PHYS 404
1st Midterm Exam
Saturday 3rd December 2016

Instructor: Dr. V. Lempesis

Student Grade: ..../20

Please answer all questions

1. If $P_2(x)=\frac{1}{2}\left(3x^2-1\right)$ and $P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$, calculate $P_3(x)$. You are given that: $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$.

(5 marks)

2. Show that the Legendre polynomials satisfy the relation: $P_n(1)=1$. Hint: Use the relation for the generating function of the Legendre polynomials: $\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n$. You will also need the relation: $\frac{1}{1-t}=\sum_{n=0}^{\infty}t^n$.

(8 marks)

3. You are given that $P_2(x)=\frac{1}{2}\left(3x^2-1\right)$.

A) Calculate the polynomials $P_2^{-1}(x)$ and $P_2^{0}(x)$ (4 marks).

B) Calculate the spherical harmonics $Y_2^{-1}(\theta,\ \phi)$, $Y_2^{0}(\theta,\ \phi)$ (3 marks).

$$P_n^m(x)=\left(1-x^2\right)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_n(x)$$

$$Y_l^m(\theta,\ \phi)=\varepsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^m(\cos\theta)e^{im\phi}$$

$$\varepsilon=\begin{cases}(-1)^m, & m\geq 0\\ 1 & m<0\end{cases}$$

Solutions of MIDTERM 1 Exam PHYS 404

1

$$(n+1)P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x) \overset{n=3}{\Longrightarrow}$$

$$4P_4(x) = 7x P_3(x) - 3P_2(x) \Rightarrow$$

$$7x P_3(x) = 4P_4(x) + 3P_2(x) \Rightarrow$$

$$7x P_3(x) = 4 \cdot \frac{1}{8}(35x^4 - 30x^2 + 3) + \frac{3}{2}(3x^2 - 1) \Rightarrow$$

$$7x P_3(x) = \frac{35}{2}x^4 - 15x^2 + \cancel{\frac{3}{2}} + \frac{9}{2}x^2 - \cancel{\frac{3}{2}} \Rightarrow$$

$$P_3(x) = \frac{35x^4}{2 \cdot 7x} - \frac{15x^2}{7x} + \frac{9x^2}{2 \cdot 7x} \Rightarrow$$

$$P_3(x) = \frac{5}{2}x^3 + \frac{x}{7}\left(-15 + \frac{9}{2}\right) \Rightarrow$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{x}{7}\,\frac{21}{2} = \frac{5}{2}x^3 - \frac{3x}{2}$$

2 $$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \overset{x=1}{\Longrightarrow} \frac{1}{\sqrt{1-2t+t^2}} = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

$$\Rightarrow \frac{1}{\sqrt{(1-t)^2}} = \sum_{n=0}^{\infty} P_n(1)\, t^n \underset{|t|<1}{\Longrightarrow} \frac{1}{1-t} = \sum_{n=0}^{\infty} P_n(1)\, t^n \quad (a)$$

But we are given that: $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \quad (b)$

$$(a)=(b) \Rightarrow \sum_{n=0}^{\infty} P_n(1)\, t^n = \sum_{n=0}^{\infty} t^n \Longrightarrow \sum_{n=0}^{\infty} P_n(1)\, t^n - \sum_{n=0}^{\infty} t^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(P_n(1) - 1\right) t^n = 0 \Rightarrow P_n(1) - 1 = 0 \Rightarrow P_n(1) = 1$$

3 A) $P_n^m(x) = (1-x^2)^{|m|/2} \dfrac{d^{|m|}}{dx^{|m|}} P_n(x) \overset{m=-1}{\underset{n=2}{\Longrightarrow}} P_2^{-1}(x) = (1-x^2)^{|-1|/2} \dfrac{d^{|-1|}}{dx^{|-1|}} P_2(x)$

$\Rightarrow P_2^{-1}(x) = (1-x^2)^{1/2} \dfrac{d}{dx} P_2(x) \Rightarrow P_2^{-1}(x) = (1-x^2)^{1/2} \dfrac{d}{dx}\left\{\dfrac{1}{2}(3x^2-1)\right\}$

$\Rightarrow P_2^{-1}(x) = (1-x^2)^{1/2} \dfrac{1}{2} 6x = 3x(1-x^2)^{1/2}$

Also

$P_2^0(x) = (1-x^2)^{0/2} \dfrac{d^0}{dx^0} P_2(x) \Rightarrow P_2^0(x) = P_2(x) = \dfrac{1}{2}(3x^2-1)$

B) $Y_\ell^m(\theta,\phi) = \varepsilon \sqrt{\dfrac{(2\ell+1)(\ell-|m|)!}{4\pi\,(\ell+|m|)!}}\, P_\ell^m(\cos\theta) e^{im\phi} \qquad \varepsilon = \begin{cases} (-1)^m, & m \geq 0 \\ 1, & m < 0 \end{cases}$

$Y_2^{-1}(\theta,\varphi) = 1 \cdot \sqrt{\dfrac{(2\cdot 2+1)}{4\pi} \dfrac{(2-|-1|)!}{(2+|-1|)!}}\, P_2^{-1}(\cos\theta) e^{-i\phi} = \sqrt{\dfrac{5}{24\pi}}\, 3\cos\theta (1-\cos^2\theta)^{1/2} e^{-i\phi}$

$$Y_2^{-1}(\theta,\phi) = \sqrt{\frac{5}{24\pi}}\, 3\cos\theta \sin\theta\, e^{-i\phi}$$

Also

$$Y_2^{0}(\theta,\phi) = (-1)^0 \sqrt{\frac{(2\cdot 2+1)\,(\ell-0)!}{4\pi\,(\ell+0)!}}\, P_2^0(\cos\theta)\, e^{i0\phi} =$$

$$= \sqrt{\frac{5}{4\pi}}\, \frac{1}{2}\left(3\cos^2\theta - 1\right)$$