PHYS 507
Lecture 9: Magnetostatics

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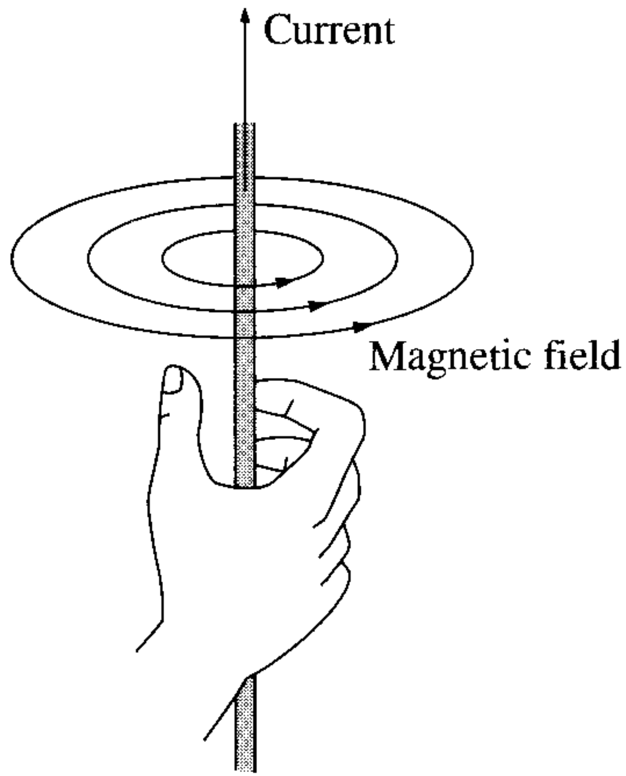
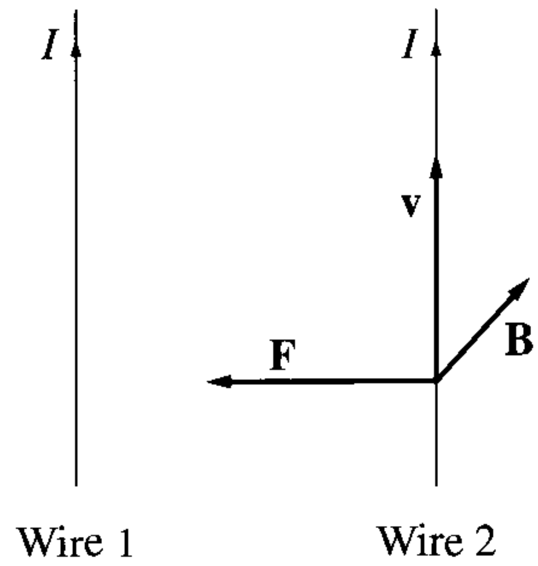
Magnetic Fields-a

- Up to now we have confined our attention to charges **at rest**. Now we will consider charges **in motion**.
- A moving charge generates, in addition to the electric field, a **magnetic field**.
- Experiment has shown that an electric current produces a magnetic field as shown in figure.



Magnetic Fields-b

- Experiment also shows that two nearby current carrying conductors may attract or repel each other due to magnetic forces.

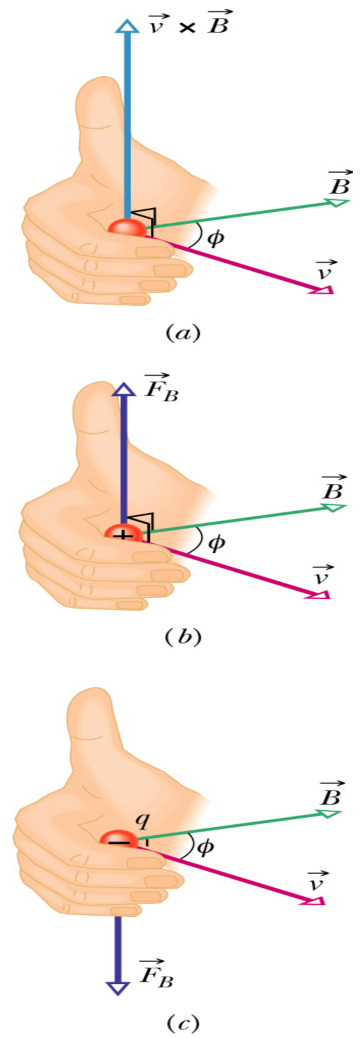


Magnetic Fields-c

- The magnetic force on a charge Q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by:

$$\mathbf{F}_B = Q\mathbf{v} \times \mathbf{B}$$

- The magnetic force is called Lorentz force. The magnetic force acting on a mobile charge is always perpendicular both to the magnetic field and the velocity. The unit of magnetic field is the Tesla (1T)



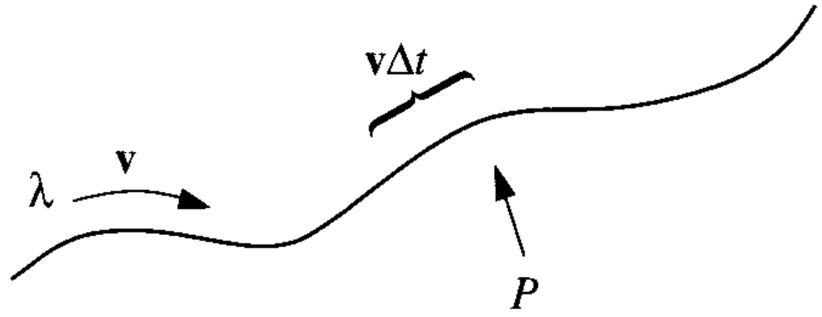
Magnetic Fields-d

- Magnetic forces do not produce work!
- In the presence of both electric and magnetic field the total force is given by:

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Electric Currents

- The current in a wire is the **charge per unit time** passing a given point: $I = \Delta Q/\Delta t$
- The unit of current is the **ampere**: 1A = 1C/s.
- A line charge λ traveling down a wire at speed $\mathbf{v}$ constitutes a current $\mathbf{I} = \lambda\mathbf{v}$.



Magnetic Forces due to Electric Currents

- The magnetic force on a segment of a current carrying wire is given by:

$$\mathbf{F}_{\text{mag}} = \int \left(\mathrm{I} \times \mathrm{B}\right) dl$$

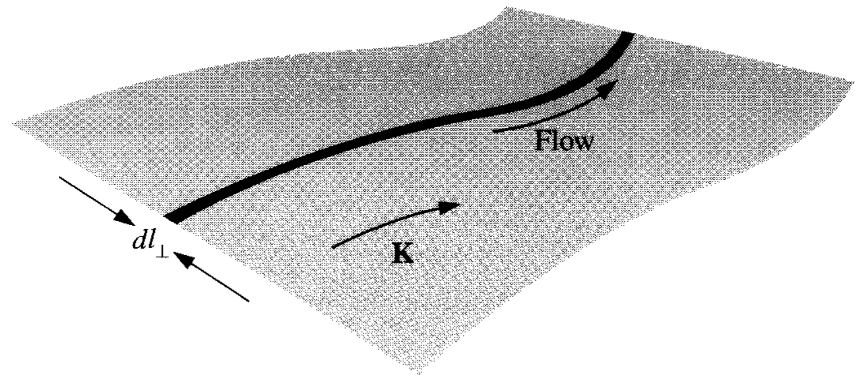
- If the current **I** is in the same direction as $d\mathbf{l}$ we can write

$$\mathbf{F}_{\text{mag}} = \int I \left(d\mathbf{l} \times \mathrm{B}\right)$$

Magnetic Forces due to Surface Electric Currents-a

- When a charge flows over a *surface*, we describe it by the **surface current density, K.**
- As the picture shows, consider a "ribbon" of infinitesimal width $dl_{\perp}$ running parallel to the flow, the surface current density is

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}}$$



Magnetic Forces due to Surface Electric Currents-b

- We could say that: K is the **current per unit width perpendicular to** flow.
- In particular, if the mobile surface charge density is σ and its velocity is **v**, then:

$$\mathbf{K} = \sigma \mathbf{v}$$

- The magnetic force on the surface current is

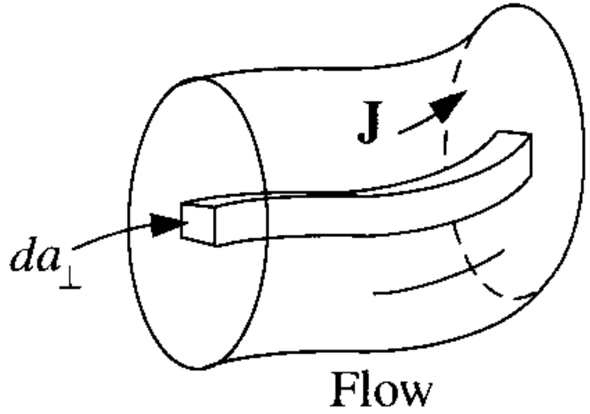
$$\mathbf{F}_{\mathrm{mag}} = \int (\mathbf{v} \times \mathbf{B}) \sigma \, da = \int (\mathbf{K} \times \mathbf{B}) \, da$$

- Just as **E** suffers from discontinuities at a surface charge, so **B** is discontinuous at a surface current. So we must be careful to use the average field.

Magnetic Forces due to Volume Electric Currents-a

- For three-dimensional flow we consider the **volume current density,** **J**. Consider a "tube" if infinitesimal cross section $da_{\perp}$ running parallel to the flow. If the current in the tube is $d\mathbf{I}$ the volume current density is

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}$$



J is *the current per unit area perpendicular to flow*

Magnetic Forces due to Volume Electric Currents-b

- If the mobile volume charge density is ρ and the velocity is $\mathbf{v}$, then:

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}$$

- The magnetic force on a volume current is therefore

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) \rho \, d\tau = \int (\mathbf{J} \times \mathbf{B}) \, d\tau$$

$$\sum_{i=1}^{n} (\quad) q_i \mathbf{v}_i \to \int_{\text{line}} (\quad) \mathbf{I} dl \to \int_{\text{surface}} (\quad) \mathbf{K} da \to \int_{\text{volume}} (\quad) \mathbf{J} d\tau$$

The continuity equation

- The current density **J** and the charge density ρ are related through the so called **continuity equation** which is given by:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

- This is the precise mathematical statement of local charge conservation.

Steady Currents

- **Stationary** charges produce electric fields that are constant in time; **Steady** currents produce magnetic fields that are constant in time. The theory of magnetic effects of steady currents is called **magnetostatics**.
- In magnetostatics we have:

$$\nabla \cdot \mathbf{J} = 0$$

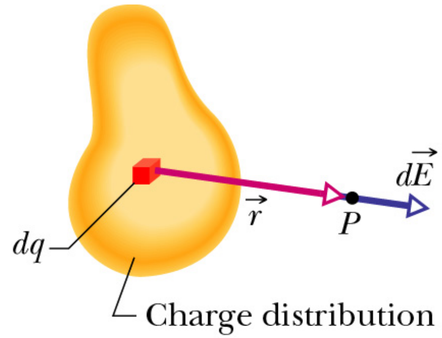
The Biot-Savart-a

- The magnetic field of a steady line current is given by the **Biot-Savart** law:

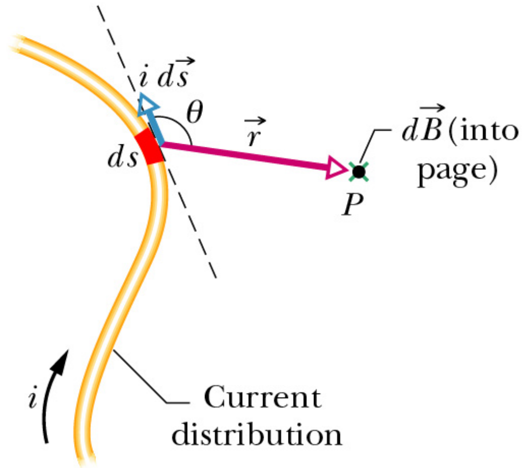
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{I d\mathbf{s} \times \mathbf{r}}{r^3}$$

- The constant μ_0 is called the **permeability of the free space**

$$\mu_0 = 4\pi \times 10^{-7} N / A^2$$



(a)



(b)

The Biot-Savart-b

- For a surface and volume charges the Biot-Savart law gives:

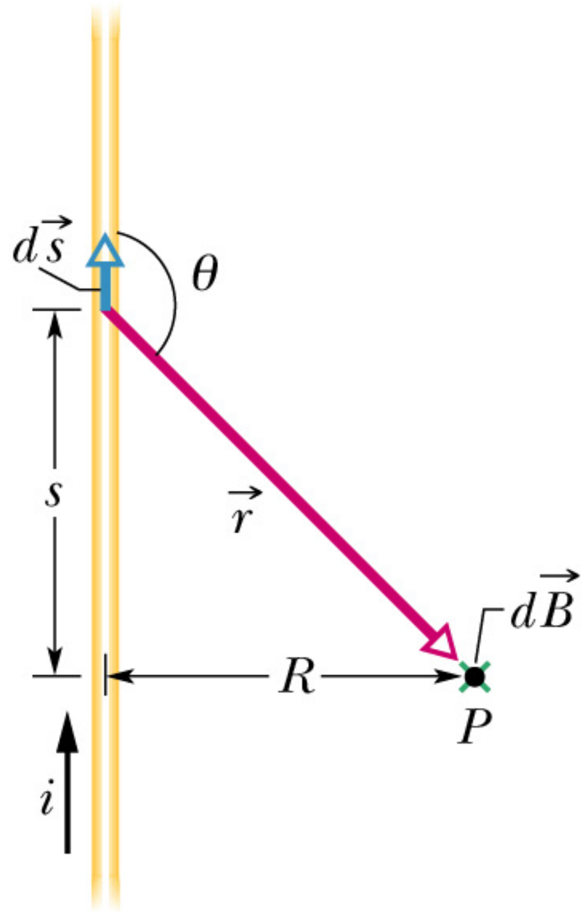
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}^{'})\times \mathbf{r}}{r^3}da^{'} \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}^{'})\times \mathbf{r}}{r^3}d\boldsymbol{\tau}^{'}$$

- The superposition principle **is valid** in magnetostatics.

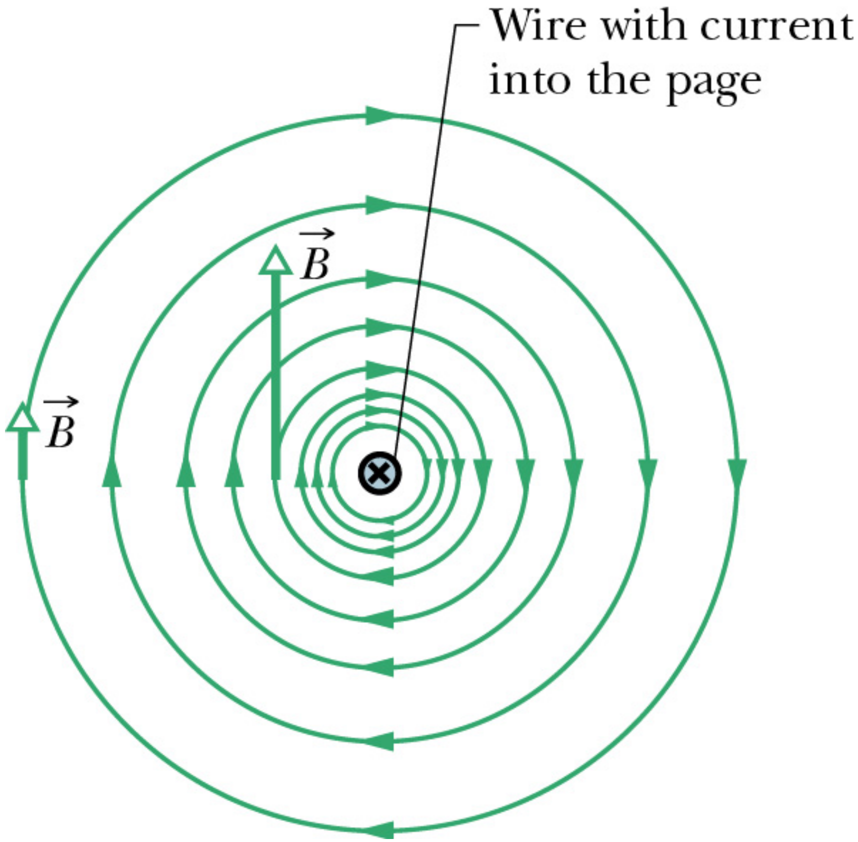
Magnetic Field due to a current in a Long Straight Wire-a

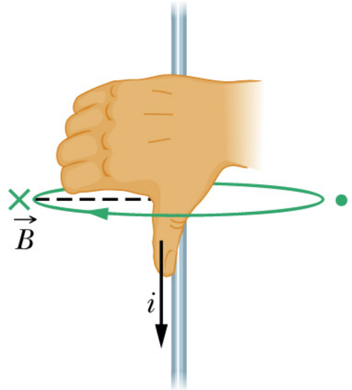
- It can be proved that the magnetic field of a very long straight wire is given by:

$$B = \frac{\mu_0 i}{2\pi R}\hat{\phi}$$

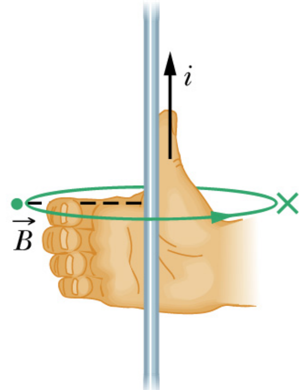


Magnetic Field due to a current in a Long Straight Wire-b





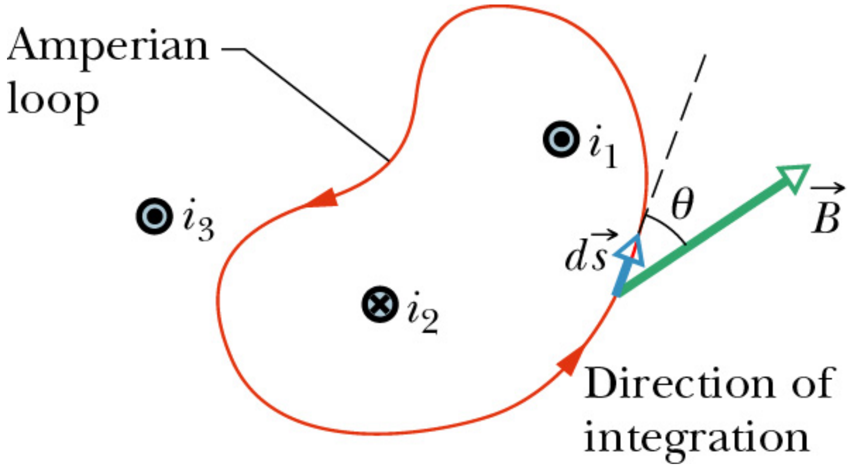
(a)

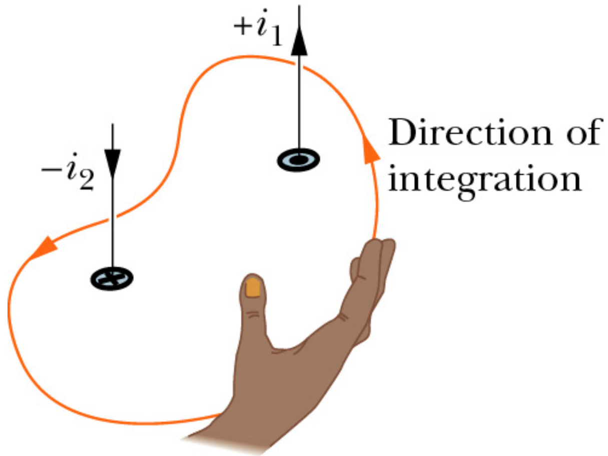


(b)

Ampere's Law-a

- Ampere's law is the equivalent of Gauss's law in magnetism. Instead of a closed surface we consider a closed loop called the *Amperian loop.*
- Curl your right hand around the American loop, with the fingers pointing in the direction of integration.
- A current through the loop in the general direction of your out-streched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.





Ampere's Law-b

- Ampere's law is expressed with the following formula:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

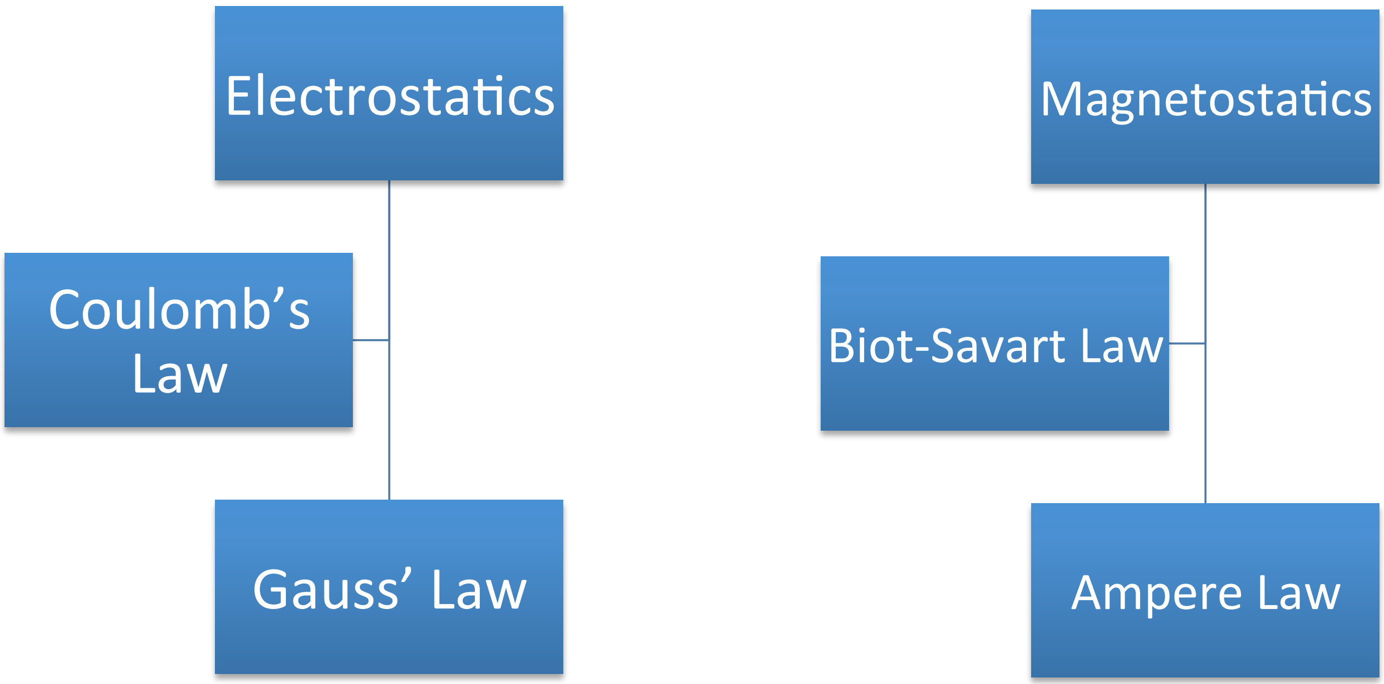
- Where I_{enc} is the total current enclosed in the loop. From Ampere's law we can derive the following relation:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

- We can prove that: $\nabla \cdot \mathbf{B} = 0$

Electrostatics & Magnetostatics

- The following scheme show the corresponding laws in electrostatics and magnetostatics:



The Vector Potential-a

- Just as $\nabla \times \mathbf{E} = 0$ permitted us to introduce a scalar potential (V) in electrostatics $\mathbf{E} = -\nabla V$ so $\nabla \cdot \mathbf{B} = 0$ guides us to introduce a **vector** potential **A** in magnetostatics:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- The vector potential satisfies the following property:

$$\nabla \cdot \mathbf{A} = 0$$

The Vector Potential-b

- The vector potential satisfies a Poisson equation given by:

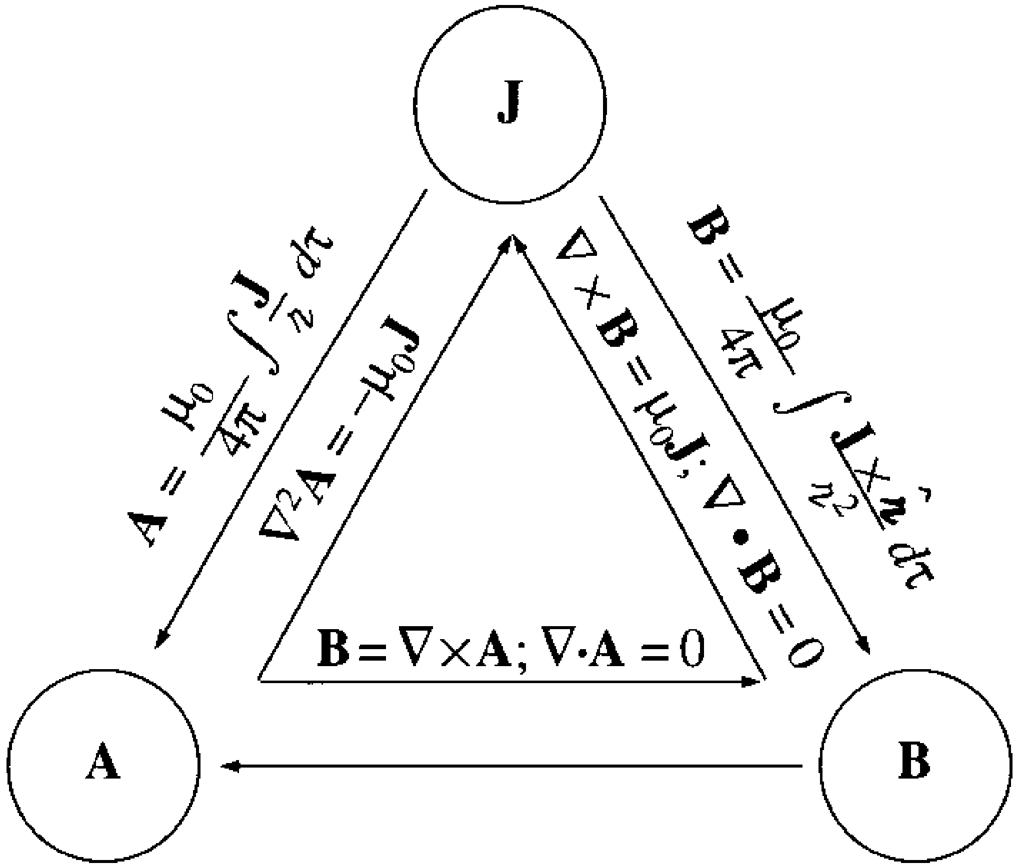
$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

- Assuming **J** goes to infinity the solution of the above equation is given by:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{s} d\tau'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{s} dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{s} dl'; \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{s} da'$$

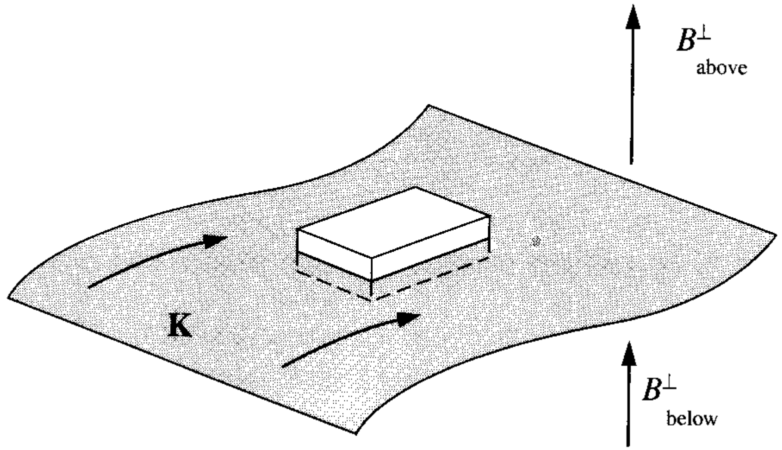
Magnetostatic Boundary Conditions-a



Magnetostatic Boundary Conditions-b

- Just as the electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a surface **current**.
- It can be shown that:

$$\left(\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}}\right) = \mu_0\left(\mathbf{K} \times \hat{\mathbf{n}}\right)$$



$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

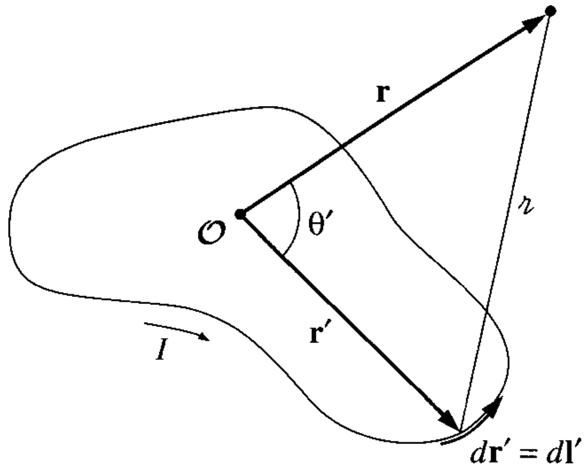
$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$$

$$\frac{\partial \mathbf{A}_{\text{above}}}{\;} - \frac{\partial \mathbf{A}_{\text{below}}}{\;} = -\mu_0 K$$

Multipole Expansion of the Vector Potential-a

- If you want an approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is in order.



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r' \cos\theta' \, d\mathbf{l}' \right.$$

$$\left. \frac{1}{r^3}\oint (r')^2 \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\mathbf{l}' + \ldots \right]$$

Multipole Expansion of the Vector Potential-b

- The leading monopole term is **always zero** since:

$$\oint d\mathbf{l}' = 0$$

- The dominant term is the magnetic dipole term given by:

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

where **m** is the **magnetic dipole term**

$$\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}$$

with **a** the “vector area” of the loop.