(A) Anti-derivatives

Anti-derivatives

Definition 1 *A function F is called an anti-derivative of f on an interval I if*

$$F^{'}(x) = f(x) \text{ for every } x \in I \ .$$

Example 1

1. *Let $F(x) = x^2 + 3x + 1$ and $f(x) = 2x + 3$.*
   Since $F^{'}(x) = f(x)$, the function $F(x)$ is an anti-derivative of $f(x)$.
2. *Let $G(x) = \sin(x) + x$ and $g(x) = \cos(x) + 1$.*
   We know that $G^{'}(x) = \cos(x) + 1$ and this means the function $G(x)$ is an anti-derivative of $g(x)$.

Generally, if $F(x)$ is an anti-derivative of $f(x)$, then every function $F(x) + c$ is also anti-derivative of $f(x)$, where c is a constant. The question that can be raised here is: Is the anti-derivative of a function f unique? In other words, does the function $f(x)$ have any other anti-derivatives that are different from $F(x)+c$. The next theorem gives the answer to this question.

Theorem 1 *If the functions $F(x)$ and $G(x)$ are anti-derivatives of a function $f(x)$ on the interval I, there exists a constant c such that $G(x) = F(x) + c$.*

The last theorem means that any anti-derivative $G(x)$, which is different from the function $F(x)$ can be expressed as $F(x) + c$ where c is an arbitrary constant. The following examples clarify this point.

Example 2 *Let $f(x) = 2x$. The functions*
$F(x) = x^2 + 2$,
$G(x) = x^2 - \frac{1}{2}$,
$H(x) = x^2 - \sqrt[3]{2}$,
and many other functions are anti-derivatives of a function $f(x)$. Generally, for the function $f(x) = 2x$, the function $F(x) = x^2 + c$ is the anti-derivative where c is an arbitrary constant.

(B) Indefinite Integrals

Indefinite Integrals

Definition 2 *Let f be a continuous function on an interval I. The Indefinite integral of $f(x)$ is the general anti-derivative of $f(x)$ on I and symbolized by $\int f(x)\ dx$.*

Remark 1 *If $F(x)$ is an anti-derivative of f, then*

$$\int f(x)\ dx = F(x) + c \ .$$

The function $f(x)$ is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of integration and c is the constant of integration.

Now, by using the previous remark, the general anti-derivatives in Example 1 are

1. $\int 2x + 3\ dx = x^2 + 3x + c.$
2. $\int \cos(x) + 1\ dx = \sin(x) + x + c.$

The following table lists basic indefinite integrals.

Derivative	Indefinite Integrals
$\frac{d}{dx}(x) = 1$	$\int 1\ dx = x + c$
$\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = 1,\ n \neq 1$	$\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}(\sin\ x) = \cos\ x$	$\int \cos\ dx = \sin\ x + c$
$\frac{d}{dx}(-\cos\ x) = \sin\ x$	$\int \sin\ x\ dx = -\cos\ x + c$
$\frac{d}{dx}(\tan\ x) = \sec^2\ x$	$\int \sec^2\ x\ dx = \tan\ x + c$
$\frac{d}{dx}(-\cot\ x) = \csc^2\ x$	$\int \csc^2\ x\ dx = -\cot\ x + c$
$\frac{d}{dx}(\sec\ x) = \sec\ x\ \tan\ x$	$\int \sec\ x\ \tan\ x\ dx = \sec\ x + c$
$\frac{d}{dx}(-\csc\ x) = \csc\ x\ \cot\ x$	$\int \csc\ x\ \cot\ x\ dx = -\csc\ x + c$

Table 1: The list of the basic integration rule.

Example 3 *Evaluate the following integrals:*

1. $\int x^{-3}\ dx$
2. $\int \frac{1}{\cos^2 x}\ dx$

Solution:

In this section, we shall list main properties of indefinite integrals and use them to integrate some functions.

Properties of Indefinite Integrals

Theorem 2 *Let f and g be integrable functions, then*

1. $\frac{d}{dx}\int f(x)\ dx = f(x)$.
2. $\int \frac{d}{dx}(F(x))\ dx = F(x) + c$.
3. $\int \big(f(x) \pm g(x)\big)\ dx = \int f(x) \pm \int g(x)\ dx$.
4. $\int kf(x)\ dx = k\int f(x)\ dx$, *where k is a constant*

In the following example, we use the previous properties and the table of the basic integrals to solve some indefinite integrals.

Example 4 *Evaluate the following integrals:*

1. $\int 4x + 3\ dx$
2. $\int 2\sin x + 3\cos x\ dx$
3. $\int \sqrt{x} + \sec^2 x\ dx$
4. $\int \frac{d}{dx}(\sin x)\ dx$
5. $\frac{d}{dx}\int \sqrt{x+1}\ dx$

Solution:

Example 5 *Solve the differential equation $f'(x) = 6x^2 + x - 5$ subject to the initial condition $f(0) = 2$.*

Solution:

$$\int f'(x)\ dx = \int (6x^2 + x - 5)\ dx$$
$$f(x) = 2x^3 + \frac{1}{2}x^2 - 5x + c\ .$$

Use the condition $f(0) = 2$ i.e., substitute $x = 0$ into the function $f(x)$. We have

$$f(0) = 0 + 0 - 0 + c = 2 \Rightarrow c = 2\ .$$

From this, the solution of the differential equation is $f(x) = 2x^3 + \frac{1}{2}x^2 - 5x + 2$.

Example 6 *Solve the differential equation $f''(x) = 5\cos x + 2\sin x$ subject to the initial condition $f(0) = 3$ and $f'(0) = 4$.*

Solution:

$$\int f''(x)\ dx = \int (5\cos x + 2\sin x)\ dx$$
$$f'(x) = 5\sin x - 2\cos x + c$$

The condition $f'(0) = 4$ yields

$$f'(0) = 0 - 2 + c = 4 \Rightarrow c = 6\ .$$

Thus, $f'(x) = 5\sin x - 2\cos x + 6$. Now, again

$$\int f'(x)\ dx = \int (5\sin x - 2\cos x + 6)\ dx$$
$$f(x) = -5\cos x - 2\sin x + 6x + c$$

Use the condition $f(0) = 3$ by substituting $x = 0$ into $f(x)$. This yields

$$f(0) = -5 - 0 + 0 + c = 3 \Rightarrow c = 8\ .$$

Thus, the solution of the differential equation is $f(x) = -5\cos x - 2\sin x + 6x + 8$.

Note that, in the previous examples, we use x as the variable of the integration. However, for this role, we can use any variable y, z, t, ...

The integration by substitution (known as u-substitution) is one technique for solving some complex integrals. The goal of changing the variable of the integration is to obtain a simple indefinite integral. In a sense that the substitution method turns the integral into a simpler integral involving the variable u that can be solved by using either the table of the basic integrals or other techniques of integration. The following definition shows how the substitution technique works.

Substitution Method

Theorem 3 *Let g be a differentiable function on the interval $[a, b]$ where the derivative is continuous. Let f be a continuous on an interval I involves the range of the function g. If F is an anti-derivative of the function f on I, then*

$$\int f(g(x))g^{'}(x)\ dx = F(g(x)) + c,\ x \in [a, b]$$

Steps of Integration by Substitution:

For simplicity, the substitution method can be summarized in the following steps:
Step 1: Choose a new variable u.
Step 2: Determine the value of du.
Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral is in terms of u.
Step 4: Evaluate the new integral.
Step 5: Return the evaluation to the initial variable x.

Example 7 *Evaluate the integral* $\int 2x(x^2+1)^3\ dx$.
Solution:

Example 8 *Evaluate the integral* $\int \frac{x^2-1}{(x^3-3x+1)^6}\ dx$.
Solution:

Example 9 *Evaluate the integral* $\int x\sqrt{x-1}\ dx$.
Solution:

The upcoming corollary simplifies the process of the integration by substitution for some functions.

Corollary 1 *If* $\int f(x)\ dx = F(x) + c$*, then for any* $a \neq 0$*,*

$$\int f(ax \pm b)\ dx = \frac{1}{a}F(ax \pm b) + c\ .$$

Example 10 *Evaluate the following integrals:*

1. $\int \sqrt{2x-5}\ dx$

2. $\int \cos(3x+4)\ dx$

Solution:

Evaluate the following integrals:

1. $\int x\sqrt{1+x^2}\ dx$
2. $\int x^2\sqrt{x-1}\ dx$
3. $\int \frac{\tan x}{\cos^2 x}\ dx$
4. $\int \sin^5 x \cos x\ dx$
5. $\int \frac{x}{\sqrt{2x^2+1}}\ dx$
6. $\int \cos t\sqrt{1-\sin t}\ dt$
7. $\int \frac{\cos^3 x}{\csc x}\ dx$
8. $\int \cos(3x+4)\ dx$
9. $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\ dx$
10. $\int sec4x \tan 4x\ dx$
11. $\int \frac{\sqrt{\cot x}}{\sin^2 x}\ dx$
12. $\int (1+\frac{1}{t})t^{-2}\ dt$
13. $\int \frac{x}{\sqrt{2x-1}}\ dx$
14. $\int x^2(4x^3-6)^7\ dx$
15. $\int \sin^2(3x)\cos(3x)\ dx$

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(A) Summation Notation

Summation is the addition of a sequence of numbers and the result is their sum or total.

Summation Notation

Definition 3 *Let $\{a_1, a_2, ..., a_n\}$ be a set of numbers. The symbol $\sum_{k=1}^{n} a_k$ represents their sum:*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n \ .$$

Example 11 *Evaluate the following sums:*

1. $\sum_{i=0}^{3}(i^3)$.
2. $\sum_{j=1}^{4}(j^2+1)$.
3. $\sum_{k=1}^{3}(k+1)k^2$.

Solution:

(B) Properties of Sum Notation

1. $\sum_{k=1}^{n} c = \underbrace{c + c + ... + c}_{\text{n-times}} = nc.$
2. $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k.$
3. $\sum_{k=1}^{n} c\ a_k = c\ \sum_{k=1}^{n} a_k$ for any $c \in \mathbb{R}$.

Example 12 *Evaluate the following sums:*

1. $\sum_{k=1}^{10} 15$.
2. $\sum_{k=1}^{4} k^2 + 2k$.
3. $\sum_{k=1}^{3} 3(k+1)$.

Solution:

In the following theorem, we present some summations of polynomial expressions. They will be used later in the Riemann sum to find the area under the graph of a function f.

Theorem 4

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
3. $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Example 13 *Evaluate the following sums:*

1. $\sum_{k=1}^{100} k$.
2. $\sum_{k=1}^{10} k$.
3. $\sum_{k=1}^{10} k$.

Solution:

Example 14 *Express the following sums in terms of n:*

1. $\sum_{k=1}^{n}(k+1)$.
2. $\sum_{k=1}^{n}(2k^2-k+1)$.

Solution:

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(C) Riemann Sum and Area

Definition 4 *A set $P = \{x_0, x_1, x_2, ..., x_n\}$ is called a partition of a closed interval $[a, b]$ if*

$$a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$$

for any positive integer n.

Note that,
1. the division of the interval $[a, b]$ by the partition P generates n sub-intervals: $[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n]$.
2. The length of each sub-interval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
3. The sub-intervals do not intersect and their union gives the main interval $[a, b]$.

Definition 5 *The norm of the partition of P is the largest length among $\Delta x_0, \Delta x_1, \Delta x_2, ..., \Delta x_n$ i.e.,*

$$\| P \| = max\{\Delta x_0, \Delta x_1, \Delta x_2, ..., \Delta x_n\} \ .$$

Example 15 *If $P = \{0, 1.2, 2.3, 3.6, 4\}$ is a partition of the interval $[0, 4]$, find the norm of the partition P.*
Solution:
We need to find the sub-intervals and their lengths.

Sub-interval $[x_{k-1}, x_k]$	Length Δx_k
$[0, 1.2]$	$1.2 - 0 = 1.2$
$[1.2, 2.3]$	$2.3 - 1.2 = 1.1$
$[2.3, 3.6]$	$3.6 - 2.3 = 1.3$
$[3.6, 4]$	$4 - 3.6 = 0.4$

From the table, the norm is $\| P \| = 1.3$.

Remark 2
1. *The partition P of the interval $[a, b]$ is called regular if $\Delta x_0 = \Delta x_1 = \Delta x_2 = ... = \Delta x_n = \Delta x$.*
2. *For any positive integer n, if the partition P is regular then*

$$\Delta x = \frac{b-a}{n} \quad and \quad x_k = a + k \, \frac{b-a}{n} \ .$$

To explain the previous result, let P be a regular partition of the interval $[a, b]$. We know that $x_0 = a$ and $x_n = b$. Then,

$$x_1 = x_0 + \Delta x,$$
$$x_2 = x_1 + \Delta x = x_0 + 2\Delta x,$$
$$x_3 = x_2 + \Delta x = x_0 + 3\Delta x \ .$$

By continuing doing so, we have

$$x_k = x_0 + k \, \Delta x = x_0 + k \, \frac{b-a}{n} \ .$$

Note that, when $n \to \infty$, the norm $\| P \| \to 0$.

Example 16 *Define a regular partition P that divides the interval $[1, 4]$ into 4 sub-intervals.*
Solution:

Riemann Sum

Definition 6 *Let f be a defined function on the closed interval $[a,b]$ and let $P=\{x_0,x_1,...,x_n\}$ be a partition of $[a,b]$. Let $\omega_k \in [x_{k-1},x_k]$, $k=1,2,3,...,n$ be a mark on the partition P. Then, the Riemann sum of f for P is*

$$R_p=\sum_{k=1}^{n} f(\omega_k)\Delta x_k .$$

Consider Figure 1, we want to explain the definition of the Riemann sum of a function f for the partition P. As shown in the figure, the amount $f(\omega_1)\Delta x_1$ is the area of the rectangular A_1, $f(\omega_2)\Delta x_2$ is the area of the rectangular A_2 and so on. The sum of these areas approximates the whole area under the graph of the function f. In other words, the area under f bounded by $x=a$ and $x=b$ can be estimated by the Riemann sum where as the number of the sub-intervals increases (i.e., $n\to\infty$), the estimation becomes better. From this,

$$A=\lim_{n\to\infty} R_p=\lim_{n\to\infty}\sum_{k=1}^{n} f(\omega_k)\Delta x_k .$$

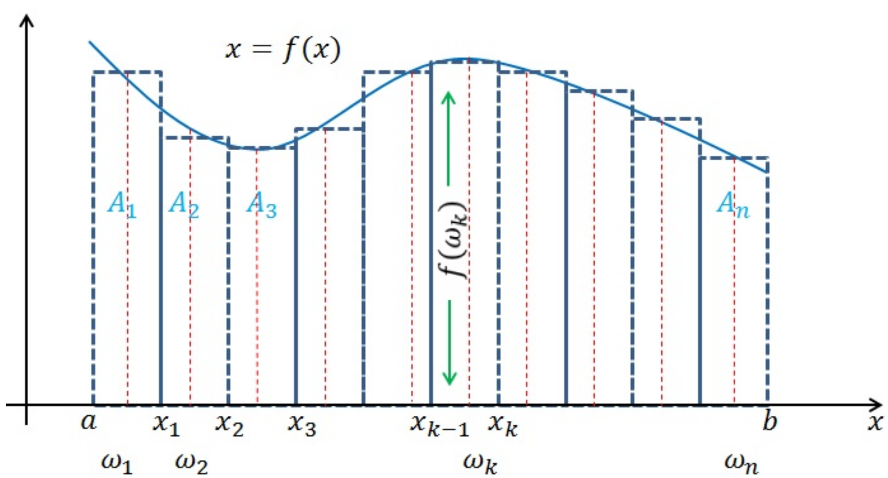

Figure 1: The Riemann sum of the function $f(x)$ for the partition P.

Example 17 *Find the Riemann sum R_p of the function $f(x)=2x-1$ for the partition $P=\{-2,0,1,3,4\}$ of the interval $[a,b]$ by choosing the mark as follows:*

1. *the left-hand end point,*
2. *the right-hand end point,*
3. *the mid point.*

Solution:

1. The left-hand end point.

Sub-intervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2,0]$	$0-(-2)=2$	-2	-5	-10
$[0,1]$	$1-0=1$	0	-1	-1
$[1,4]$	$4-1=3$	1	1	3
$[4,6]$	$6-4=2$	4	7	14
$R_p=\sum_{k=1}^{n} f(\omega_k)\Delta x_k$				6

2. The right-hand end point.

Sub-intervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2,0]$	$0-(-2)=2$	0	-1	-2
$[0,1]$	$1-0=1$	1	0	0
$[1,4]$	$4-1=3$	4	7	21
$[4,6]$	$6-4=2$	6	11	22
$R_p=\sum_{k=1}^{n} f(\omega_k)\Delta x_k$				41

3. The mid point.

Sub-intervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2,0]$	$0-(-2)=2$	-1	-3	-6
$[0,1]$	$1-0=1$	0.5	0	0
$[1,4]$	$4-1=3$	1.5	2	6
$[4,6]$	$6-4=2$	5	9	18
$R_p=\sum_{k=1}^{n} f(\omega_k)\Delta x_k$				18

Example 18 *Let A be the area under the graph of $f(x) = x + 1$ from $x = 1$ to $x = 3$. Find the area A by taking the limit of the Riemann sum such that the partition P is regular and the mark ω_k is the right end point of each sub-interval.*

Solution:

For regular partition, we have

1. $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$, and
2. $x_k = x_0 + k\ \Delta x$ where $x_0 = 1$.

Since the mark ω_k is the right end point of the sub-intervals $[x_{k-1}, x_k]$, then $\omega_k = x_k = 1 + \frac{2k}{n}$. From this,

$$f(\omega_k) = \frac{2k}{n} + 2 = \frac{2}{n}(k+1) .$$

Now,

$$R_p = \sum_{k=1}^{n} f(w_k)\Delta x_k = \frac{4}{n^2}\sum_{k=1}^{n}(n+k) = \frac{4}{n^2}\left[n^2 + \frac{n(n+1)}{2}\right] = 4 + \frac{4n^2+n}{2n^2} .$$

Remember: $\sum_{k=1}^{n}(n + k) = \sum_{k=1}^{n} n + \sum_{k=1}^{n} k$ also $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Therefore, $\lim_{n\to\infty} R_p = 4 + 2 = 6$.

Exercises

1 - 8 ■ If P is a partition of the interval $[a, b]$, find the norm of the partition P:

1. $P = \{-1, 0, 1.3, 4, 4.1, 5\}$, $[-1, 5]$
2. $P = \{0, 0.5, 1, 2.5, 3.1, 4\}$, $[0, 4]$
3. $P = \{-3, 0, 2.3, 4.6, 4.8, 5.5, 6\}$, $[-3, 6]$
4. $P = \{-2, 0, 2.3, 3, 3.5, 4\}$, $[-2, 4]$
5. $P = \{3, 3.5, 3.6, 4, 4.9, 7\}$, $[3, 7]$
6. $P = \{-1, 0, 1.3, 4, 4.1, 5\}$, $[-1, 5]$
7. $P = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$, $[-1, 2]$
8. $P = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$, $[0, \pi]$

1 - 4 ■ Define a regular partition P that divides the interval $[a, b]$ into n sub-intervals:

1. $[a, b] = [0, 3]$ $n = 5$
2. $[a, b] = [-1, 4]$ $n = 6$
3. $[a, b] = [-4, 4]$ $n = 8$
4. $[a, b] = [0, 1]$ $n = 4$

5 - 7 ■ Find the Riemann sum R_p of the function $f(x) = x^2 + 1$ for the partition $P = \{0, 1, 3, 4\}$ of the interval $[a, b]$ by choosing the mark as follows:

5. the left-hand end point,
6. the right-hand end point,
7. the mid point.

8 - 11 ■ Let A be the area under the graph of $f(x)$ from a to b. Find the area A by taking the limit of the Riemann sum such that the partition P is regular and the mark ω_k is the right end point of each sub-intervals:

8. $f(x) = x/3$ $a = 1$, $b = 2$
9. $f(x) = x - 1$ $a = 0$, $b = 3$
10. $f(x) = 5 - x^2$ $a = -1$, $b = 1$
11. $f(x) = x^3 - 1$ $a = 0$, $b = 4$

Definite Integrals

In this section, we are going to define the definite integral and how it is calculated. The following definition shows that the definite integral of a function f on the interval $[a, b]$ is the Riemann sum when $\| P \| \to 0$.

Definite Integrals

Definition 7 *Let f be a function defined on a closed interval $[a, b]$. If f is integrable on that interval, the definite integral of f is*

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_k f(\omega_k)\Delta x_k = A .$$

The numbers a and b are called the limits of the integration.

Example 19 *Evaluate the following integral* $\int_2^4 x + 2\ dx$.

Solution:

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The following remark simplifies the process of calculating the definite integrals.

Remark 3 *To find the value of a definite integral* $\int_a^b f(x)\ dx$*, we first find the value of the indefinite integral* $\int f(x)\ dx = F(x) + c$ *as shown in Chapter* **??**. *Then, we substitute* a *and* b *into* $F(x)$ *as follows:*

$$\int_a^b f(x)\ dx = F(b) - F(a)\ .$$

Example 20 *Evaluate the following integrals:*

1. $\int_{-1}^{2} 2x + 1\ dx$
2. $\int_0^3 x^2 + 1\ dx$
3. $\int_1^2 \frac{1}{\sqrt{x^3}}\ dx$
4. $\int_0^{\frac{\pi}{2}} \sin(x) + 1\ dx$
5. $\int_{\frac{\pi}{4}}^{\pi} \sec^2(x) - 4\ dx$
6. $\int_0^{\frac{\pi}{3}} \sec(x)\ tan(x) + x\ dx$

Solution:

One application of the definite integrals is to find the area under the graph of a function f on the interval $[a, b]$. This is clear from Definition 7, if f is integrable on the interval $[a, b]$, then

$$A = \int_a^b f(x)\ dx\ .$$

The application of the definite integrals will be discussed in details in Chapter **??**.

Properties of Definite Integrals

Theorem 5

1. $\int_a^b c\ dx = c(b - a)$,
2. $\int_a^a f(x)\ dx = 0.$

Reversed Interval of Definite Integrals

Theorem 6 *If* f *is integrable on* $[a, b]$*, then*

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx\ .$$

Linearity of Definite Integrals

Theorem 7

1. *If* f *and* g *are integrable on* $[a, b]$*, then* $f + g$ *and* $f - g$ *are integrable on* $[a, b]$ *and*

$$\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x) \pm \int_a^b g(x)\ dx\ .$$

2. *If* f *is integrable on* $[a, b]$ *and* $k \in \mathbb{R}$*, then* $k\ f$ *is integrable on* $[a, b]$ *and*

$$\int_a^b k\ f(x)\ dx = k\ \int_a^b f(x)\ dx\ .$$

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Comparison of Definite Integrals

Theorem 8

1. *If f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for all $x \in [a,b]$ then*

$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx\ .$$

2. *If f is integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$ then*

$$\int_a^b f(x)\ dx \geq 0\ .$$

Additive Interval of Definite Integrals

Theorem 9 *If f is integrable on the intervals $[a,c]$ and $[c,b]$, then $f(x)$ is integrable on $x \in [a,b]$ and*

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx\ .$$

Example 21 *Evaluate the following integrals:*

1. $\int_0^2 3\ dx$.

2. $\int_2^2 x^2 + 4\ dx$.

Solution:

Example 22 *If* $\int_0^2 f(x)\ dx = 4$ *and* $\int_0^2 g(x)\ dx = 2$, *then find* $\int_0^2 3f(x) - \frac{g(x)}{2}\ dx$.

Solution:

Example 23 *Prove that* $\int_0^2 (x^3 + x^2 + 2)\ dx \geq \int_0^2 (x^2 + 1)\ dx$ *without evaluating the integrals.*

Solution:

Put $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 + 1$. We find that $f(x) - g(x) = x^3 + 1 > 0$ for all $x \in [0,2]$. From Theorem 8, we have

$$\int_0^2 (x^3 + x^2 + 2)\ dx \geq \int_0^2 (x^2 + 1)\ dx\ .$$

Example 24 *Evaluate the integral* $\int_0^2 |\ x - 1\ |\ \ dx$

Solution: