

Partial Derivatives

Linear Algebra and Vector Analysis

Introduction

- If f is a function of two variables x and y , suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.
- Then we are really considering a function of a single variable x , namely, $g(x) = f(x, b)$.
- If g has a derivative at a , then we call it the *partial derivative of f with respect to x at (a, b)* and denote it by $f_x(a, b)$.

Notations

- There are many alternate notations for partial derivatives:

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Finding Partial Derivatives

- The partial derivative with respect to x is just the ordinary derivative of the function g of a single variable that we get by keeping y fixed:

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x .
2. To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .

Example

- If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.
- Solution Holding y constant and differentiating with respect to x , we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Solution (cont'd)

- Holding x constant and differentiating with respect to y , we get

$$f_x(x, y) = 3x^2y^2 - 4y$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Example

- If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$.
- Solution We have

$$f_x(x, y) = -2x$$

$$f_y(x, y) = -4y$$

$$f_x(1, 1) = -2$$

$$f_y(1, 1) = -4$$

Example

- If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
- Solution Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x} \left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y} \left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

Example

- Find f_x , f_y , and f_z if $f(x, y, z) = e^{xy} \ln z$.
- Solution Holding y and z constant and differentiating with respect to x , we have

$$f_x = ye^{xy} \ln z$$

- Similarly,

$$f_y = xe^{xy} \ln z \quad \text{and} \quad f_z = e^{xy}/z$$

Higher Derivatives

- If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives

$$(f_x)_x, (f_x)_y, (f_y)_x, \text{ and } (f_y)_y,$$

which are called the *second partial derivatives* of f .

Higher Derivatives (cont'd)

- If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Mixed Partial Derivatives

- Note that $f_{xy} = f_{yx}$ in the preceding example, which is not just a coincidence.
- It turns out that $f_{xy} = f_{yx}$ for most functions that one meets in practice:

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example

- Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

- Solution Earlier we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \quad f_y(x, y) = 3x^2y^2 - 4y$$

- Therefore

$$f_{xx} = \frac{\partial}{\partial x} (3x^2 + 2xy^3) = 6x + 2y^3$$

$$f_{xy} = \frac{\partial}{\partial y} (3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x} (3x^2y^2 - 4y) = 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y} (3x^2y^2 - 4y) = 6x^2y - 4$$

Example

- Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.
- Solution

$$f_x = 3 \cos(3x + yz)$$

$$f_{xx} = -9 \sin(3x + yz)$$

$$f_{xxy} = -9z \cos(3x + yz)$$

$$f_{xxyz} = -9 \cos(3x + yz) + 9yz \sin(3x + yz)$$