PHYS 507
Lecture 2: Mathematical Description of Vector Fields

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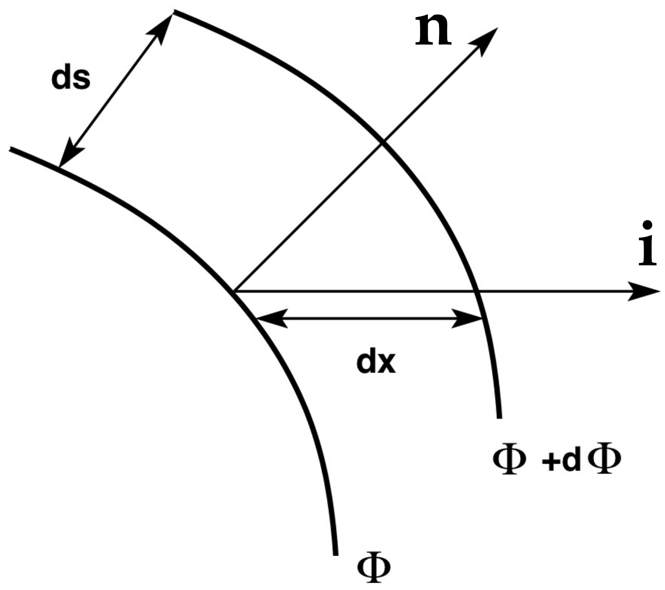
Introduction

- The study of electromagnetic theory requires considerable knowledge of vector analysis. In this lecture, we will introduce vector operations such as gradient, divergence and curl that we will need for our study of electromagnetic theory. As we shall see, these vector operations are very convenient to determine the properties of electromagnetic field, also considerably simplify the formulation of electromagnetic theory and allow get a better insight into electromagnetic phenomena.

Gradient of a Scalar Function-a

Assume that a function Φ represents a scalar field and that Φ is a single valued, continuous, and differentiable function of position. Let $d\Phi$ represents a change of Φ with a distance ds.

The gradient of the scalar function Φ is defined as:



$$grad\Phi \equiv \nabla\Phi = \frac{\partial\Phi}{\partial s}\mathbf{n}$$

Gradient of a Scalar Function-b

where **n** is a unit vector in the direction the rate $\partial\Phi/\partial s$ has its maximum value. In other words, **gradient tells us in which direction the change in $\boldsymbol{\Phi}$ is maximal.**

- In Cartesian coordinates the gradient is defined as:

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k}$$

Divergence of a Vector Function-a

- The divergence is the scalar function which results from operation of ∇ upon a vector **F** in a fashion analogous to the dot product of two vectors. The result is a scalar function.
- In Cartesian coordinates the divergence is written as follows:

$$div\mathbf{F} \equiv \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Divergence of a Vector Function-b

Properties: A positive divergence means that there is a **source** of a vector field, and a negative divergence means the presence of a **sink** of the field.

When $\nabla \cdot \mathbf{F} = 0$ everywhere, the field $\mathbf{F}$ is called **solenoidal**, since no starting points or sources can be assigned to the lines describing the field. In other words, it has no sources or sinks.

The Flux of a Vector Field-a

- A vector field propagating in space may cross some surfaces not necessary normal to the field direction. In this case, we may speak about a flux of the field through the surface. The flux is measured by the number of field lines crossing the surface.

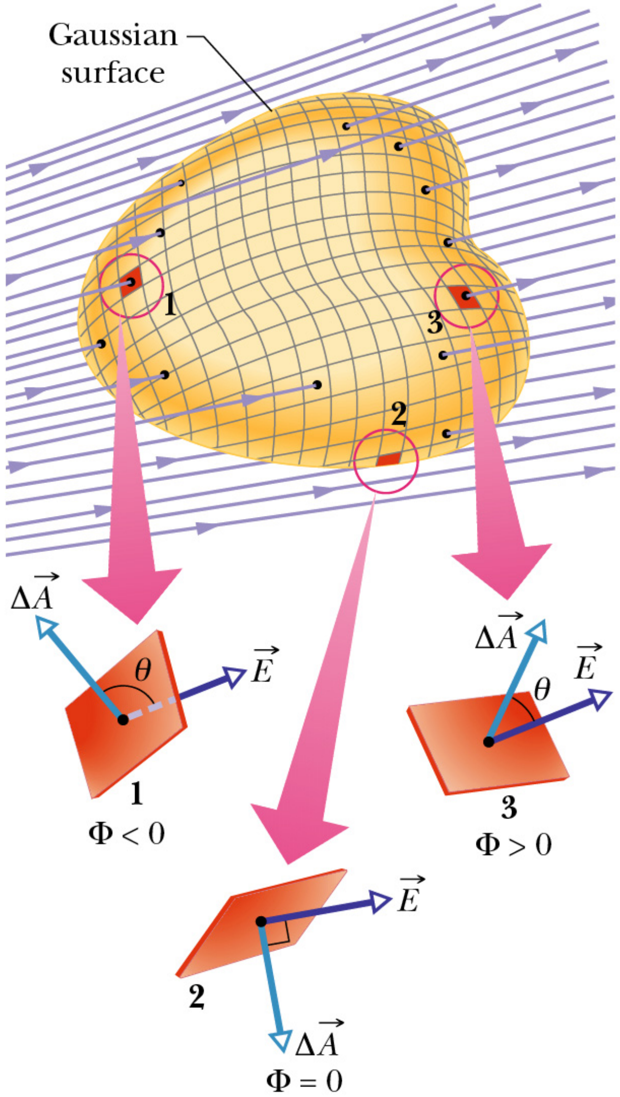
The Flux of a Vector Field-b

- Suppose a field **E** crossing a surface *A* of area the flux of this field through the surface is given by:

$$\Phi = \sum \mathbf{E} \cdot \Delta \mathbf{A}$$

or by letting surface element ΔA being very small

$$\Phi = \int_A \mathbf{E} \cdot d\mathbf{A} = \int_A \mathbf{E} \cdot \mathbf{n}\, dA$$



Gauss's Divergence Theorem-a

- Consider a vector field **F** crossing a closed surface bounding a volume V , as shown in previous slide. Then:

$$\int_V \nabla \cdot \mathbf{F}\, dV = \oint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

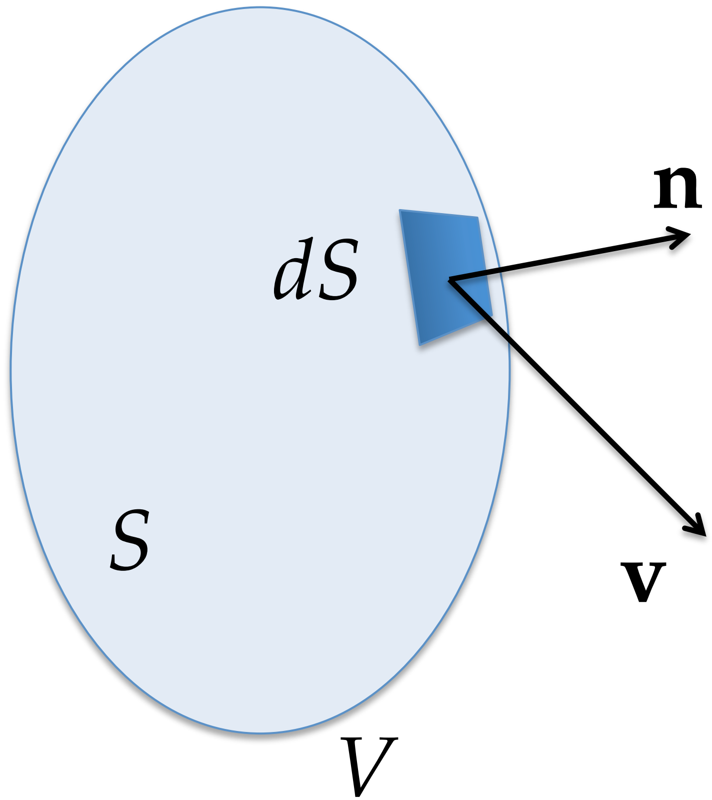
- The Gauss's law states that the volume integral of the divergence of a vector field over a volume V is equal to the closed surface integral of the vector over the surface bounding the volume V .

Gauss’s Divergence Theorem-b

- Mathematically, the Gauss’s divergence theorem converts a volume integral of the divergence of a vector to a closed surface integral of the vector, and vice versa. Physically, the Gauss’s divergence theorem says that the number of the field lines flowing through the surface S is equal to the ”strength” of the field source contained inside the volume V .

The Continuity Equation

- Suppose we have some charge of density ϱ in a volume *V* enclosed by a surface *S*. Let **v** is a macroscopic velocity of the charge. Then, the rate of decrease of the total charge in the volume V is equal to the rate of transport of the charge out through the surface S. This is expressed by the **continuity equation**:



$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Curl (Rotation) Function-a

- Curl is the operation of ∇ operator upon a vector in a fashion analogous to the cross product of two vectors. The result is a vector that in Cartesian coordinates is written as:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

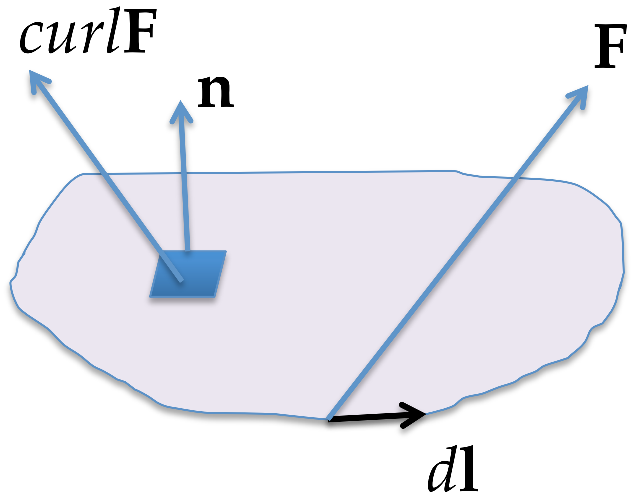
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ F_x & F_y & F_z \end{vmatrix}$$

Curl (Rotation) Function-b

- **Properties:** Curl is nonzero when the field increases (or decreases) in a different direction that the field pointed. If the field is pointed in the same direction as that in which is increased, the curl is zero. So the curl is related to how the field changes as you move across the field.
- When $\nabla \times \mathbf{F} = 0$ everywhere, the field F is called **irrotational**.

Stokes's Theorem

- The Stokes's theorem states that the closed line integral of a vector field **F** along the contour bounding an open surface S is equal to the surface integral of the **curl** of the vector field over the surface.



$$\oint_l \mathbf{F}\cdot d\mathbf{l} = \int_S \left(\nabla \times \mathbf{F}\right)\cdot \mathbf{n}\, dS$$

Successive Application of ∇-a

$$\nabla \cdot \nabla \Phi = \nabla \cdot \left(\frac{\partial \Phi}{\partial x}\mathbf{i} + \frac{\partial \Phi}{\partial y}\mathbf{j} + \frac{\partial \Phi}{\partial z}\mathbf{k} \right) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

The above defines a new **scalar** operator the so called **Laplacian**:

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Laplacian is applied also to a vector

$$\nabla^2 \mathbf{F} \equiv \nabla \cdot \nabla = \frac{\partial^2 \mathbf{F}_x}{\partial x^2} + \frac{\partial^2 \mathbf{F}_y}{\partial y^2} + \frac{\partial^2 \mathbf{F}_z}{\partial z^2}$$

Successive Application of ∇-b

$$\nabla \times \nabla \Phi = 0$$

$$\nabla \cdot \nabla \times \mathbf{F} = 0$$

These two properties are very useful in the vector field theory, in particular in the electromagnetic theory.

The first relation shows that an irrotational field can always be expressed as gradient of an arbitrary scalar field.

The second relation shows that any solenoidal field can always be expressed as a curl of an arbitrary vector field.

Vector Operators in Cylindrical Coordinates

Unit vectors $\left\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}\right\}$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{z}}$, $\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_z\,\hat{\mathbf{z}}$.

$$
\begin{aligned}
x &= r\,\cos\theta & r &= \sqrt{x^2+y^2} \\
y &= r\,\sin\theta & \tan\theta &= \frac{y}{x} \\
z &= z & z &= z
\end{aligned}
$$

$$
\boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\psi}{\partial z}\,\hat{\mathbf{z}}
$$

$$
\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}\,(r\,A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}
$$

$$
\boldsymbol{\nabla}\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\,(r\,A_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\mathbf{z}}
$$

Note that

$$
\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ A_r & r\,A_\theta & A_z \end{vmatrix}.
$$

$$
\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}
$$

Vector Operators in Spherical Coordinates

Unit vectors $\left\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\right\}$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$, $\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}}$.

$$x = r\,\sin\theta\,\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r\,\sin\theta\,\sin\phi \qquad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$z = r\,\cos\theta \qquad \tan\phi = \frac{y}{x}$$

$$0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\,\pi$$

$$\boldsymbol{\nabla}\,\psi = \frac{\partial\psi}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\,\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\theta\right) + \frac{1}{r\,\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{A} = {} & \frac{1}{r\,\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} \\ & + \left[\frac{1}{r\,\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\,A_\phi\right)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,A_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}\end{aligned}$$

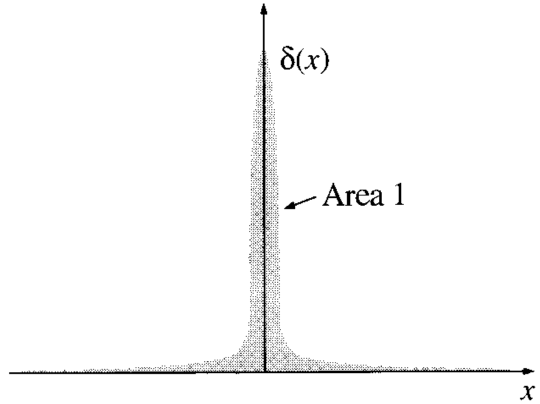
Note that

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r^2\,\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\,\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & r\,A_\theta & r\,\sin\theta\,A_\phi \end{vmatrix}.$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\,\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

The one-dimensional Dirac Delta function-a

- The one dimensional Dirac function $\delta(x)$ can be pictured as an infinitely high, infinitesimal narrow "spike", with area 1. That is:



$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\,dx = f(0)$$

The one-dimensional Dirac Delta function-b

- Of course the spike of delta function can be shifted to any real number *a*.

$$\int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1$$

$$\delta(x-a) = \begin{cases} 0, & \text{if} \quad x \neq a \\ \infty, & \text{if} \quad x = a \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\,dx = f(a)$$

The three-dimensional Dirac Delta function

- The Delta function can be generalized to three dimensions as follows:

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

$$\int_{\substack{\text{all}\\\text{space}}} \delta^3(\mathbf{r})\, dV = \int_{x=-\infty}^{x=+\infty} \int_{y=-\infty}^{y=+\infty} \int_{z=-\infty}^{z=+\infty} \delta(x)\delta(y)\delta(z) = 1$$

$$\int_{\substack{\text{all}\\\text{space}}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\, dV = f(\mathbf{a})$$