MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE 5

COMPLEX INTEGRATION - A

Properties of analytic functions

- Let two functions $f(z)$ and $g(z)$ analytic in a region R. Then:
- $f(z) + g(z)$
- $f(z)g(z)$

  are analytic in R.
- $f(z)/g(z)$ analytic in R at points where $g(z)$ is not equal to zero.
- $f(g(z))$ analytic in a region S such that $g(z)$ is in R

Integration of complex functions of a real variable-a

- Let a complex function *f* of a real variable *t* in an interval $a \le t \le b$

$$f(t) = u(t) + iv(t)$$

Where $u(t)$ and $v(t)$ are real functions. We say that this function is **integrable** in the interval $[a, b]$ if the functions $u(t)$ and $v(t)$ are integrable in this interval. In this case we have:

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Integration of complex functions of a real variable-b

- The usual rules of integration do apply here. Especially the fundamental theorems of the integral calculus:

$$\frac{d}{dt}\int_a^b f(\tau)\,d\tau = f(\tau)$$

$$\int_a^b f^{'}(\tau)\,d\tau = f(b) - f(a)$$

There are theorems of real integral calclulus that do not apply here: For exampe the "average value theorem"

Integration of complex functions of a real variable-c

- The following properties are valid:

$$\int_a^b f(\tau)d\tau = \int_a^c f(\tau)d\tau + \int_c^b f(\tau)d\tau$$

- If $f(t) = u(t) + iv(t)$, $F(t) = U(t) + iV(t)$

and $F^{'}(t) = f(t)$ then

$$\int_a^b f(\tau)d\tau = F(b) - F(a)$$

Integration of complex functions of a real variable-d

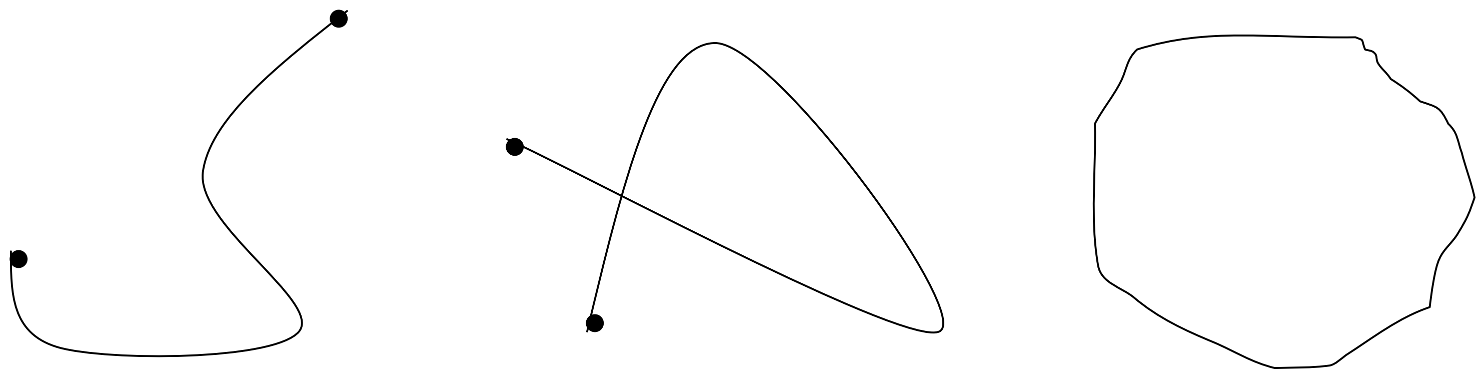
- Now we extend the idea of complex integration to an integration along a curve on the complex plane which can be described in a parametric form:

$$z(t) = x(t) + iy(t), \qquad a \le t \le b$$

- The above relation means that for any t in the interval [a, b] there is a point ($x(t)$,$y(t)$) which "maps" the point $z(t)$. These points $z(t)$ are ordered in increasing z. The curve is said to be continuous (differentiable) if $x(t)$,$y(t)$ are continuous (differentiable).

Integration of complex functions of a real variable-e

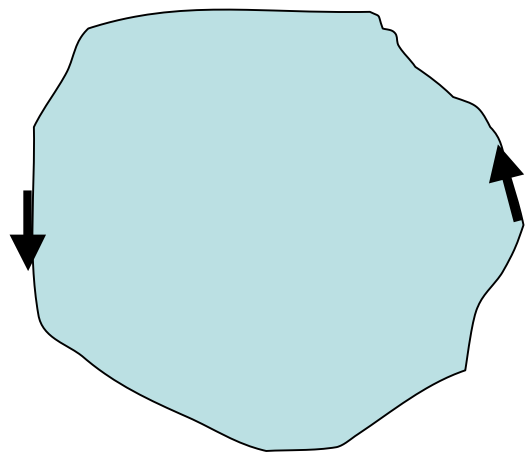
- A curve C is said to be **simple** (or **Jordan arc**) if it does not intersect itself, i.e. when $z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$ for $t \in [a,b]$. (unless $z(b) = z(a)$ which is allowed and in this case we say that C is a **simple closed curve**, or **Jordan curve**).



Simple, non closed (Jordan arc) **Non simple, non closed** **Simple, closed (Jordan curve)**

Integration of complex functions of a real variable-f

- A closed curve can be “traveled” in two opposite directions. By convention we consider as positive direction the one which has the interior of the curve at the left of the curve.



positive direction

Integration of complex functions of a real variable-h

- A **smooth** curve *C* is a curve for which the function $z'(t)$ is continuous.
- A **contour** is a curve made up by a finite number of smooth curves.
- A simple closed contour is also known as **Jordan loop**.
- The contour integral along a smooth contour C is defined as:

$$\int_C f(z)dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$$

The value of the integral depends on the loop *C*.