,$ $P(\frac{\pi}{8},1)$.
15. $y = 1 - x^2,$ P(0,1).
16. $y = x^2,$ P(0,0).
17. $xy = 1,$ P(1,1).
18. $y = x + \frac{1}{x},$ at $x = 2$.
19. Find a point on $y = x^2 - x + 1$, where the curvature is 2.

11.6 Tangential and Normal components of Acceleration

$$a = a_T T + a_N N$$

$$\|a\|^2 = a_T{}^2 + a_N{}^2$$

Tangential Component $$a_T = \frac{r'(t).r''(t)}{\|r'(t)\|}$$

Normal Component $$a_N = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|}, \quad a_N = \sqrt{\|a\|^2 - a_T{}^2}$$

Curvature $$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = a_N \frac{1}{\|r'(t)\|^2}$$

Example : The position vector of a moving point at time t is $r(t) = 3t\, i + t^3 j + 3t^2 k$, find the tangential and normal components of acceleration at time t. Also find curvature

Solution:

$$r(t) = 3t\, i + t^3 j + 3t^2 k$$
$$r'(t) = 3\, i + 3\, t^2 j + 6t\, k$$
$$r''(t) = 6\, t\, j + 6\, k$$
$$r'(t) . r''(t) = 18t^3 + 36t$$
$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 3 & 3t^2 & 6t \\ 0 & 6t & 6 \end{vmatrix} = -18t^2 i - 18j + 18tk$$
$$\|r'(t)\| = \sqrt{9 + 9t^4 + 36t^2} = 3\sqrt{t^4 + 4t^2 + 1}$$
$$\|r'(t) \times r''(t)\| = \sqrt{342t^4 + 342t^2 + 342} = 18\sqrt{t^4 + t^2 + 1}$$

Tangential Component $$a_T = \frac{r'(t).r''(t)}{\|r'(t)\|} = \frac{18t^3 + 36t}{3\sqrt{t^4 + 4t^2 + 1}} = \frac{6t^3 + 12t}{\sqrt{t^4 + 4t^2 + 1}}$$

Normal Component $$a_N = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|} = \frac{18\sqrt{t^4 + t^2 + 1}}{3\sqrt{t^4 + 4t^2 + 1}} = \frac{6\sqrt{t^4 + t^2 + 1}}{\sqrt{t^4 + 4t^2 + 1}}$$

Curvature $$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{18\sqrt{t^4 + t^2 + 1}}{9\sqrt{t^4 + 4t^2 + 1}.(t^4 + 4t^2 + 1)} = \frac{2}{3} . \frac{\sqrt{t^4 + t^2 + 1}}{(t^4 + 4t^2 + 1)^{\frac{3}{2}}}$$

Ex. The position vector of a moving point at time t is $r(t) = \langle 4\cos t, 9\sin t, t\rangle$, Find the tangential and normal components of acceleration and curvature at time t.

Solution.

$$r(t) = \langle 4\cos t, 9\sin t, t\rangle$$

$$r'(t) = \langle -4\sin t, 9\cos t, 1\rangle$$

$$r''(t) = \langle -4\cos t, -9\sin t, 0\rangle$$

$$\|r'(t)\| = \sqrt{16\sin^2 t + 81\cos^2 t + 1}$$

$$r'(t)\cdot r''(t) = +16\sin t\cos t - 81\sin t\cos t = -65\sin t\cos t$$

$$r'(t)\times r''(t) = \begin{vmatrix} i & j & k \\ -4\sin t & 9\cos t & 1 \\ -4\cos t & -9\sin t & 0 \end{vmatrix} = 9\sin t\, i - 4\cos t\, j + 36k$$

$$a_T = \frac{r'\cdot r''}{\|r'\|} = \frac{-65\sin t\cos t}{\sqrt{16\sin^2 t + 81\cos^2 t + 1}}$$

$$a_N = \frac{\|r'\times r''\|}{\|r'\|} = \frac{\sqrt{81\sin^2 t + 16\cos^2 t + 1296}}{\sqrt{16\sin^2 t + 81\cos^2 t + 1}}$$

$$K = \frac{\|r'\times r''\|}{\|r'\|^3} = \frac{\sqrt{81\sin^2 t + 16\cos^2 t + 1296}}{\sqrt{(16\sin^2 t + 81\cos^2 t + 1)^3}}$$

Curvature

Let a space curve C have the parametrization $x = f(t)$, $y = g(t)$, $z = h(t)$, where f'', g'', and h'' exist. The curvature K at the point $P(x, y, z)$ on C is

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = a_N \frac{1}{\|\mathbf{r}'(t)\|^2}.$$

EXAMPLE 4

(a) Find the curvature K of the twisted cubic $x = t$, $y = t^2$, $z = t^3$ at the point (x, y, z).

(b) Find four-decimal-place approximations for K at the points corresponding to $t = 1, 2, 3$, and 4.

SOLUTION

(a) If we let

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},$$

then the curve is the same as that considered in Example 1. Substituting the expressions obtained there for $\mathbf{r}'(t)$ and $\mathbf{r}'(t) \times \mathbf{r}''(t)$ into the formula for K

$$K = \frac{2(9t^4 + 9t^2 + 1)^{1/2}}{(9t^4 + 4t^2 + 1)^{3/2}}.$$

We could also find K by substituting for a_N and $\|\mathbf{r}'(t)\|$

(b) Substituting $t = 1, 2, 3$, and 4 into the formula for K obtained in part (a), we obtain the following approximations for K (compare with the table on page 782).

t	1	2	3	4
(x, y, z)	(1, 1, 1)	(2, 4, 8)	(3, 9, 27)	(4, 16, 64)
K	0.1664	0.0132	0.0027	0.0009

Note IF the normal component is difficult to find by above formula, then we can use

$$a_N = \sqrt{\|a\|^2 - a_T^2}$$

Example Use above formula to find a_N in Example page 11.13

Solution

$$a_T = \frac{4t + 18t^3}{(1+4t^2+9t^4)^{\frac{1}{2}}}$$

$$a = r''(t) = 2j + 6tk \qquad \|a\| = \sqrt{4 + 3t^2}$$

$$a_N = \sqrt{\|a\|^2 - a_T^2}$$

$$= \sqrt{(4+3\cancel{6}t^2 - \frac{(4t+18t^3)^2}{(1+4t^2+9t^4}} = 2\left(\frac{9t^4+9t^2+1}{9t^4+4t^2+1}\right)^{\frac{1}{2}}$$

Summary

1. vector valued function $r(t) = f(t)i + g(t)j + h(t)k$

   of curve C. $x = f(t)$, $y = g(t)$, $z = h(t)$

2. derivative $\dfrac{dr}{dt} = r'(t) = \dfrac{df}{dt}i + \dfrac{dg}{dt}j + \dfrac{dh}{dt}k$

   $= f'(t)i + g'(t)j + h'(t)k$

3. velocity vector $v(t) = r'(t)$

   speed $\|v(t)\| = \|r'(t)\| = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}$

   acceleration vector $a(t) = r''(t) = f''(t)i + g''(t)j + h''(t)k$

4. Unit Tangent vector $T(t) = \dfrac{r'(t)}{\|r'(t)\|}$

   Principal unit normal vector $N(t) = \dfrac{T'(t)}{\|T'(t)\|}$

5. curvature in the plane (i) C is given by $y = f(x)$ at Point P

   $$K = \frac{|y''_P|}{[1 + (y'_P)^2]^{\frac{3}{2}}}$$

   (ii) C is given by $x = f(t)$, $y = g(t)$

   $$K = \frac{|f'g'' - g'f''|}{[f'^2 + g'^2]^{3/2}} \text{ at } P.$$

   Radius of curvature $\rho = \dfrac{1}{K}$

   center of curvature $h = x - \dfrac{y'(1+y'^2)}{y''}$

   $k = y + \dfrac{1 + y'^2}{y''}$

6. Curvature in space $K = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

7. Components of acceleration.

   Tangential component $a_T = \dfrac{r'(t) \cdot r''(t)}{\|r'(t)\|}$

   Normal component $a_N = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|}$

EXERCISES For Curvature and Tangential and Normal Components of acceleration

In Exercises 1- 10 , find the tangential and normal components of acceleration and the curvature of a particle moving along the curve C at the point P:

1. $x = t, y = \frac{1}{2}t^2, z = \frac{1}{3}t^3,$ $\quad$ $P(1, \frac{1}{2}, \frac{1}{3}).$
2. $r(t) = \cos 4ti + \sin 4tj + 3tk,$ $\quad$ $t = \frac{\pi}{8}.$
3. $r(t) = \frac{1}{2}t^2 i + \frac{1}{4}t^4 j + \frac{1}{3}t^3 k,$ $\quad$ $t = 1.$
4. $r(t) = 2ti + 7tj + 3tk,$ $\quad$ $t = 1.$
5. $r(t) = (t^2 - 1)i + (t + 1)j + \frac{1}{2}(t^2 + 1)k,$ $\quad$ $t = 1.$
6. $r(t) = 3ti + t^2 j + tk,$ $\quad$ $t = 2.$
7. $r(t) = (\cos t + t\sin t)i + (\sin t - t\cos t)j + k,$ $t = 0.$
8. $r(t) = < 3t, t^2, t^2 >,$ $\quad$ $t = 2.$
9. $r(t) = < t^2, 4\cos t, 3\sin t >,$ $\quad$ $t = \frac{\pi}{2}.$
10. $r(t) = (3t - t^3)i + 3t^2 j + (3t + t^3)k,$ $\quad$ $t = 1.$

CALCULUS

OF

SEVERAL

VARIABLES