MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE 4

Complex Functions

Multivalued complex functions-a

- Let's get a complex number z in polar form. The following holds because we can add to the phase any integral multiple of 2π.

$$z = re^{i\theta} = re^{i(\theta+2n\pi)}, \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

- Let's calculate its logarithm

$$\ln z = \ln re^{i(\theta+2n\pi)} = \ln r + i(\theta + 2n\pi)$$

- This means that the logarithm is a *multivalued* function having an infinite number of values for a single pair of real values r and θ.

Multivalued complex functions-b

- To avoid any ambiguity, we usually agree to set $n=0$ and limit the phase to an interval $(-\pi,\pi)$.
- The value of ln$z$ with $n=0$ is called the *principal value* of ln$z$
- A multivalued function can be considered as a set of singlevalued functions. Any such a singlevalued function member of this set is called a *branch.*

Elementary complex functions-a

- Polynomial functions

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 , \quad n \in N, \; a_n \neq 0$$

- Rational functions

$$f(z) = \frac{P(z)}{Q(z)}$$

where *P(z), Q(z)* are polynomials

- Exponential functions

$$f(z) = e^z$$

Elementary complex functions-b

- Trigonometric functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{\left(e^{iz} - e^{-iz}\right)}{i\left(e^{iz} - e^{-iz}\right)} \qquad \cot z = \frac{\cos z}{\sin z} = \frac{i\left(e^{iz} + e^{-iz}\right)}{\left(e^{iz} - e^{-iz}\right)}$$

Quiz: The functions sin*z* and cos*z* are entirely analytical. Can you explain why? Could you claim the same for tan*z* or cot*z*?

Elementary complex functions-c

- We may show that the following relations are valid for complex trigonometric functions

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos(z), \quad \tan(\text{-}z) = -\tan z$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\tan\left(z_1 \pm z_2\right) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \cdot \tan z_2}$$

Elementary complex functions-d

- The hypebolic functions

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cos z = \frac{e^z + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{\left(e^z - e^{-z}\right)}{\left(e^z + e^{-z}\right)} \quad \coth z = \frac{\cosh z}{\sinh z} = \frac{\left(e^z + e^{-z}\right)}{\left(e^z - e^{-z}\right)}$$

Elementary complex functions-e

- We may show that the following relations are valid:

$$\cosh^2 - \sinh^2 z = 1$$

$$\sinh(-z) = -\sinh z, \;\; \cosh(-z) = \cosh(z), \quad \tanh(\text{-}z) = -\tanh z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \mp \sinh z_1 \sinh z_2$$

$$\tanh\left(z_1 \pm z_2\right) = \frac{\tanh z_1 \pm \tanh z_2}{1 \mp \tanh z_1 \cdot \tanh z_2}$$

Elementary complex functions-f

- Between the trigonometric and the hyperbolic functions we have the following relations:

$$\sin iz = i\sinh z \quad \cos iz = \cosh z \quad \tan iz = i\tanh z$$

$$\sinh iz = i\sin z \quad \cosh iz = \cos z \quad \tanh iz = i\tan z$$

Elementary complex functions-g

- Logarithmic functions

$$z = a^{w} \Rightarrow w = \log_{a} z$$

$$a > 0, \quad a \neq 0,\ 1$$

- Functions with complex exponents. For any complex number *c*, the function z^{c} is defined by:

$$z^{c} = \exp\left(c \log z\right), \quad z \neq 0$$

Elementary complex functions-h

- Inverse trigonometric functions
- These are all multivalued functions and we give here only their principal values:

$$\sin^{-1} z = \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right) \qquad \cos^{-1} z = \frac{1}{i}\ln\left(z + \sqrt{1-z^2}\right)$$

$$\tan^{-1} z = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right) \qquad \cot^{-1} z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$$

Elementary complex functions-i

- Inverse hyperbolic functions.
- These are all multivalued functions and we give here only their principal values

$$\sinh^{-1} z = \ln\left(z + \sqrt{1+z^2}\right) \quad \cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \qquad \coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

Branching points-a

- A point z_0 of a function $f(z)$ is called a **branching point** if a complete turn of the variable z around z_0 does not return the function to its original value.
- A function has branching points when it is multivalued. In this case it has several branches. The branching points are those points where different branches meet.
- The order of a branching point is n if the minimum number of turns that are needed to return the function at its original value is $n+1$.

Branching points-b

- **Example:** Multivalued functions arise naturally as the inverse functions of single valued functions. For example let the function, $z = w^2$ in this case $w = z^{1/2}$.
- We can show that as z goes around a closed path centered at $z = 0$, then w does not return to its original value. We need to perform another one rotation in order to return to the original value. The point $z = 0$ is a branching point of w.
- We can also show that $z = \infty$ is also a branching point of w.

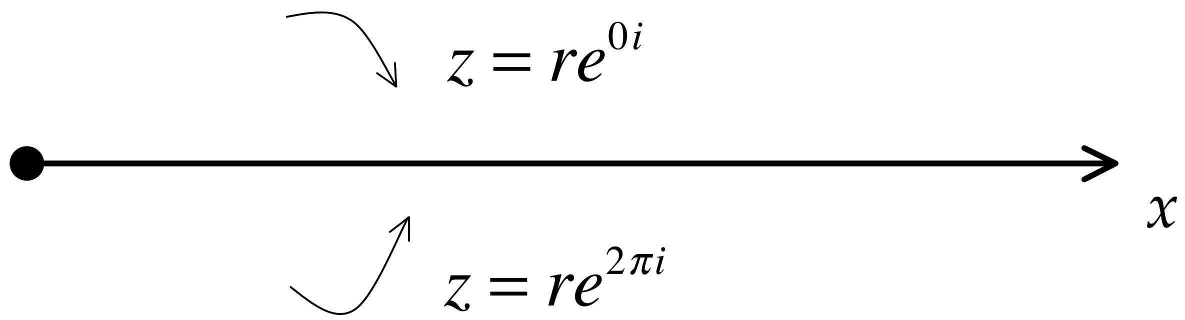
Branching points-c

- A multivalued function, as we said in the beginning, can be considered as a set of singlevalued functions. Any such a singlevalued function member of this set is called a *branch.*
- The study of multivalued functions is considerably facilitated when they are expressed as singlevalued functions.

Sections

- Section of a complex function is called any line on the complex plane which connects two branching points.
- A function is fully defined if we define its value at a point *which does not lie on a section.* If we do not cross the section the function remains single valued.
- In the squared root function in the previous slide the real axis x is a section since it connects the points $z = 0$ and $z = \infty$.

Turning a multivalued function in a single valued one.



- We limit the multivalued function in a certain region of x-y plane. For the square root function we take out the real axis number and the points $z = 0$ and $z = \infty$