PHYS 507
Lecture 4: Electrostatics-b
Electric Potential

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Introduction

- The electric field is not an ordinary vector. It is a special kind of vector with a curl equal to zero.
- Because of this property the line integral of **E** around any closed loop is zero.
- From vector analysis we also know that any vector whose curl is zero is equal to the gradient of some scalar.
- The above help us to define the function:

$$\mathbf{V}(\mathbf{r}) = -\int_{\wp}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

where $\wp$ is a reference point on which we have agreed beforehand.

Electric Potential-a

- The function $V(\mathbf{r})$ depends only on the position $\mathbf{r}$.
- It is called **electric potential**.
- The potential difference between two points **a** and **b** is given by:

$$\mathbf{V}(\mathbf{r}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

- The relation between the electric field **E** and the potential $V(\mathbf{r})$ is given by

$$\mathbf{E} = -\nabla \mathbf{V}(\mathbf{r})$$

Do not confuse potential with potential energy!

Electric Potential-b

- The dependence on a reference point indicates that potential itself has no physical meaning. It is the potential difference which has a physical content.
- Normally we consider the potential equal to zero at infinity.
- There is a (theoretical) exception to this, when the charge distribution itself is extended to infinity.

Electric Potential-c

- The potential obeys the superposition principle. That is, the potential at any given point is the sum of the potentials due to all the source charges separately.

$$V = V_1 + V_2 + \dots$$

- But this is an ordinary algebraic sum and not a vectorial one. Thing are easier with potential

Poisson and Laplace equations

- What do the fundamental equations for **E** give for the potential?
- We can show that we get the so called **Poisson equation**:

$$\nabla^2 V(\mathbf{r}) = \rho / \varepsilon_0$$

which in regions where there are no charges ($\rho=0$) gives us the so called **Laplace equation**:

$$\nabla^2 V(\mathbf{r}) = 0$$

It takes only one equation (Poisson) to determine V because is scalar. For **E** we need two: the div and the curl

Potential of a point like charge

- Taking our reference point to the infinity (where V=0) the potential at a distance r away from a charge q is given by:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

- For a collection of discrete charges the potential is given by (superposition principle):

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}$$

Potential of a continuous charge distribution

- In the case of continuous charge distributions (volume, surface and line) the potential is given respectively by:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda\left(\mathbf{r}'\right)}{r}dl'$$

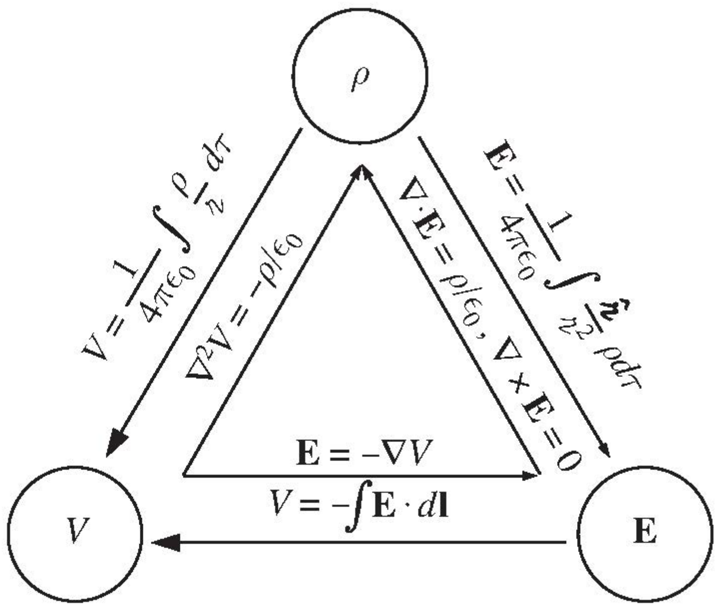
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma\left(\mathbf{r}'\right)}{r}da'$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\left(\mathbf{r}'\right)}{r}d\tau'$$

Warning: All the above hold provided that the potential is zero at infinity

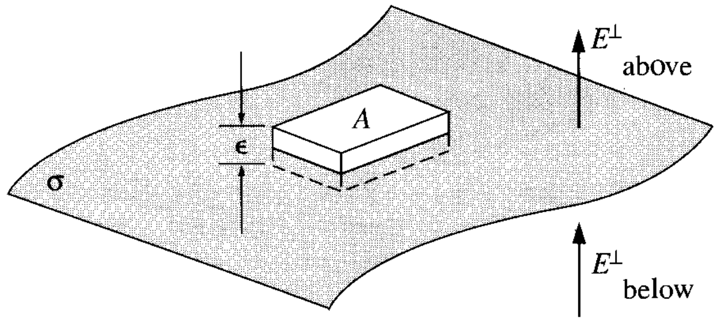
Electrostatic boundary conditions-a

- Our working has been based on two experimental facts: a) superposition principle, and b) Coulomb's law.
- The figure shows us the ways, charge distribution, electric field and potential are related to each other.
- If symmetry allows it, then from charge distribution we calculate the field. Otherwise it is better to calculate the potential.



Electrostatic boundary conditions-b

- In the worked examples you will notice that the electric field always undergoes a discontinuity when you cross a surface charge σ. We can prove that the normal **component of E is discontinuous by an amount σ/ε_o at any boundary.**
- The tangential component of E by contrast is always continuous.



$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\varepsilon_0}\sigma$$

$$E^{\parallel}_{\text{above}} = E^{\parallel}_{\text{below}}$$

Electrostatic boundary conditions-c

- The boundary conditions for the electric field can be combined into a single one.
- The vector $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface from "below" to "above".

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

Electrostatic boundary conditions-d

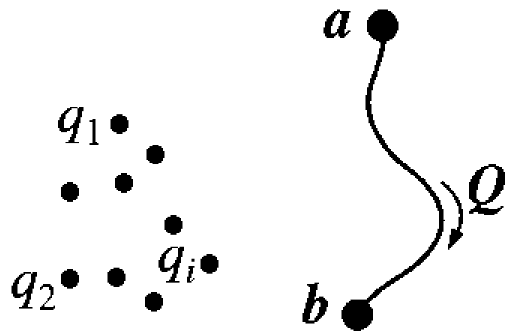
- The potential, on the contrary, is continuous across any boundary.
- However, **the gradient** of the potential inherits the discontinuity of **E**. We can show that:

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

Where $\partial V / \partial n$ is the derivative of the potential in a direction perpendicular to the surface.

Work and Energy in Electrostatics

- Suppose that we have a stationary configuration of source charges and we want to move a test charge *Q* from point **a** to point **b**.
- **Question:** How much work we have to do?



$$W = Q\left(V_{\mathbf{b}} - V_{\mathbf{a}}\right)$$

Potential Difference

- The expression for the work leads us to the definition of the **potential difference** between two points in an electrostatic field:

$$V_{\mathbf{b}} - V_{\mathbf{a}} = W / Q$$

- The potential difference between points **a** and **b** is equal to the work per unit charge required to carry a charged particle from **a** and **b**.

Potential Energy

- In particular if you want to bring the charge from far away and to stick it at a given point **r** then:

$$W = Q\left(V\left(\mathbf{r}\right) - V\left(\infty\right)\right)$$

- So if we set our reference point at infinity:

$$W = QV\left(\mathbf{r}\right)$$

- This work is the **potential energy** (the work needed to create the system).

The Energy of a Point Charge Distribution-a

- How much work is required to assemble an entire collection of point charges?
- The general rule is: Take the product of each pair of charges, divide by their separation distance and add it all up.

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{r_{ij}}$$

The stipulation $j > i$ is just to remind you not to count the same pair twice!

The Energy of a Point Charge Distribution-b

- If we intentionally count the same pair twice then we have to divide by 2 and we get:

$$W = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{r_{ij}}$$

- If we pull out the factor q_i then we get:

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_j}{r_{ij}}\right)}_{\substack{\text{potential at point}\\ \mathbf{r}_i\text{ due to all}\\ \text{other charges}}}$$

- We thus get the expression

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V\left(\mathbf{r}_i\right)$$

- Note that this is also the work you get if you dismantle the system.

The energy of a continuous charge distribution

- In the case of a continuous charge distribution the energy is given by

$$W = \frac{1}{2}\int \rho V \, d\tau$$

- This can be shown to be

$$W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau$$

- We can show that the superposition principle does not hold for electrostatic energy.