MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE - 2

Powers and roots of complex functions - Limits of complex functions - Continuity of complex functions

POWERS & ROOTS - a

- The integer powers of a non-zero complex number $z = re^{i\theta}$ are given by the equation:

$$z^n = r^n e^{in\theta} \qquad (n = 0, \ \pm 1, \ \pm 2, \ldots)$$

- A useful application of the above formula is the famous *de Moivre formula.*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad (n = 0, \ \pm 1, \ \pm 2, \ldots)$$

POWERS & ROOTS - b

- With the help of the concept of the complex power we can define the concept of a complex root.
- The complex roots of 1 are those complex numbers which satisfy the relation:

$$z^n = 1 \qquad (n=0,\ \pm 1,\ \pm 2,\ldots)$$

- We can show, using the polar form, that the **different** n-th roots of 1 are:

$$z = \exp\left(i\frac{2k\pi}{n} \right) \qquad (k = 0,\ 1,\ 2,\ldots,n-1)$$

POWERS & ROOTS - c

- Let a complex number $z = |z| e^{i\theta_0}$ we can show that the **different** *n*-th roots of this number are given by:

$$w_k = z^{1/n} = \sqrt[n]{|z|}\exp\left[i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right] \quad (0 \le k \le n-1)$$

Complex Functions -a

- Using complex variables we may construct complex functions. Any complex function may be resolved into real and and imaginary parts.

$$w = f(z) = u(x,y) + iv(x,y)$$

- With $u(x,y)$ and $v(x,y)$ real functions.
- Example: find the real and imaginary parts of the complex function

$$f(z) = (x + iy)^2$$

Elementary complex functions-a

- Polynomial functions

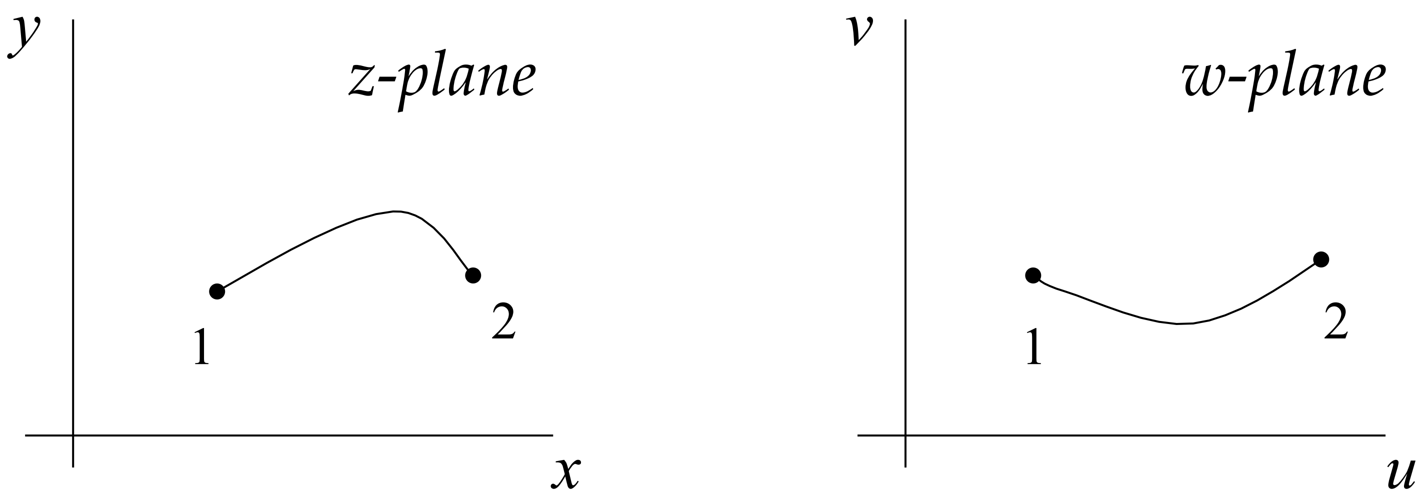
$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0\,, \qquad n \in N,\;\; a_n \neq 0$$

- Rational functions

$$f(z) = \frac{P(z)}{Q(z)}$$

where *P(z), Q(z)* are polynomials

Complex Functions and Mapping



- The function $w(z) = u(x,y) + iv(x,y)$

maps points in the *xy*-plane into points in the *uv*-plane

Example: Check the mapping achieved by function *iz*.

Limit of a complex function-1

- We may define the concept of the limit of a complex function in a way similar, but not identical, to the concept of limit in real functions.
- Let's assume a complex function $f(z)$ and a complex number $z_0=x_0+iy_0$. Then we may define the following equivalent statements.

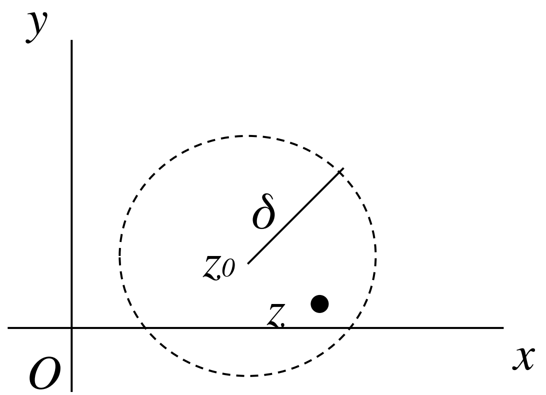
Limit of a complex function-2

$$f(z) = u(x,y) + iv(x,y), \qquad w_0 = u_0 + iv_0$$

$i)\ \lim_{z \to z_0} f(z) = w_0$

$ii)\ \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) \to u_0 \qquad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) \to v_0$

Limit of a complex function-3

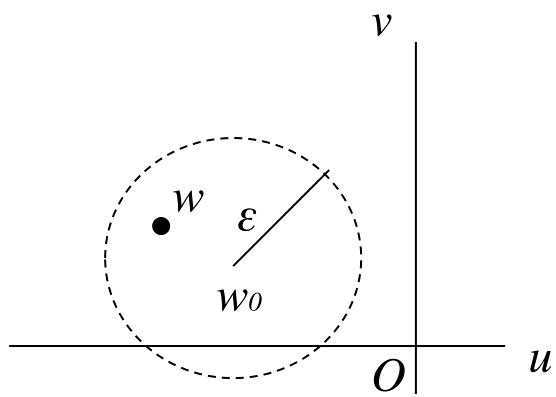


- In a more mathematical language. For any positive number ε, there is a positive number δ such that:

$$\left|f(z) - w_0\right| < \varepsilon$$

when

$$0 < \left|z - z_0\right| < \delta$$



Limit of a complex function-4

theorems

- Let: $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$

then

$$\lim_{z \to z_0} \left[f(z) + F(z) \right] = w_0 + W_0$$

$$\lim_{z \to z_0} \left[f(z) \cdot F(z) \right] = w_0 W_0$$

and if $W_0 \neq 0$

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$$

Limit of a complex function-5

theorems

$$\lim_{z \to z_0} z = z_0 \qquad \lim_{z \to z_0} z^n = z_0^n \qquad (n = 1, 2, ..)$$

$$\lim_{z \to z_0} c = c$$

$$\text{If } P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

$$\lim_{z \to z_0} P(z) = P(z_0)$$

$$\text{If } \lim_{z \to z_0} f(z) = w_0 \quad \text{then} \quad \lim_{z \to z_0} |f(z)| = |w_0|$$

Limit of a complex function-6

theorems

- An interesting thing in the theory of limits in complex analysis is the treatment of infinity.
- What actually is the meaning of $z \to \infty$?
- In such a case the treatment is based on the following relation:

$$\lim_{z \to \infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z \to 0} f(1/z) = w_0$$

Show that $\lim_{z \to \infty} (2z + i)/(z + 1) = 2$

Continuity of a complex function-a

- A complex function $f(z)$ is said to be continuous at a point $z=z_0$ if;

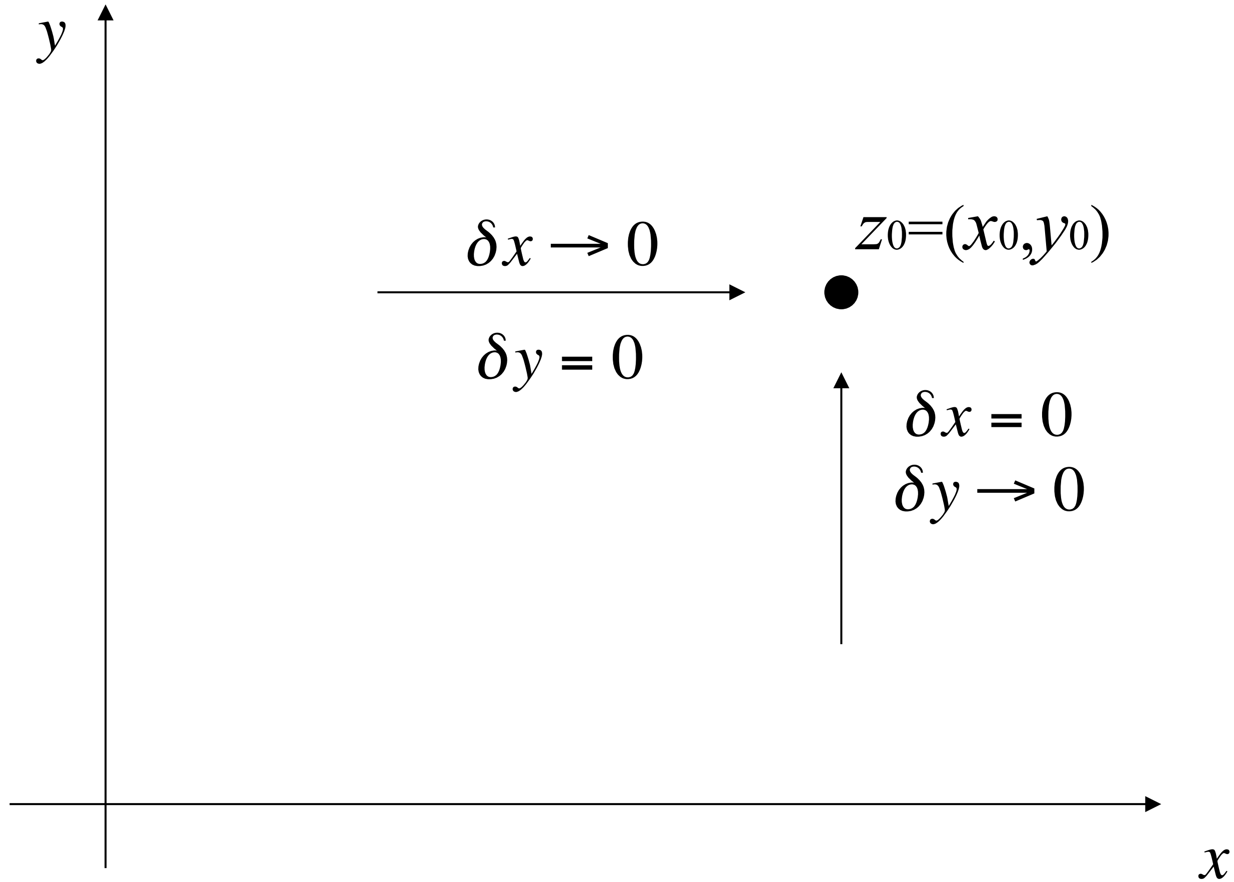
I) The function exists (can be defined) at that point

II) $\lim_{z \to z_0} f(z) = f(z_0)$

Conversely if the $\lim_{z \to z_0} f(z)$ exists independently of

the direction of approach to z_0 then the function is continuous

Approaching a complex number



Continuity of a complex function-b

- A complex function is said to be continuous in a region R if it is continuous at any point of this region.
- If two complex functions are continuous at a common point then their sum and product are also continuous at this point.
- Their ratio is also is continuous at this point if the denominator is non-zero at this point.
- The composition of two continuous functions is a continuous function as well.