PHYS 404
HANDOUT 5 – *Bessel Functions*

1. Show that: $J_{-n}(x) = \left(-1\right)^n J_n(x)$ (n, integer).

(Arf. p. 576)

2. Show that: $J_n(x) + J_{n+2}(x) = 2(n+1)J_{n+1}(x)/x$

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(Arf. p. 576)

3. Show that: $J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$.

4. Show that:

$d/dx\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ and $d/dx\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

(Arf. p. 577)

5. Verify the following: $1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)$.

(Arf. p. 613)

6. Derive the Jacobi-Anger relation: $e^{iz\cos\theta} = \sum_{m=-\omega}^{\infty} i^m J_m(z)e^{im\theta}$. This is an expansion of a plane wave in a series of cylindrical waves.

(Arf. p. 585)

7. Show that the recurrence relation

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

follows directly from differentiation of

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos\left(n\theta - x\sin\theta\right)d\theta$$

(Arf. p. 589)

8. Show that $\frac{d}{dx}\left(xJ_n(x)J_{n+1}(x)\right) = x\left(J_n^2(x) - J_{n+1}^2(x)\right)$.

9. Evaluate $\int x^{-1} J_4(x)\,dx$.

10. From the product of the generating function $g(x,t)=g(x,-t)$ show that $1=\left[J_0(x)\right]^2+2\left[J_1(x)\right]^2+2\left[J_2(x)\right]^2+\ldots$

11. Using the generating function prove the addition theorem of Bessel equations: $J_n(u+v)=\sum_{m=-\infty}^{\infty} J_m(u)J_{n-m}(v)$.

12. Show that

$$\text{(a) } \cos x = J_0(x)+2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x),$$

$$\text{(b) } \sin x = 2\sum_{n=1}^{\infty}(-1)^{n+1} J_{2n+1}(x).$$

13. Using mathematical induction derive the relation:

$$J_n(x)=(-1)^n x^n\left(\frac{d}{xdx}\right)^n J_0(x).$$