Planes and Surfaces

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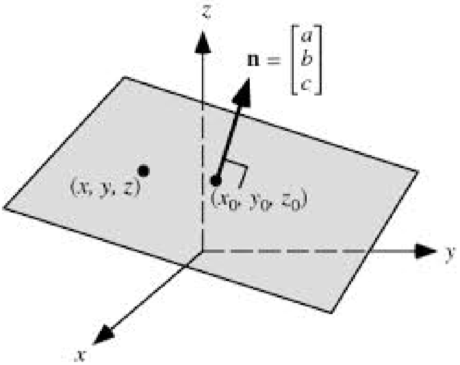
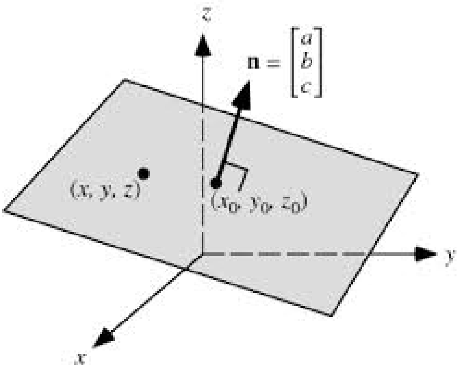
1. Planes

2. Surface

3. Quadric Surface
   - Ellipsoid
   - Hyperbolids
   - Paraboloid
   - Hyperbolic Paraboloid


z
$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
(x, y, z)
(x_0, y_0, z_0)
y
x



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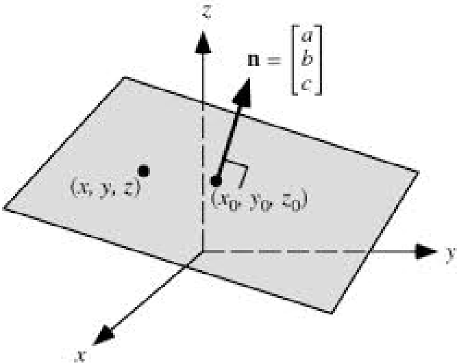

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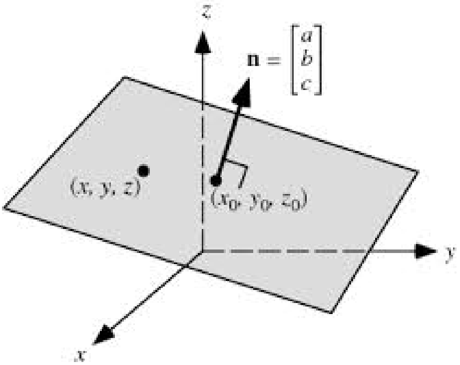

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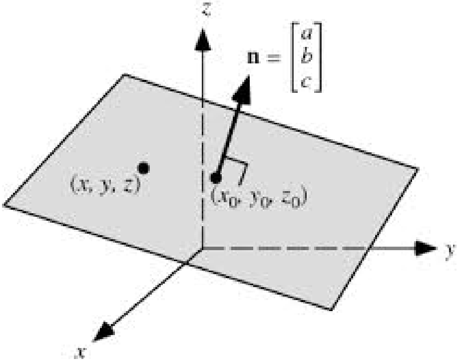

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$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

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$$\begin{aligned} -31(x - 4) - 20(y + 3)7(z - 1) &= 0 \\ \Rightarrow -31x - 20y + 7z + 57 &= 0. \end{aligned}$$

Note(1): Distance from a point $P(x_0, y_0, z_0)$ to the plane $ax + by + cz + d = 0$ is:

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Note(2): The shortest distance d between skew lines ℓ_1 and ℓ_2 is

$$d = \frac{|(\overrightarrow{P_1Q_2} \times \overrightarrow{P_1Q_2}).\overrightarrow{P_1P_2}|}{\left\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\right\|}.$$

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Q(1): Find Parametric equations for the line of intersection of the planes

$$\begin{aligned} x + 2y - 9z &= 7 \rightarrow P_1 \\ 2x - 3y + 17z &= 0 \rightarrow P_2. \end{aligned}$$

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$$d = \frac{\sqrt{49+100+185}}{\sqrt{16+1+9}} = 2.67.$$

Q(3): Find the shortest distance between the lines ℓ_1 through the points $A(1,-2,3)$, $B(2,0,5)$ and line ℓ_2 through $C(4,1,-1)$, $D(-2,3,4)$. ℓ_1, ℓ_2 are skew.

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$$(\overrightarrow{AB} \times \overrightarrow{CD}).\overrightarrow{AC} = 18 - 51 - 56 = -89.$$

Q(3): Find the shortest distance between the lines ℓ_1 through the points $A(1,-2,3)$, $B(2,0,5)$ and line ℓ_2 through $C(4,1,-1)$, $D(-2,3,4)$. ℓ_1, ℓ_2 are skew.

Solution: the shortest distance is given by

$$h = \frac{|(\overrightarrow{AB} \times \overrightarrow{CD}).\overrightarrow{AC}|}{\left\|\overrightarrow{AB} \times \overrightarrow{CD}\right\|}.$$

Now $\overrightarrow{AB} = \langle 1, 2, 2\rangle$, $\overrightarrow{CD} = \langle -6, 2, 5\rangle$ and $\overrightarrow{AC} = \langle 3, 3, -4\rangle$. So

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ -6 & 2 & 5 \end{vmatrix} = 6i - 17j + 14k.$$

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Now we copmute:

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Finally

$$h = \frac{|-89|}{\sqrt{36 + 289 + 196}} = 3.9.$$

Q(4): Use the dot product to find the dstance from $A(2,-6,1)$ to the line through $B(3,4,-2)$ and $C(7,-1,5)$.

Q(4): Use the dot product to find the dstance from $A(2, -6, 1)$ to the line through $B(3, 4, -2)$ and $C(7, -1, 5)$.

Solution:

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Solution:

We have $\overrightarrow{BA}$

Q(4): Use the dot product to find the dstance from $A(2, -6, 1)$ to the line through $B(3, 4, -2)$ and $C(7, -1, 5)$.

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We have $\overrightarrow{BA} = \langle -1, -10, 3 \rangle$ and $\overrightarrow{BC}$

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Solution:

We have $\overrightarrow{BA} = \langle -1, -10, 3\rangle$ and $\overrightarrow{BC} = \langle 7, -1, 5\rangle$.

Q(4): Use the dot product to find the dstance from $A(2,-6,1)$ to the line through $B(3,4,-2)$ and $C(7,-1,5)$.

Solution:

We have $\overrightarrow{BA} = \langle -1, -10, 3\rangle$ and $\overrightarrow{BC} = \langle 7, -1, 5\rangle$. We have

$$|AD| =$$

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We have $\overrightarrow{BA} = \langle -1, -10, 3 \rangle$ and $\overrightarrow{BC} = \langle 7, -1, 5 \rangle$. We have

$$|AD| = \text{Comp}_{\overrightarrow{BC}}^{\overrightarrow{BA}} = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{\left\|\overrightarrow{BC}\right\|}$$

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Solution:

We have $\overrightarrow{BA} = \langle -1,-10,3\rangle$ and $\overrightarrow{BC} = \langle 7,-1,5\rangle$. We have

$$|AD| = \text{Comp}_{\overrightarrow{BC}}^{\overrightarrow{BA}} = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{\left\|\overrightarrow{BC}\right\|} = \frac{67}{\sqrt{90}}.$$

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The distance is:

$$|AD|$$

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The distance is:

$$|AD| = \sqrt{110 - \frac{67^2}{90}} = 7.75.$$

Check that:

$$|AD| = \frac{\left\|\overrightarrow{BA} \times \overrightarrow{BC}\right\|}{\left\|\overrightarrow{BC}\right\|}.$$

Definition

let c be a curve in a plane and ℓ be a line that is not a parallel to the plane. The set of points on all lines that are parallel to ℓ and intersect with c is a *cylinder*.

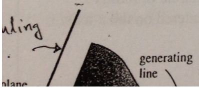
98.

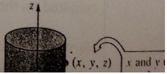
10.6 SURFACES

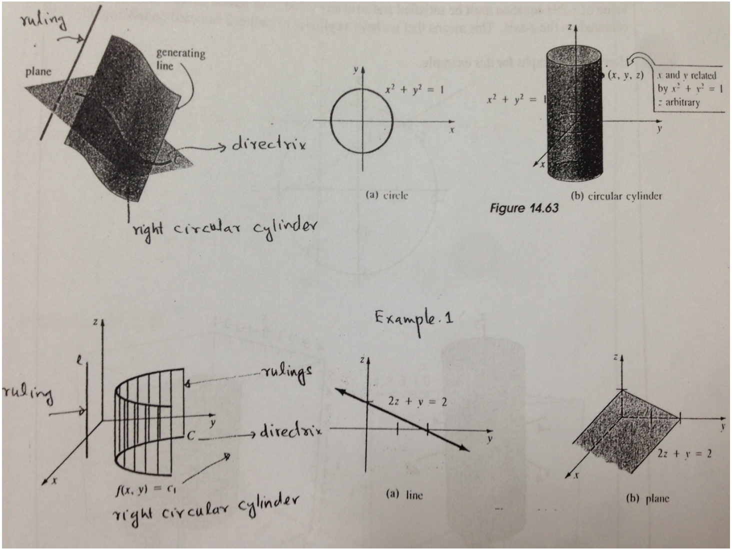
Cylinder

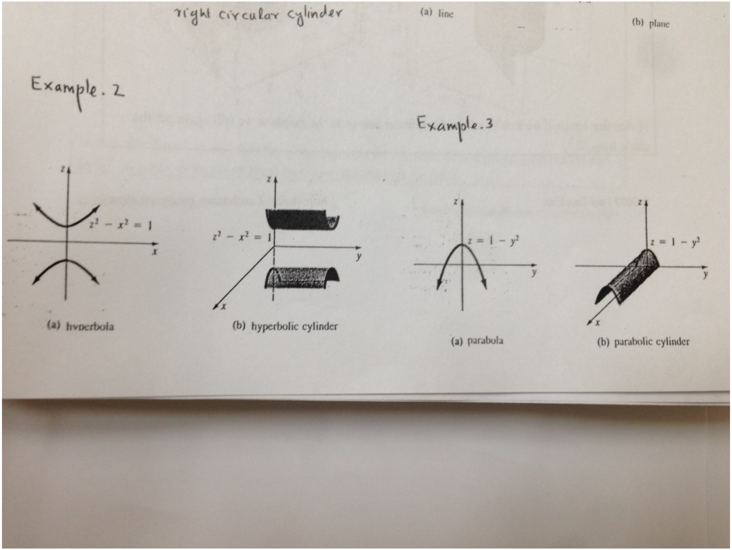
Def.

Let C be a curve in a plane, and let l be a line that is not in a parallel plane. The set of points on all lines that are parallel to l and intersect C is a cylinder.






ruling
plane
generating line
directrix
right circular cylinder
$x^2 + y^2 = 1$
(a) circle
$x^2 + y^2 = 1$
(x, y, z)
x and y related by $x^2 + y^2 = 1$
z arbitrary
(b) circular cylinder
Figure 14.63
Example. 1
ruling
rulings
directrix
C
$f(x, y) = c_1$
right circular cylinder
$2z + y = 2$
(a) line
$2z + y = 2$
(b) plane


right circular cylinder
(a) line
(b) plane
Example. 2
Example. 3
$z^2 - x^2 = 1$
$z^2 - x^2 = 1$
$z = 1 - y^2$
$z = 1 - y^2$
(a) hyperbola
(b) hyperbolic cylinder
(a) parabola
(b) parabolic cylinder

Example 3 Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

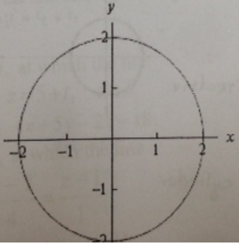
Solution

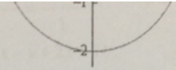
As with the previous example this won't have a 1-D graph since there are two variables.

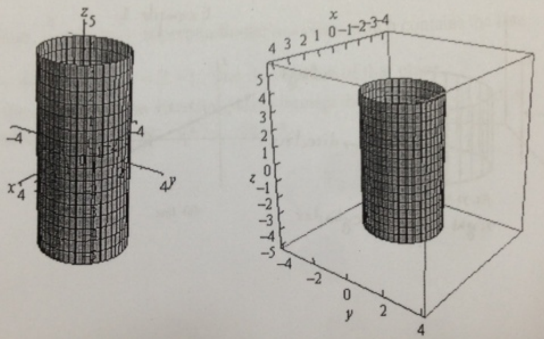
In $\mathbb{R}^2$ this is a circle centered at the origin with radius 2.

In $\mathbb{R}^3$ however, as with the previous example, this may or may not be a circle. Since we have not specified z in any way we must assume that z can take on any value. In other words, at any value of z this equation must be satisfied and so at any value z we have a circle of radius 2 centered on the z-axis. This means that we have a cylinder of radius 2 centered on the z-axis.

Here are the graphs for this example.







Notice that again, if we look to where the cylinder intersects the *xy*-plane we will again get the circle from $\mathbb{R}^2$.

 7 http://tutorial.math.lamar.edu/terms.aspx

Example 2 Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

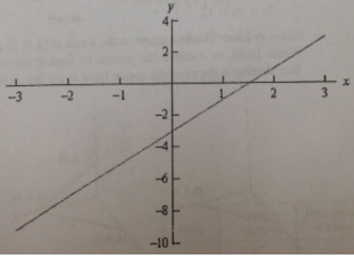
Solution

Of course we had to throw out $\mathbb{R}$ for this example since there are two variables which means that we can't be in a 1-D space.

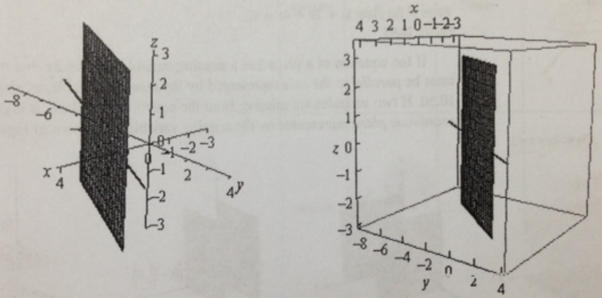
In $\mathbb{R}^2$ this is a line with slope 2 and a y intercept of -3.

However, in $\mathbb{R}^3$ this is not necessarily a line. Because we have not specified a value of z we are forced to let z take any value. This means that at any particular value of z we will get a copy of this line. So, the graph is then a vertical plane that lies over the line given by $y = 2x - 3$ in the xy-plane.

Here is the graph in $\mathbb{R}^2$.



here is the graph in $\mathbb{R}^3$.



Notice that if we look to where the plane intersects the *xy*-plane we will get the graph of the line in $\mathbb{R}^2$ as noted in the above graph by the red line through the plane.

Sketching Planes in Space:

Sketching Planes in Space: If a plane in space intersect one of the coordinate planes, we call *the line of intersection* *the trace of* the given plane in the coordinate plane.

Sketching Planes in Space

If a plane in space intersects one of the coordinate planes, we call the line of intersection the **trace** of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane given by

$$3x + 2y + 4z = 12. \qquad \text{Equation of plane}$$

We find the xy-trace by letting $z = 0$ and sketching the line

$$3x + 2y = 12 \qquad xy\text{-trace}$$

in the xy-plane. This line intersects the x-axis at $(4, 0, 0)$ and the y-axis at $(0, 6, 0)$. In Figure 10.49, we continue this process by finding the yz-trace and the xz-trace, and then shading in the triangular region lying in the first octant.

Traces of the plane $3x + 2y + 4z = 12$
Figure 10.49

If the equation of a plane has a missing variable such as $2x + z = 1$, the plane must be *parallel to the axis* represented by the missing variable, as shown in Figure 10.50. If two variables are missing from the equation of a plane, it is *parallel to the coordinate plane* represented by the missing variables, as shown in Figure 10.51.

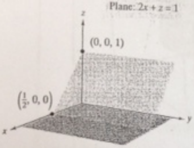
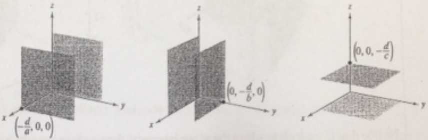


Figure 10.50

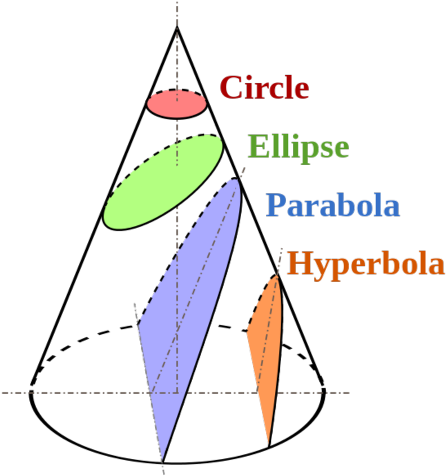


Plane $ax + d = 0$ is parallel to yz-plane.

Plane $by + d = 0$ is parallel to xz-plane.

Plane $cz + d = 0$ is parallel to xy-plane.

Figure 10.51


Circle
Ellipse
Parabola
Hyperbola

The equations for Ellipse is:

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \quad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

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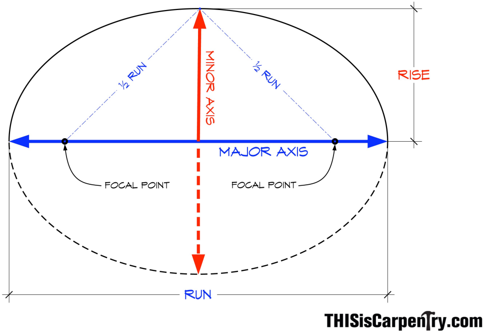
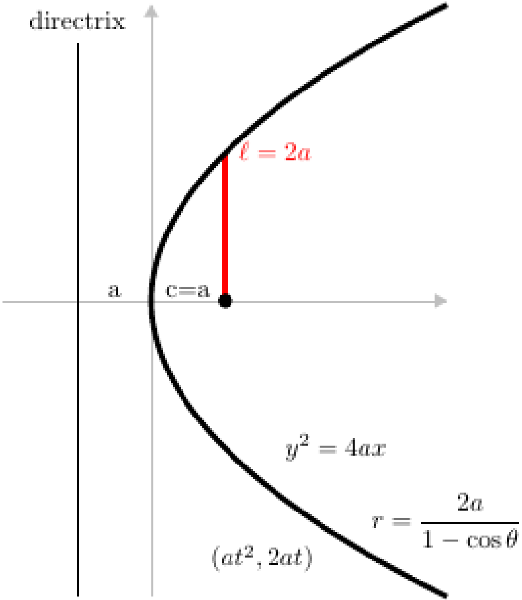
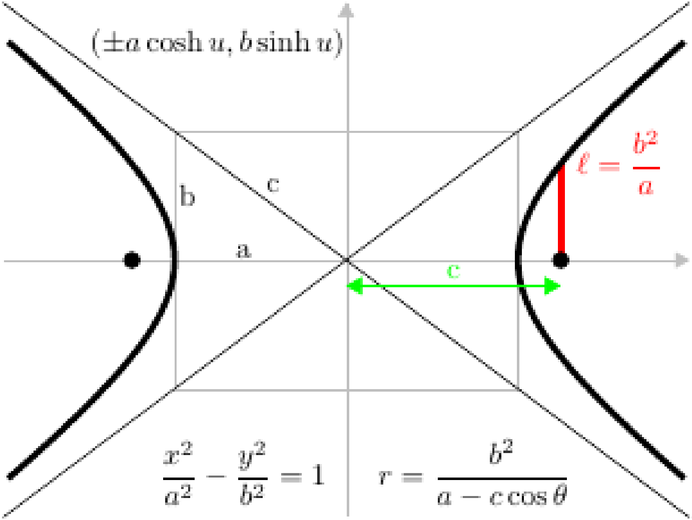
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \quad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$




directrix
$\ell = 2a$
a
c=a
$y^2 = 4ax$
$r = \dfrac{2a}{1 - \cos\theta}$
$(at^2, 2at)$



$(\pm a\cosh u, b\sinh u)$
b
c
a
c
$\ell = \dfrac{b^2}{a}$
$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
$r = \dfrac{b^2}{a - c\cos\theta}$

For more information check:
`http://tutorial.math.lamar.edu/Classes/CalcIII/QuadricSurfaces.aspx`

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The graph of a second-degree equation in x, y, z:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Cx + Hy + Iz + J = 0$$

is a *quadric surface*.

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There are three types of quadric surfaces:

1. Ellipsoids.

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2. Hyperboloids.

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The graph of a second-degree equation in x, y, z:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Cx + Hy + Iz + J = 0$$

is a *quadric surface*.
There are three types of quadric surfaces:

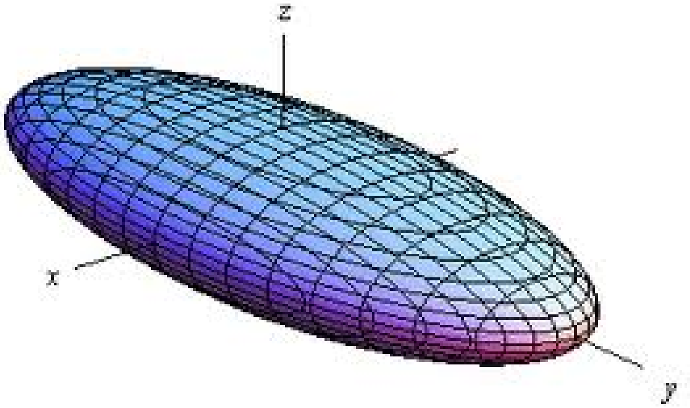
1. Ellipsoids.
2. Hyperboloids.
3. Paraboloids.

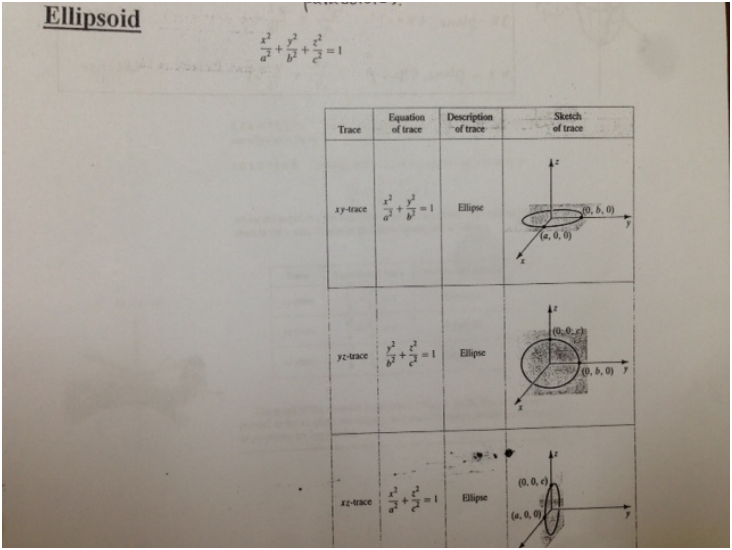
Elipsoid surface equation has the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Elipsoid surface equation has the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$





Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Ellipse	z, $(0, b, 0)$, y, $(a, 0, 0)$, x
yz-trace	$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Ellipse	z, $(0, 0, c)$, $(0, b, 0)$, y, x
xz-trace	$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$	Ellipse	z, $(0, 0, c)$, $(a, 0, 0)$, y

100.

Ex. Let a surface be $36x^2 + 16y^2 + 9z^2 = 144$

(i) write the name of the surface,

(ii) write the names and the equation of the traces of the surface on the coordinate planes and

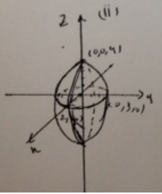
(iii) Sketch the surface.

Solution Dividing both sides of the equation by 144, we obtained

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

(i) This is an ellipsoid, axis is z-axis.

(ii)



Trace	Equation of Trace	Discription
xy-plane (z=0)	$\frac{x^2}{4} + \frac{y^2}{9} = 1$	Ellipse
yz-plane (x=0)	$\frac{y^2}{9} + \frac{z^2}{16} = 1$	Ellipse
xz-plane (y=0)	$\frac{x^2}{4} + \frac{z^2}{16} = 1$	Ellipse

[a] Hyperbolids of One Sheet:

The equation has the form:

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The equation has the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

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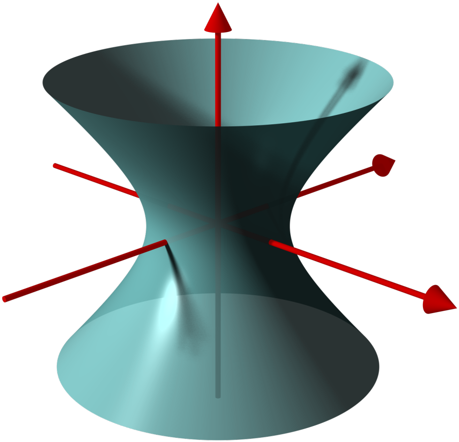


[b] Hyperbolids of Two Sheets:

The equation has the form:

[b] Hyperbolids of Two Sheets:

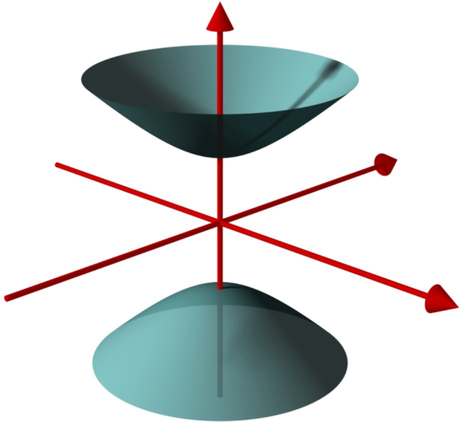
The equation has the form:

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, a>0, b>0, c>0$$

[b] Hyperbolids of Two Sheets:

The equation has the form:

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, a>0, b>0, c>0$$



[c] Cone:

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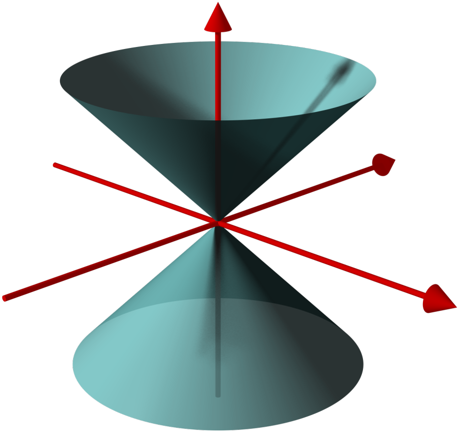
The equation has the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, a > 0, b > 0, c > 0$$

[c] Cone:

The equation has the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, a > 0, b > 0, c > 0$$



The equation has the form:

The equation has the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$$

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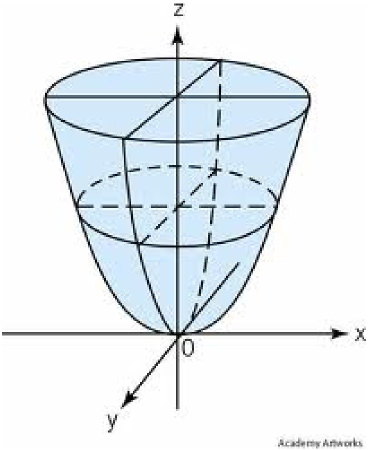
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$$



The equation has the form:

The equation has the form:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = cz$$

The equation has the form:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = cz$$