MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE - 1

"The imaginary numbers are a wonderful flight of God's spirit; they are almost an amphibian between being and not being",

LEIBNITZ 1702

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Why we care about complex variables in physics?-1

- In physics we may encounter pairs of functions u and v which both satisfy Laplace equation.

$$\nabla^2\psi = \frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi(x,y)}{\partial y^2} = 0$$

- For example if u is the electrostatic potential then v is the electric field. Or, in the hydrodynamics of irrotational flow of an ideal fluid, u might describe the velocity potential, whereas the function v *the stream function.* In many cases in which these functions are unknown, mapping or transforming in the complex plane permits us to create a coordinate system tailored to the particular problem.

Why we care about complex variables in physics?-2

- Second order differential equations of interest in physics may be solved in power series in a complex plane. The use of complex analysis gives us greater insight into the behavior of our solution and a powerful tool (analytic continuation) for extending the reason in which the solution is valid.
- The change of the parameter k from real to imaginary, $k \to ik$, transforms the Helmholtz equation into the diffusion equation. The same change transforms the Helmholtz equation solutions (Bessel and spherical Bessel) into the diffusion equation solutions (modified Bessel and spherical Bessel).

Why we care about complex variables in physics?-3

- Integrals in the complex plane have a wide variety of useful applications:

1. Evaluating definite integrals
2. Inverting power series
3. Forming finite products
4. Obtaining solutions of differential for large values of the variable (asymptotic solutions)
5. Investigating the stability of potentially oscillatory systems.
6. Inverting integral transforms.

Basics of complex algebra-1

- A complex number is an ordered pair of two numbers (*a, b*) or *a+ib,* where $i = \sqrt{-1}$.
- Similarly a complex variable is an ordered pair of two real variables

$$z = (x, y) = x + iy$$

- The real numbers x and y are known as the *real* and *imaginary* parts of z, respectively:

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y$$

- The ordering is significant since

$$x + iy \neq y + ix$$

Basics of complex algebra-2

- The addition and multiplication of complex numbers are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2,\ y_1 x_2 + x_1 y_2)$$

- The following properties (known from real numbers) do hold:

$$z_1 + z_2 = z_2 + z_1, \quad z_1 \cdot z_2 = z_2 \cdot z_1$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

$$z(z_1 + z_2) = z z_1 + z z_2$$

Basics of complex algebra-3

- The identity elements with respect to addition and multiplication are the numbers:

$$0 = (0,\ 0), \quad 1 = (1,\ 0)$$

- For which:

$$z + 0 = z, \quad z \cdot 1 = z$$

- For any complex number $z = (x, y)$ there is a number $-z = (-x, -y)$ such that:

$$z + (-z) = 0$$

Basics of complex algebra-4

- For any **non-zero** complex number z there is a number z^{-1} such that:

$$z \cdot z^{-1} = 1$$

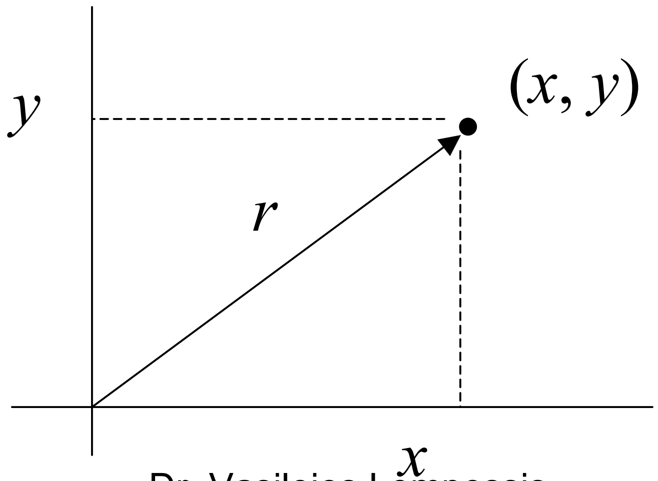
- The division of two complex numbers is defined as:

$$\frac{z_1}{z_2} = z_1 \left(z_2 \right)^{-1}$$

Basics of complex algebra-5

The complex plane or Argand

- Complex variable can be graphically represented if we plot the real part x of z as the abscissa and the imaginary part y of z as the ordinate.



Basics of complex algebra-6
The complex plane or Argand

- The geometric representation shows us that to each complex number corresponds a vector. The magnitude of this vector is the *modulus* of the complex number:

$$|z| = \sqrt{x^2 + y^2}$$

Basics of complex algebra-7

complex conjugation

- For a given complex number $z = x + iy$

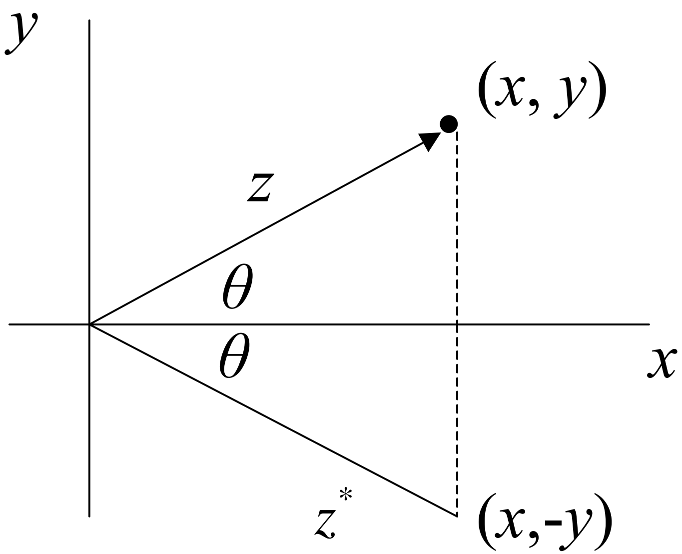
we may define a conjugate number given by

$$z^* = x - iy$$

The complex variable and its conjugate are mirror images of each other reflected in the x-axis.

Basics of complex algebra-8

complex conjugation



- Example: Show that the product of a complex number and its conjugate gives the following important relation: $z \cdot z^{*} = |z|^{2}$

Basics of complex algebra-9

The polar form of a complex number

- The vector form of complex numbers is reflected on the so called *triangle inequality*

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$$

- This inequality can be generalized to:

$$|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n| \quad (n \ge 2)$$

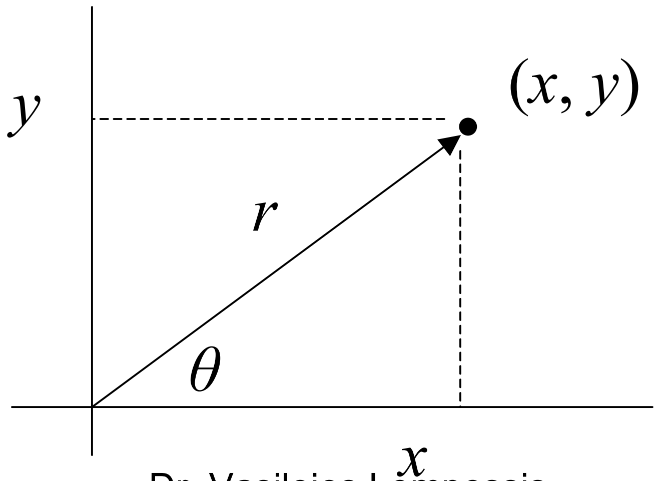
Basics of complex algebra-10

The polar form of a complex number

- If we use polar coordinates we know that

$$x = r\cos\theta$$

$$y = r\sin\theta$$



$$r = \left(x^2 + y^2\right)^{1/2}$$

$$\theta = \arctan(y/x)$$

or $\theta = \arg z$

Basics of complex algebra-11
The polar form of a complex number

- Then we may write for the complex number *z*:

$$z = r(\cos\theta + i\sin\theta)$$

- For a complex number $z \neq 0$ we may write also

$$z = r\left[\cos\left(\theta + 2n\pi\right) + i\sin\left(\theta + 2n\pi\right)\right] \qquad \left(n = 0,\ \pm 1,\ \pm 2,\ldots\right)$$

- As *principal value* of the *arg z,* which we denote it *Arg z,* we define the unique value of *arg z* which lies in the interval (-π, π], thus

$$\arg z = Argz + 2n\pi \qquad \left(n = 0,\ \pm 1,\ \pm 2,\ldots\right)$$

Basics of complex algebra-12
the exponential form

- The relation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is known as *Euler's formula.*

- With the help of this formula we can write a complex number z in polar representation as follows:

$$z = re^{i\theta}$$

Basics of complex algebra-13

product of complex numbers

- Using the exponential form of a complex number we may find the following relations

$$\left|z_1 \cdot z_2\right| = \left|z_1\right| \cdot \left|z_2\right|$$

$$\arg\left(z_1 \cdot z_2\right) = \arg z_1 + \arg z_2$$

Basics of complex algebra-14

The polar form of a complex number

- The choice of polar or cartesian representation is a matter of convenience. Addition and subtraction of complex variables are easier in cartesian form. Multiplication, division, powers and roots are easier to handle in polar form.