MATHEMATICAL PHYSICS II
COMPLEX ALGEBRA
LECTURE 9

Singular Points, Residues and Poles

Residues-a

- As we have seen from Cauchy-Goursat's theorem, when a function is analytic at all points interior to and on a simple closed contour C, then the value of the integral of the function around the contour is zero. If, however, the function fails to be analytic at a finite number of points interior to C, there is, as we shall see a specific number, called a **residue**, which each of those points contributes to the value of the integral.

Residues-b

- A point z_0 is called a **singular point** of a function *f* if *f* fails to be analytic at z_0 but is analytic at some points in every neighborhood of z_0.
- A singular point is said to be **isolated** if, in addition, there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which *f* is analytic.

Problems 1-4 of Hand 9

Residues-c

- When z_0 is an isolated singular point of a function *f*, there is a positive number *R* such that *f* is analytic in $0 < |z - z_0| < R$. In this region there is a Laurent series for *f*.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \ldots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}}$$

Residues-c

- For n=1 we have that $\int_C f(z)dz = 2\pi i b_1$.
- The number b_1, which is the coefficient of $1/(z-z_0)$ in the Laurent series is called the **residue** of f around the isolated singular point z_0 and we often use the notation:

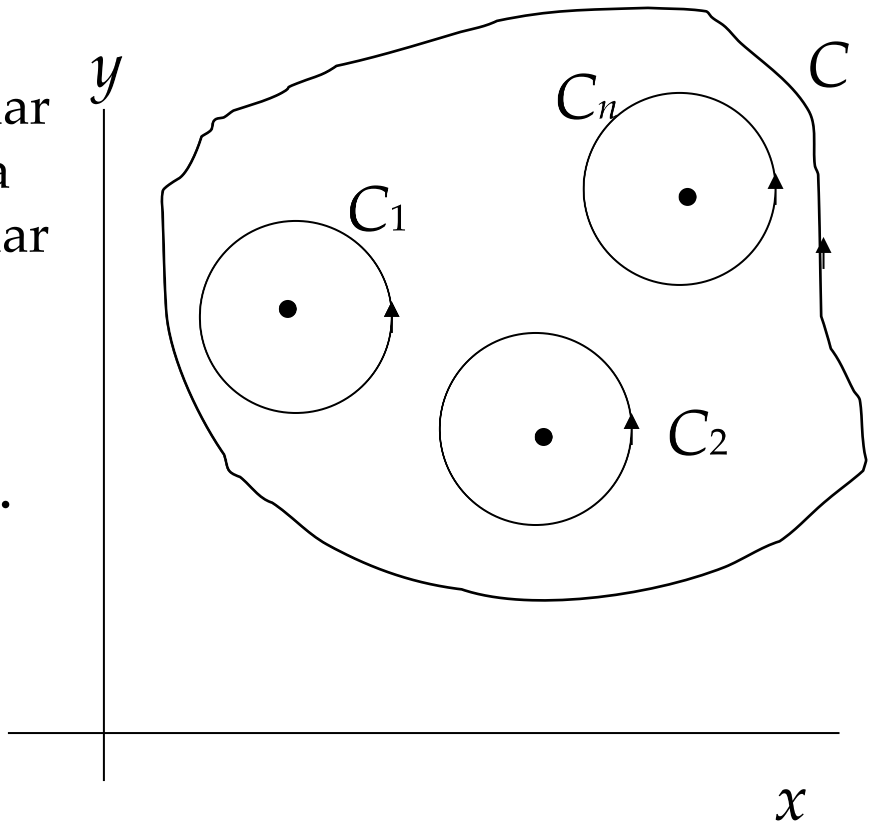
$$b_1 = \operatorname{Res}_{z=z_0} f(z)$$

- The above offers us a powerful method of calculating integrals.

Cauchy's Residue Theorem

- If, except for a *finite* number of singular points, a function *f* is analytic inside a simple closed contour *C*, those singular points must *isolated.*
- **Theorem:** Let *C* a positively oriented simple closed contour. If a function *f* is analytic inside and on *C* except for a finite number of points $z_k \left(1 \le k \le n\right)$ inside *C*, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$



See for example problems 10, 11, 12 of Handout 9

Using a Single Residue

- If the function *f* of Cauchy's residue theorem is, in addition, analytic at each point in the finite plane exterior to *C*, it is sometimes more efficient to evaluate the integral of *f* around *C* by finding a *single* residue of a certain related function.
- **Theorem:** If a function *f* is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour *C* then

$$\int_C f(z)dz = 2\pi i \operatorname{Res}_{z=z_k} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

See for example problem 13 of Handout 9

The Principal Part of a Complex Function

- We have seen that if z_0 is an isolated singular point of a function *f*, there is a positive number *R* such that *f* is analytic in $0 < |z - z_0| < R$ In this region there is a Laurent series for *f*.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_n}{(z - z_0)^n} + \ldots}_{\text{principal part}}$$

- The part of the series with the negative powers is called the **principal part** of *f* at z_0 .

Problems 5, 13,

The Principal Part and the Nature of Singular Points-a

- If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is finite, then there exists a positive integer m such that $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \ldots = 0$

then the series takes the form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \ldots + \frac{b_m}{(z - z_0)^m}$$

In this case z_0 is called **a pole of order *m*.** If $m=1$ is usually referred as a **simple pole**.

The Principal Part and the Nature of Singular Points-b

- There remain two extremes, the case in which all of the coefficients in the principal part are zero and the one in which an infinite number of them are nonzero.
- When all the b_n are zero the point z_0 is known as **removable singular point**. At such a point the residue is always zero.

The Principal Part and the Nature of Singular Points-c

- When an infinite number of them are nonzero then the point z_0 is said to be an **essential singular point** of f.
- An important result concerning the behavior of a function near an essential singular point is due to Picard. It states that **in each neighborhood of an essential singular point, a function assumes every finite value, with one possible exception an infinite number of times**

Residues at Poles

The following theorem is very useful because it provides an alternative characterization of poles and another way of finding the corresponding residues.

- **Theorem:** An isolated singular point of a function *f* is a pole of order *m* if and only if *f*(*z*) can be written in the form:

$$f(z) = \frac{\varphi(z)}{(z - z_0)^m}$$

where $\varphi(z)$ is analytic and nonzero at z_0 . Moreover

$$\operatorname*{Res}_{z=z_0} f(z) = \varphi(z_0) \qquad (\text{if } m = 1)$$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} \qquad (\text{if } m \geq 2)$$

Roots and Poles-b

- For the residue at a point z_0 which is a pole of order k we can find it from the formula:

$$b_1 = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left\{ (z - z_0)^k f(z) \right\}$$