MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE 7

Taylor and Laurent Series

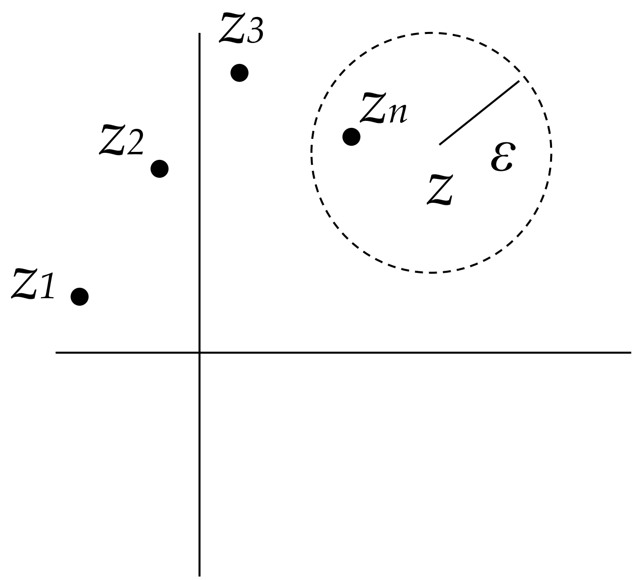
Convergence of Sequences & Series

- An infinite *sequence* $z_1, z_2, \dots z_n, \dots$ of complex numbers has as its limit the number *z*, if for every $\varepsilon > 0$, there is a positive integer n_0 such that the following inequality holds:

$$\left| z_n - z \right| < \varepsilon \quad \text{when} \quad n > n_0$$

Convergence of Sequences & Series

- Geometrically this means that for large values of *n* the points z_n are in a "neighborhood" of *z* with radius ε. If we choose it as small as we like the points z_n come as closer to *z* as we like.
- The sequence can have only one limit. When this limit exists we say that the sequence *converges to z.*



Convergence of Sequences & Series

- **Theorem 1:** Let $z_n = x_n + iy_n \quad (n = 1, 2, \ldots)$ and $z = x + iy$ then

$$\lim_{n \to \infty} z_n = z$$

if and only if

$$\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y$$

Convergence of Sequences & Series

- An infinite complex number series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \ldots + z_n + \ldots.$$

converges to a *sum* S if the sequence

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \ldots + z_n \quad (N \text{ positive integer})$$

of the partial sums converges to S. In this case we write

$$\sum_{n=1}^{\infty} z_n = S$$

Convergence of Sequences & Series

- **Theorem 2:** Assume that $z_n = x_n + iy_n \quad (n = 1,2,\ldots)$ and $S = X + iY$. Then

$$\sum_{n=1}^{\infty} z_n = S$$

if and only if:

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

Convergence of Sequences & Series

- If we recall from Calculus that the n-th term of a converging sequence converges to zero when n tends to infinity. Then from the two previous theorems we conclude that the same must hold for complex series.
- A **necessary** condition for the convergence of the series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \ldots + z_n + \ldots.$$

is:

$$\lim_{n \to \infty} z_n = 0$$

Convergence of Sequences & Series

- Another important property of the complex numbers is that the absolute convergence of a complex series implies the convergence of the series itself:

If $\sum_{n=1}^{\infty} |z_n|$ converges then $\sum_{n=1}^{\infty} z_n$ converges.

Another useful concept is the **residue** of a sequence, which converges to a sum *S*, and is defined by:

$$\rho_N = S - S_N$$

A series converges to a number *S* if and only if the sequence of the residues converges to zero

Taylor Series-a

- Functions that are analytic at a point z_0 can be represented in the form of a power series around point z_0 (Taylor Series).
- *Theorem*: Let a function $f(z)$ analytic in an open disk C centered at point z_0 and with radius r, then at any point in this disk, i.e at any point such that $|z - z_0| \leq r$ the function can be written as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} = \frac{1}{n!} \frac{d^n f}{dz^n}\bigg|_{z=z_0}$$

Taylor Series-B

- If the function is entire then the radius of the disk could be chosen as infinitely large. Then the series converges to *f*(*z*) for any *z*.
- we develop the function in a power series around a new point *z*2 the series will converge in a circle C2 which can extend beyond the circle C1.
- The values of the new function *f*2(*z*) are fully determined by the values of *f*1(*z*) in the common region of convergence of *f*1(*z*) and *f*2(*z*).
- In the case where $z_0 = 0$ we talk about a **Maclaurin series**

Taylor Series-C

- If we consider the case of a disk with a generic centre z_0 and a radius *r*, then the complex function $f\left(z+z_0\right)$ is again analytic in the disk $\left|\left(z+z_0\right)-z_0\right|<r$ we have that $g\left(z\right)=f\left(z+z_0\right)$ is analytic in the disk $|z|<r$. So we have achieved the representation of $g\left(z\right)$ through a Maclaurin series:

$$g\left(z\right)=\sum_{n=0}^{\infty}\frac{g^{(n)}\left(0\right)}{n!}z^n, \quad \left(|z|<r\right) \quad \text{or}$$

$$f\left(z+z_0\right)=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(z_0\right)}{n!}z^n \quad \left(|z|<r\right)$$

Taylor Series-D

- As we shall see later in the course if there are constants $a_n \ (n \geq 0)$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$$

for any z inside a any circle centered at z_0 then this series is the Taylor series of f around z_0 no matter how we got these constants. This will be very useful in finding the coefficients of the Taylor series in a way much more easier than with the formula $a_n = f^{(n)}\left(z_0\right)/n!$

Laurent Series-a

- If a series $f(z)$ is not analytic at a point z_0, we cannot apply the Taylor's theorem there. We can though find a representation in the form of a series.
- If it is analytic in the circular section $r_1 < |z - z_0| < r_2$ again it can be represented with a power series of positive and negative powers of $(z - z_0)$. Such a series is called a *Laurent series*.

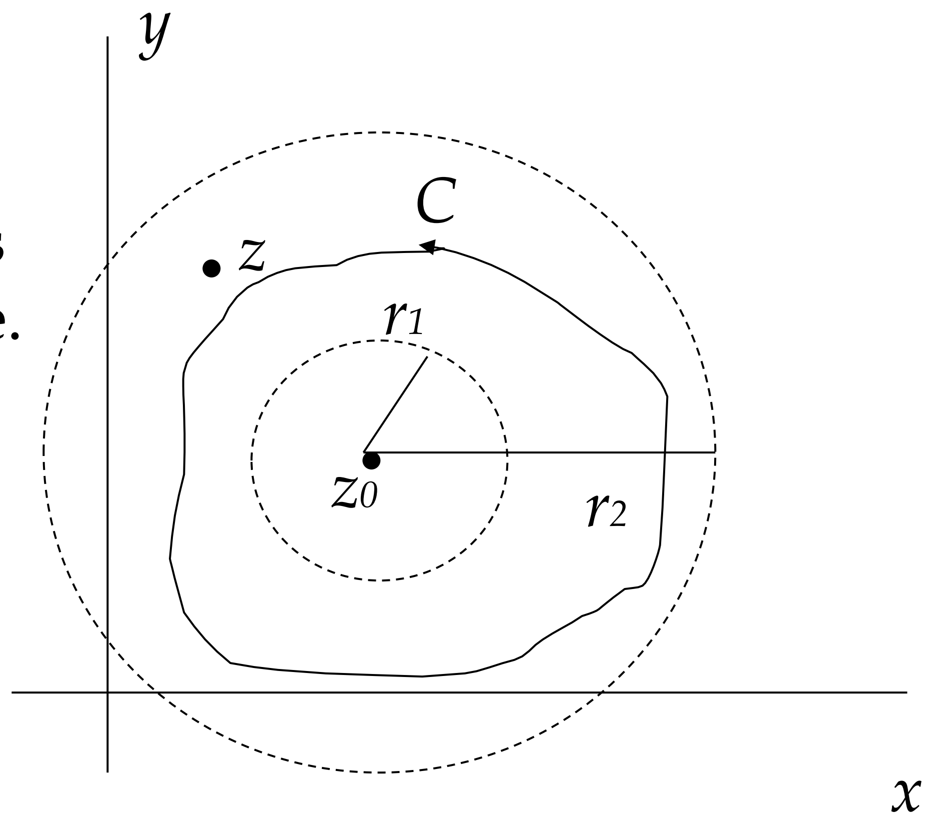
Laurent Series-b

- **Theorem:** Let a function *f*(*z*) analytic in the circular section between the circles C_1 and C_2 centered at z_0, i.e. at points such

$$r_1 < |z - z_0| < r_2$$

- Then at any *z* in this disk

$$f(z) = \sum_{n=-\infty}^{\infty} d_n \left(z - z_0\right)^n$$



$$d_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\left(\zeta - z_0\right)^{n+1}} d\zeta$$

Laurent Series-c

- The Laurent series can take also the form:

$$f(z) = \sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n + \sum_{n=1}^{\infty} \frac{b_n}{\left(z - z_0\right)^n}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f\left(\zeta\right)}{\left(\zeta - \zeta_0\right)^{n+1}} dz \qquad \left(n \geq 0\right)$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f\left(\zeta\right)}{\left(\zeta - \zeta_0\right)^{-n+1}} dz \qquad \left(n \geq 1\right)$$