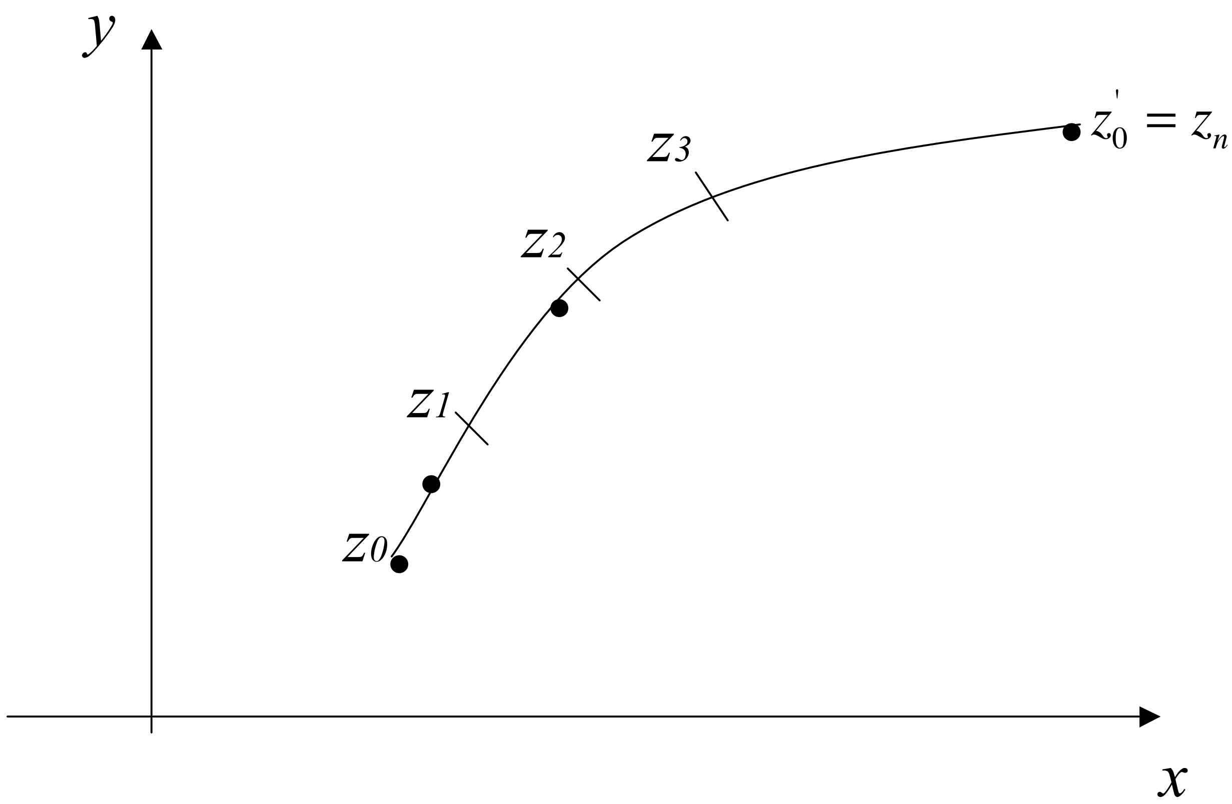
MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE 6

COMPLEX INTEGRATION - B

Cauchy integral theorem-a

- The integral of a complex variable over a contour in the complex plane may be defined in a closed analogy to the Riemann integral of a real function integrated along the real axis.
- We divide a contour into n intervals by picking n-1 intermediate points on the contour and we consider the sum:

Cauchy integral theorem-b



Cauchy integral theorem-c

$$S_n = \sum_{j=1}^{n} f(\zeta_j)\left(z_j - z_{j-1}\right)$$

- Where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with $\left|z_j - z_{j-1}\right| \to 0$ for all *j*. If the $\lim_{n\to\infty} S_n$ exists and is independent of ζ_j the details of choosing the points z_j, then

Cauchy integral theorem-d

$$\lim_{n\to\infty}\sum_{j=1}^{n} f(\zeta_j)\left(z_j - z_{j-1}\right) = \int_{z_0}^{z_0'} f(z)dz$$

- The right hand side is called the *contour integral* of $f(z)$ along the specified contour.

Cauchy integral theorem-e

- An alternative definition of the contour integral may be defined by

$$\int_{z_1}^{z_2} f(z)dz = \int_{x_1,y_1}^{x_2,y_2} \left[u(x,y)+iv(x,y)\right][dx+idy]$$

$$= \int_{x_1,y_1}^{x_2,y_2} \left[u(x,y)dx - v(x,y)dy\right] + i\int_{x_1,y_1}^{x_2,y_2} \left[v(x,y)dx + u(x,y)dy\right]$$

- with the path joining (x_1,y_1) and (x_2,y_2) specified

Cauchy integral theorem-f

- Let a function $f(z)$ which is defined in a region of the complex plane. If there is a function $F(z)$ such that

$$f(z) = \frac{dF(z)}{dz}$$

for every point in the region, then $F(z)$ is called the *primitive* of $f(z)$ in this region.

Cauchy integral theorem-g

- If the primitive of a function in a region is known and the integration contour is in this region then the integral of the function along this contour is given by:

$$I = \int_C f(z)dz = \int_{t_a}^{t_b} dt\left(\frac{dF}{dt}\right) = F(z(t_b)) - F(z(t_a))$$

- This integral is independent from the integration contour

Cauchy integral theorem-g

- The previous integral has the following properties

$$\int_C af(z)dz = a\int_C f(z)dz$$

$$\int_C \left[f(z) \pm g(z)\right]dz = \int_C f(z)dz \pm \int_C g(z)dz$$

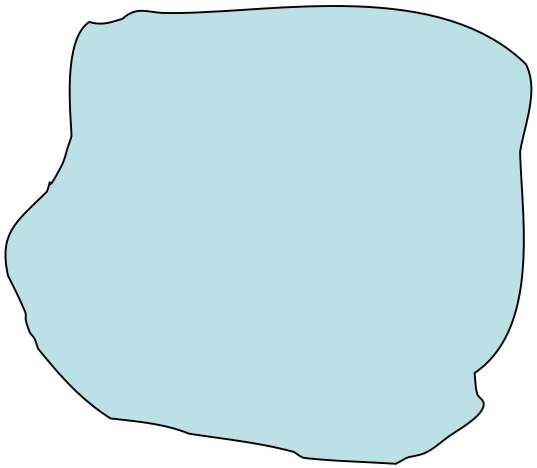
$$\int_a^b f(z) = \int_a^c f(z)dz + \int_c^b f(z)dz$$

$$\int_a^b f(z) = -\int_b^a f(z)dz$$

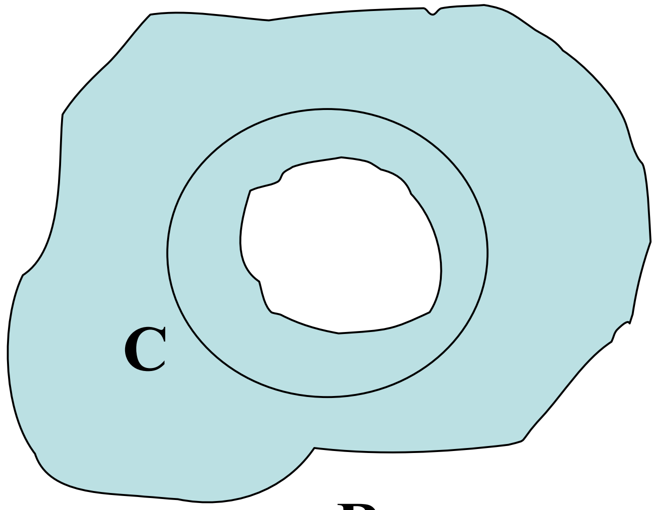
$$\int_a^b f^{'}(z)g(z) = f(z)g(z)\Big|_a^b - \int_a^b f(z)g^{'}(z)dz$$

Connected regions

- **Definition**: A *simply* connected region or domain is one in which every closed contour in that region encloses only the points contained in it. If a region is not simply connected, it is called a *multiply* connected region. As an example of a multiply connected region, consider the z-plane with the interior of the unit circle *excluded*.



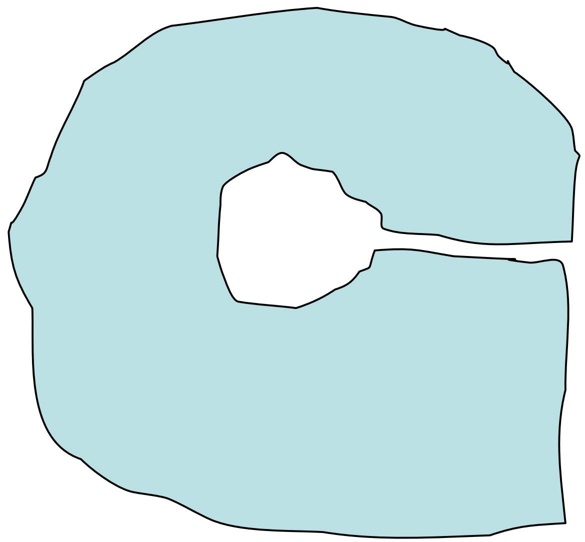
R_1
(simply connected)



R_2
(multiply connected)

R_3

(simply connected)



Connected regions

- The region R_1 is simply connected
- The region R_2 is multiply connected. The contour C contains points (those of the hole) which do not belong to the region.
- The region R_3 is simply connected.
- A simply connected region has no holes!
- By “cutting a ribbon” from a multiply connected region we can reduce it to a simply connected region.

Cauchy-Goursat theorem-a

- Let a closed contour *C* on the complex plane and a complex function *f*(z) which is analytic (therefore single-valued) on the contour *C* and is enclosed by this contour, then:

$$\int_C f(z)dz = 0$$

or

$$\int_a^b f(z)dz = \text{independent from the integration contour}$$

Cauchy-Goursat theorem-b

- The inverse of this theorem is not always true. It is possible, even if the function *f*(*z*) is not everywhere on contour C analytic, to have

$$I = \int_C f(z)dz = 0$$

Cauchy formula-a

- Let a function $f(z)$ which is analytic at all the points of a simply connected region R and let a contour C which is continuous by parts. Let a point which belongs to region R but not on C. Then:

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = \begin{cases} f(z) & \text{if } z \text{ is inside C} \\ 0 & \text{if } z \text{ is outside C} \end{cases}$$

Cauchy formula-b

- Cauchy's formula allows the calculation of an analytic function $f(z)$ everywhere in simply connected region if we know the values of the function at the borders of this region!
- This has tremendous consequences: the differentiable of a function (a local property) has drastic non-local properties.
- There is no such a formula in real numbers.

Cauchy formula-c
Derivatives of analytic functions

- Let a function

$$f(z) = \frac{1}{2\pi i}\int_C \frac{g(\zeta)}{\zeta - z}\,d\zeta$$

- where C is any contour of finite length and $g(\zeta)$ a function continuous on C. Then the function $f(z)$ is analytic at all the points on contour C and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{g(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

Cauchy formula-d

Derivatives of analytic functions

- A consequence of the above formula is that a function which is analytic on a region of the complex plane has derivatives of any order at any point in the region.

Morera's theorem

- Morera's theorem is the inverse of Cauchy's theorem:
- Let a function $f(z)$ which is continuous in a region R and for any closed contour in R we have

$$\int_C f(z)dz = 0$$

- Then $f(z)$ is analytic in R.

Cauchy-Liouville theorem

- An entire function which is bound is necessarily a constant function
- Also, if a function is analytic in a region R it cannot have a local maximum in R.