MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE 8

Absolute and uniform convergence of series -a

- **Theorem 1:** If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges when $z = z_1$ $(z_1 \neq 0)$, then, it converges absolutely at any point of the open disk $|z| < |z_1|$.
- **Theorem 2:** If $z = z_1$ is a point inside the circle of convergence, $|z| = R$, of the power series $\sum_{n=0}^{\infty} a_n z^n$, then the series converges uniformly at the closed disk $|z| \leq |z_1|$

Integration & Differentiation of Power Series

- **Theorem 1:** Let C a loop in the interior of a convergence circle of the power series $S(z) = \sum_{n=0}^{\infty} a_n z^n$ and let $g(z)$ a function continuous on the loop C. The series which is formed by the multiplication of the terms of the power series with $g(z)$ could be integrated term by term, on the loop C.

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)z^n dz$$

Integration & Differentiation of Power Series

- **Theorem 2:** The power series $S(z) = \sum_{n=0}^{\infty} a_n z^n$ could be differentiated term by term. That means, for any z in the interior of a convergence circle of the power series we have that:

$$S^{'}(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

The uniqueness of the development into power series

- **Theorem 1:** If the series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges to $f(z)$ for all the points inside a disk $|z - z_0| = R$, then this is the Taylor series of f around the point z_0.

The uniqueness of the development into power series

- **Theorem 1:** If a series of the form

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

converges to $f(z)$ for all the points inside a disk $|z - z_0| = R$, then this is the Laurent series of f around the point z_0 in the given disk.

Multiplication and Division of power series

- Let' s consider that the following power series,

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

converge in the interior of a circle $|z| = R$. The sums of these series: $f(z)$ and $g(z)$ are analytic functions in the circular disk $|z| < R$. Thus, their product has a Maclaurin development in the circular disk of the form

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n \qquad \left(|z| < R\right)$$

Multiplication and Division of power series

- The general formula for the coefficient c_n is given by:

$$c_n = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z)$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0,\ 1,\ \ldots, n)$$

Analytic Continuation-a

- **Identity theorem:** if two functions are each analytic in a region R and have the same values for all points in some subregion or along some curve within R , the two functions are identical everywhere within R.
- We can immediately use this result to extend functions originally defined on the real axis to the complex plane for example:

$$e^z = 1 + z + z^2 / 2! + \ldots$$

is the unique function f(z) that is equal to e^x on the real line.

Analytic Continuation-b

- Analytic continuation is a characteristic of complex functions. To understand this concept we study the following example:
- Expand in series around the points $z = 0, 1, i$, the function:

$$f(z) = \frac{1}{z(z-1)}$$

- *In class we show that:*

Analytic Continuation-c

$$\frac{1}{z(z-1)} = -\sum_{n=-1}^{\infty} z^n, \qquad 0 < |z| < 1$$

$$\frac{1}{z(z-1)} = \sum_{n=-1}^{\infty} (-)^{n+1} (z-1)^n, \qquad 0 < |z-1| < 1$$

$$\frac{1}{z(z-1)} = \sum_{n=-1}^{\infty} (-)^n \left\{ -\frac{1}{i^{n+1}} + \frac{1}{(i-1)^{n+1}} \right\} (z-i)^n, \quad 0 < |z-i| < 1$$

Analytic Continuation-d

- This is an example of the *analytic continuation*.
- A power series around a point z_1 represents an analytic function $f_1(z)$ in a circle C_1 which extends up to the nearest singular point.
- If we develop the function in a power series around a new point z_2 the series will converge in a circle C_2 which can extend beyond the circle C_1.
- The values of the new function $f_2(z)$ are fully determined by the values of $f_1(z)$ in the common region of convergence of $f_1(z)$ and $f_2(z)$.

Analytic Continuation-e

- In this case $f_2(z)$ is called *analytic continuation* of $f_1(z)$ in the new sector.
- This process could be continued to cover all the domain (except singular points) where a more general function $f(z)$ is to be defined.
- In this process we could not approach the singular points neither we can cross lines which are formed by singular points.

Analytic Continuation-f

- An example of analytic continuation in physics is in the study of scattering of two particles in quantum mechanics.
- By using complex analysis and analytic continuation then the branches and poles of the scattering amplitude have a direct physical meaning:
- The imaginary poles at the upper half-plane correspond to the bound states of the system.