Limit Theorems For Random Triangular Urn Schemes

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Abstract. In this paper we study a generalized Pólya urn with balls of two colors and a random triangular replacement matrix. We extend some results of Svante Janson [11, 12] to the case when the largest eigenvalue of the mean of the replacement matrix is not in the dominant class. Using some useful martingales and the embedding method introduced in Athreya and Karlin [2], we describe the asymptotic composition of the urn after the n^{th} draw, for large n.

1 Introduction

We consider a generalized Pòlya urn with balls of two colors, say "white" (W) and "black" (B). The urn is initially non empty. At each time n, a ball in the urn is drawn uniformly at random, its color is observed (thus the ball is drawn, looked at and then placed back into the urn): if it is white then we add in the urn X_n white balls ; if it is a black one then Y_n white and Z_n black balls are inserted. The random variables X_n, Y_n and Z_n are independent copies respectively of some nonnegative and integer-valued random variables X, Y and Z.

The evolution rule in time n is then summarized by a 2×2 random matrix defined by:

Drawn↓	Added →	
	W	B
W	X_n	0
B	Y_n	Z_n

Thus, the composition of the urn after n draws is represented as a vector (W_n, B_n), where W_n (resp. B_n) is the number of white [resp. black] balls in

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the urn. The urn starts with a given vector (W_0, B_0), which we assume is non-random.

The assumption that X, Y, Z are nonnegative integer-valued random variables, guarantee the non-extinction of the urn. Furthermore, in order to avoid any explosion of the urn we suppose that X, Y and Z have finite variances. We define

$$\mu_X = \mathbf{E}(X),\ \mu_Y = \mathbf{E}(Y),\ \mu_Z = \mathbf{E}(Z), \sigma_X^2 = \mathbf{V}(X), \sigma_Y^2 = \mathbf{V}(Y),\ \sigma_Z^2 = \mathbf{V}(Z),$$

and for every integer n, the ball addition matrix or the replacement matrix is

$$R(n) = \begin{pmatrix} X_n & 0 \\ Y_n & Z_n \end{pmatrix}.$$

One of the first studies is developped in Eggenberger and Pólya [7], which deals with the fixed schemata $\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$. This urn is known as the Pólya Eggenberger urn. In 1931 this model was discussed by Pólya [17]. Next, Bernstein [6], Savkevich [18] generalizes this model, to the case where s balls of the same color and a balls of the antithecal color are added:

$$\begin{pmatrix} s & a \\ a & s \end{pmatrix}.$$

These two models have been studied and generalized in later papers (see [5, 9, 4, 15, 16]).
Several Pólya urn models with various settings for the ball addition matrix have been studied by many authors. In particular, when the mean of the replacement matrix is irreducible, Janson [11] carries out a study in which he characterizes the number of balls of each colors. When the replacement matrix is triangular, not irreducible and non random, Janson [12] characterizes the limit law and the almost sure limits of the numbers of balls of each colors.
The aim of this paper is to extend these results to the case when the replacement matrix is assumed to be random.

To attempt our goal, we will use the embedding method of Athreya and Karlin [2, 3] and study the urn process by embedding it into a multitype continuous time Markov branching process $\mathcal{X}(t) = (W(t), B(t))$ with initial condition $\mathcal{X}(0) = (W(0), B(0)) = (W_0, B_0)$. In the continuous process we

assume that every ball of type i (type 1: white balls, type 2: black balls) has an exponentially distributed lifetime with mean 1, i.e. it dies with intensity 1, and when it dies, it is replaced by a set of balls with distribution $(\delta_{i,j})_{j=1,2}$, where, if we note by $\mathcal{L}$ the distribution

$$\delta_{1,1} = \mathcal{L}(X+1),\ \delta_{1,2} = \mathcal{L}(0),\ \delta_{2,1} = \mathcal{L}(Y),\ \delta_{2,2} = \mathcal{L}(Z+1).$$

We shall assume that all life times and offspring compositions are independent.

Let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ be the splitting times of the process $\mathcal{X}$. Athreya and Karlin [2] proved that the process $(\mathcal{X}(\tau_n))_{n\geq 0}$ has the same law as $((W_n,\, B_n))_{n\geq 0}$; hence since $\tau_n \overset{a.s}{\to}_n +\infty$ limit theorems for $(W_n,\, B_n)$ can be derived from limit theorems for $\mathcal{X}(t)$.

In order to study the process $\mathcal{X}(t) = (W(t), B(t))$, we define a suitable martingale $\mathcal{Y}(t)$. The martingale that we use, is a standard one in branching process theory.
Let $A := \mathbf{E}(R^T(n)) = \begin{pmatrix} \mu_X & \mu_Y \\ 0 & \mu_Z \end{pmatrix}$ and define $\mathcal{Y}(t) = e^{-tA}\mathcal{X}(t)^T$. Here, M^T denotes the transpose of the matrix M.
Using the Markov property, we extend a fundamental well-known result [Lemma 9.8 [11], Theorem V.7.2 [3]]. Let $\mathcal{F}_t$ be the σ-algebra generated by the family $\{(W(s),\, B(s)),\, s \leq t\}$. Theorem 1 is proved in section 3.

Theorem 1 *If $\mu_X \geq \mu_Z$, the Martingale $\{\mathcal{Y}(t),\, \mathcal{F}_t,\, t \geq 0\}$ is an L^2 bounded martingale, and hence converges almost surely and in L^2. Moreover, $\mathcal{Y}(t) = \mathbf{E}(\widetilde{\mathcal{Y}}/\mathcal{F}_t)$.*
Where $\widetilde{\mathcal{Y}} := \begin{pmatrix} W \\ B \end{pmatrix}$ is the almost sure and L^2 limit of $\mathcal{Y}(t)$.

Remarks:

(1) I think that we still get an almost sure and L^1 convergence of the martingale $\mathcal{Y}(t)$ if we relax this assumption on X, Y and Z and to replace it by

$$\mathbf{E}(X\ln X) + \mathbf{E}(Y\ln Y) + \mathbf{E}(Z\ln Z) < \infty.$$

But, for our main results we need the L^2 assumption.

(2) Theorem 1 proves that,

- if $\mu_X > \mu_Z$

$$e^{-t\mu_X} W(t) + \frac{\mu_Y}{\mu_X - \mu_Z}(e^{-t\mu_X} - e^{-t\mu_Z})B(t) \overset{(a.s, L^2)}{\longrightarrow} W, \quad (1)$$

$$e^{-t\mu_Z} B(t) \overset{(a.s, L^2)}{\longrightarrow} B. \quad (2)$$

As proved in Theorem 3.1 [11], we have the fonctionnel limit Theorem. In $D[0,\infty)$, the space of all right-continuous functions $[0,\infty) \to \mathbb{C}$ with left-hand limits,

$$e^{-tx\mu_X} W(tx) + \frac{\mu_Y}{\mu_X - \mu_Z}(e^{-tx\mu_X} - e^{-tx\mu_Z})B(tx) \overset{\mathcal{L}}{\longrightarrow} W,$$

$$e^{-tx\mu_Z} B(xt) \overset{\mathcal{L}}{\longrightarrow} B.$$

- if $\mu_X = \mu_Z$

$$e^{-t\mu_X} W(t) - \mu_Y t e^{-t\mu_X} B(t) \overset{a.s., L^2}{\longrightarrow} W, \quad (3)$$

$$e^{-t\mu_X} B(t) \overset{a.s., L^2}{\longrightarrow} B. \quad (4)$$

and, in $D[0,\infty)$

$$e^{-xt\mu_X} W(xt) - \mu_Y xt e^{-xt\mu_X} B(xt) \overset{\mathcal{L}}{\longrightarrow} W,$$

$$e^{-xt\mu_X} B(xt) \overset{\mathcal{L}}{\longrightarrow} B.$$

(3) By Theorem 1, B is a nonnegative random variable with $\mathbf{E}(B) = B_0$. From [3] (Theorems III.4.1 and III.7.2), we deduce that

$$\mathbf{P}(B = 0) = 0 \ .$$

(4) Let $g_Z(s) = s\mathbf{E}(s^Z)$ be the probability generating function of $Z+1$ and $\varphi_B(u) = \mathbf{E}(e^{-uB})$. Then [10, 13]

$$\varphi_B^{-1}(v) = (1-v)\exp\int_v^1 \Big[\frac{1}{s-1} - \frac{\mu_Z}{g_Z(s) - s}\Big] ds, \ \ 0 < v \leq 1.$$

(5) In this paper we have not enable to characterize the distribution of W in all cases.

Theorem 1 is the basis of all our results for the branching process and generalized Pólya urns.

The paper is organized as follows. In Section 2 we give our main results after having specified the notations. Section 3 deals with the proof of Theorem 1. Finally, the proofs of the main results are given in section 4.

2 Main results

In the following, $\mathcal{P}(\lambda)$ denotes a Poisson random variable variable with parameter λ and $\mathcal{N}(\mu, \sigma^2)$ a normal law with mean μ and variance σ^2. Let B_n [resp. W_n] be the number of black [resp. white] balls after n draws, B and W denotes the limiting random variables defined in Theorem 1. For $n \geq 1$ we denote by S_n the total number of balls in the urn after the n^{th} draws. We have $S_n = W_n + B_n$ and $S_0 = W_0 + B_0$. The quantities N_n^W (resp. N_n^B) denote the number of splits of white (resp. black) balls among the first n splits, then $N_n^B + N_n^W = n$. Let $\tau_1^W < \cdots < \tau_{N_n^W}^W$ (resp. $\tau_1^B < \cdots < \tau_{N_n^B}^B$) be the split times, before τ_n, of the N_n^W (resp. N_n^B) white (resp. black) balls. The basic hypothesis of this section is:

Hypothesis H

(H1) X, Y and Z are nonnegative independent integer-valued random variables satisfying

(*) $\sigma_X + \sigma_Y + \sigma_Z < \infty$,

(**) $\mu_X \geq \mu_Z$,

(***) $\mu_X(\mu_Y + \mu_Z) > 0$.

(H2) The initial composition of the urn is (W_0, B_0), with $B_0 > 0$.

Our main results are the following ones

Theorem 2 *Consider a generalized Pólya urn with two colors having a triangular random replacement matrix*

$$\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix},$$

assume ***H*** *and* $\mu_Y > 0$. *Then,*

1. *if* $\mu_X > \mu_Z > 0$, *let* $\rho = \frac{\mu_Z}{\mu_X}$ *and* $K = (\mu_X)^\rho B \left(W + \frac{\mu_Y}{\mu_X - \mu_Z} B \right)^{-\rho}$,

   (a) *almost surely*

$$\begin{aligned} W_n &= \mu_X n + o(n), & B_n = K n^\rho + o(n^\rho) \\ N_n^W &= n - \frac{K}{\mu_Z} n^\rho + o(n^\rho), & N_n^B = \frac{K}{\mu_Z} n^\rho + o(n^\rho). \end{aligned}$$

(b) If $\rho \le \frac{1}{2}$, then,

$$\frac{W_n - n\mu_X}{\sqrt{n}} - \frac{K}{\mu_Z}(\mu_Y - \mu_X)\mathbb{1}_{\{\rho=\frac{1}{2}\}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_X^2),$$

(c) if $\frac{1}{2} < \rho < 1$,

$$\frac{W_n - n\mu_X}{n^\rho} \xrightarrow{\mathbf{P}} \frac{K}{\mu_Z}(\mu_Y - \mu_X).$$

2. *If $\mu_X = \mu_Z > 0$, we have almost surely*

$$\begin{aligned} W_n &= \mu_X n + o(n), \\ B_n &= \frac{\mu_X^2}{\mu_Y}\frac{n}{\ln n} + o\Big(\frac{n}{\ln n}\Big). \end{aligned}$$

Remark : Kotz, Mahmoud and Robert [14] give exact formulas and some asymptotics for 2-type urns. They comment that the case $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ gives asymptotics "of an essentially different character". They prove, "heuristically", that $\mathbf{E}(B_n)$ is of order $\frac{n}{\ln n}$. This is expressed by Theorem (2.2).

An interesting exceptional case is when $Z \overset{a.s.}{=} 0$. In this case the number of black balls remains unchanged B_0. The number $W(t)$ of white balls in the branching process is a generalized Yule-type process with immigration. We obtain the following limit results

Theorem 3 *Consider a generalized Pólya urn with two colors having a triangular random replacement matrix*

$$\begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix}.$$

*Under **H** we have,*

1. *almost surely: $W_n = \mu_X n + o(n)$,*

2. *if $\sigma_X \neq 0$, then we have the central limit theorem,*

$$\frac{W_n - \mu_X n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_X^2),$$

3. if $X = \alpha \neq 0$ is non random, then

$$\frac{1}{\sqrt{\ln n}}\Big(W_n - n\alpha - \frac{\mu_Y - \alpha}{\alpha} B_0 \ln n\Big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, \Big(\sigma_Y^2 + (\mu_Y - \alpha)^2\Big)\frac{B_0}{\alpha}\Big).$$

Remarks :

- The case $\mu_X < \mu_Z$ has been studied by S. Janson [11].
- In the case (3), if almost surely, $Y = \gamma =$ *constant*, we recover the result of S. Janson [12](Remark IV.4).

The next result deals with the diagonal case: $Y \overset{a.s}{=} 0$.

Theorem 4 *Consider a generalized Pólya urn with diagonal replacement matrix*

$$\begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix},$$

1. if $\mu_X > \mu_Z > 0$, let $\rho = \frac{\mu_Z}{\mu_X}$ and $D = \mu_X^\rho \, B \, W^{-\rho}$,

(a) almost surely as $n \to +\infty$

$$W_n = \mu_X \, n + o(n), \quad B_n = D \, n^\rho + o(n^\rho),$$

(b) if $0 < \rho \leq \frac{1}{2}$, we have as $n \to +\infty$

$$\frac{W_n - n\mu_X}{\sqrt{n}} + \sqrt{\mu_X}\frac{\mu_X}{\mu_Z} B W^{-1/2} 1\!\!1_{\{\rho = \frac{1}{2}\}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_X^2),$$

(c) if $\frac{1}{2} < \rho < 1$, we have as $n \to +\infty$

$$\frac{W_n - n\mu_X}{n^\rho} \xrightarrow{\mathbf{P}} -\mu_X^\rho \frac{\mu_X}{\mu_Z} B W^{-\rho}.$$

2. if $\mu_X = \mu_Z$ almost surely as $n \to +\infty$

$$W_n = \mu_X \frac{W}{W + B} n + o(n), \qquad B_n = \mu_X \frac{B}{W + B} n + o(n).$$

Remarks:

1. The branching process $(W(t),\, B(t))$ consists of two independent generalized Yule processes, thus, the limits W and B are independent and $\mathbf{P}(B>0)\mathbf{P}(W>0)>0$ (recall that $\mu_X.\mu_Z>0$).

2. Let $f_X(s)=s\mathbf{E}(s^X)$ (resp. $g_Z(s)=s\mathbf{E}(s^Z)$) be the probability generating function of $X+1$ (resp $Z+1$) and $\varphi_W(u)=\mathbf{E}[e^{-uW}]$, $\varphi_B(u)=\mathbf{E}[e^{-uB}]$. Since conditions $\mathbf{E}[X\ln X]<\infty$ and $\mathbf{E}[Z\ln Z]<\infty$ are fulfilled we deduce by [10, 13]
$$\varphi_W^{-1}(v)=(1-v)\exp\int_1^v\Big[\frac{\mu_X}{f_X(s)-s}-\frac{1}{s-1}\Big]ds,\quad 0<v\le 1. \tag{5}$$
$$\varphi_B^{-1}(v)=(1-v)\exp\int_1^v\Big[\frac{\mu_Z}{g_Z(s)-s}-\frac{1}{s-1}\Big]ds,\quad 0<v\le 1. \tag{6}$$
These two expressions give the characterization of W's and B's distributions. As a consequence, in the case 2) W and B have the same law. In the deterministic case where X and Z are constants, we find that W and B are **Beta** distributed.

We present also some results about the split times $(\tau_n)_{n\ge1}$. It's known (see [3]) that τ_n tends almost surely to infinity as $n\to\infty$. We define, for all positive integers n , the process $\{(T_n,\,\mathcal{G}_n)\}_{n\ge1}$ by
$$T_n=\tau_n-\sum_{j=0}^{n-1}(W(\tau_j)+B(\tau_j))^{-1}$$
where $\mathcal{G}_n$ is the σ-algebra generated by $\tau_1,\cdots,\tau_n,\ W(\tau_1),\ B(\tau_1),\cdots,W(\tau_{n-1})$ and $B(\tau_{n-1})$. We have the following result:

Theorem 5 *Under hypothesis **H**, the process $(T_n,\,\mathcal{G}_n)_{n\ge1}$ is an L^2 bounded martingale, consequently, it converges almost surely, and in L^2 to some random variable $(T_\infty,\mathcal{G}_\infty)\in L^2$.*

Using Theorem 5, we obtain an asymptotic expansion of τ_n,

Proposition 1 *Under **H** , one has, almost surely*
$$\frac{\tau_n}{\ln n}=\frac{1}{\mu_X}+o(1).$$

This extends results by Athreya and Ney [3] (Theorem V.7.3) and S. Janson [11] (Lemma 11.1).

3 Proof of Theorem 1

Let

$$f(t) = \mathbf{E}(W(t)), \quad g(t) = \mathbf{E}(B(t)) \quad and \quad \mathcal{H}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \mathbf{E}(\mathcal{X}(t)).$$

Markov property and Chapman-Kolmogorov equation imply, this is a classical result, that

$$\mathcal{H}'(t) = \frac{d}{dt}\mathcal{H}(t) = A\mathcal{H}(t) \tag{7}$$

and that $\mathcal{Y}(t)$ is an $\mathcal{F}_t$-martingale. By integration, equation (7) implies

$$\mathcal{H}(t) = \exp(tA)\mathcal{H}(0) = \exp(tA)(W_0, B_0)^T. \tag{8}$$

Case when $\mu_X > \mu_Z$

Consider the process $\{B(s),\, s \geq 0\}$ (resp. $\{W(s),\, s \geq 0\}$) at time $t + \delta t$, where δt is an infinitesimal increment of time. Given $B(t)$ and $W(t)$, the number of black (resp. white) balls at time $t + \delta t$ is what it was at time t, plus the number of black (resp. white) balls added during the infinitesimal period $(t, t + \delta t)$. For each ball among the $S(t) := B(t) + W(t)$ black and white ones, the number of splittings that happen in the time interval $(t, t + \delta t)$ follows a $\mathcal{P}(\delta t)$ distribution; some of the resulting children may themselves give additional children but their number has $O(\delta t)^2$ average and square mean. Further more, any two different balls have independent birth processes. Therefore, given $B(t)$ and $W(t)$, $B(t + \delta t)$ and $W(t + \delta t)$ satisfy:

$$B(t + \delta t) = B(t) + \sum_{j=1}^{B(t)} \sum_{k=1}^{\mathcal{P}_j(\delta t)} Z_{k,j} + O(\delta t)^2 \tag{9}$$

and

$$W(t + \delta t) = W(t) + \sum_{j=1}^{W(t)} \sum_{k=1}^{\mathcal{P}_j(\delta t)} X_{k,j} + \sum_{j=1}^{B(t)} \sum_{k=1}^{\mathcal{P}_j(\delta t)} Y_{k,j} + O(\delta t)^2 \tag{10}$$

where $\mathcal{P}_j(\delta t)$ are iid random variables that have $\mathcal{P}(\delta t)$ distribution, all $Z_{k,j}$, $X_{k,j}$, $Y_{k,j}$ are iid r.v. that have respectively Z's, X's, Y's distribution and independent of $B(t)$, $W(t)$. Squaring equations (9) and (10) we obtain equations of the conditional second moments:

$$\begin{aligned}\mathbf{E}(B^2(t+\delta t)|B(t)) &= B^2(t) + 2B^2(t)\delta t\mu_Z \\ &\quad + \mu_{Z^2}B(t)\delta t + \mu_Z^2\delta t B(t)\Big(\delta t B(t) - 1\Big) + O((\delta t)^2).\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}(W^2(t+\delta t)|B(t), W(t)) &= W^2(t) + 2\Big[W^2(t)\mu_X + W(t)B(t)\mu_Y\Big]\delta t \\ &\quad + \Big[W(t)\mu_{X^2} + B(t)\mu_{Y^2}\Big]\delta t \\ &\quad + O((\delta t)^2).\end{aligned}$$

Take expectations, and let $\delta t \to 0$ to get the differential equations:

$$\frac{d}{dt}\varphi(t) = 2\mu_Z\varphi(t) + g(t)\sigma_Z^2 \tag{11}$$

and

$$\frac{d}{dt}\chi(t) = 2\mu_X\chi(t) + 2\mu_Y\psi(t) + \mu_{X^2}f(t) + \mu_{Y^2}g(t) \tag{12}$$

where $\varphi(t) = \mathbf{E}(B^2(t)), \chi(t) = \mathbf{E}(W^2(t))$ and $\psi(t) = \mathbf{E}(B(t)W(t))$. The solution of (11) is easily seen to be

$$\varphi(t) = B_0\Big(1 + \frac{\sigma_Z^2}{\mu_Z}\Big)e^{2\mu_Z t} - B_0\frac{\sigma_Z^2}{\mu_Z}e^{\mu_Z t}. \tag{13}$$

We prove similarly that $\psi(t)$ satisfies the following differential equation

$$\frac{d}{dt}\psi(t) = (\mu_Z + \mu_X)\psi(t) + \mu_Y\varphi(t).$$

After computation of $\psi(t)$ and $\chi(t)$, we easily conclude that $(\mathcal{Y}(t),\, t \geq 0)$ is L^2 bounded.

Case when $\mu_X = \mu_Z$:

By (13), also available in this case, the process $(e^{-t\mu_X}B(t),\, t \geq 0)$ is L^2 bounded.
As in the first case, we prove that, the process

$$M(t) := e^{-t\mu_X}W(t) - \mu_Y t e^{-t\mu_X}B(t)$$

is L^2 bounded.

4 Proofs of the main results

4.1 Proof of Theorem 5

For all integer $n \geq 1$, let $\upsilon_n = \tau_n - \tau_{n-1}$: be the holding time between the $(n-1)$-st and the n-th splits and $\upsilon_0 = 0$, $S_n = W_n + B_n$. Given $\mathcal{G}_{n-1}$, υ_n is an exponential random variable with mean $(S_{n-1})^{-1}$. But, for all integer n, we can write $\tau_n = \sum_{j=0}^{n} \upsilon_j$, this yields the martingale property.

On other hand, we have , $\mathbf{E}(T_n^2) = \sum_{j=0}^{n-1} \mathbf{E}(S_j^{-2})$ (see [3]).
Define for each $i \in \mathbb{N}$, $R_i = \min\{X_i, Y_i + Z_i\}$. The random variables $\{R_i\}_{\{i\geq 1\}}$ are independent with the same law as $R = \min\{X, Y + Z\}$.
By $(H1)$, $\mathbf{P}(R > 0) = \mathbf{P}(X > 0)\mathbf{P}(Y + Z > 0) > 0$.
Then $\mu_R = \mathbf{E}(R) > 0$ ($\mathbf{P}(R = 0) < 1$).
From [3] (Theorem III.9.4) and the strong law of large numbers

$$\lim_n \mathbf{E}\Big(\frac{n^2}{(S_0 + \sum_{i=1}^n R_i)^2}\Big) = \frac{1}{\mu_R^2}.$$

But, since the initial total number of balls equals S_0, one gets $S_n \geq S_0 + \sum_{i=1}^n R_i$, so that

$$\mathbf{E}\Big(\frac{n^2}{S_n^2}\Big) \leq \mathbf{E}\Big(\frac{n^2}{(S_0 + \sum_{i=1}^n R_i)^2}\Big)$$

is bounded above. As a matter of consequence, the sequence $\mathbf{E}(T_n^2)$ converges.
$\square$

Proof of Proposition 1: We prove this Proposition in two steps, assuming at first $\mu_X > \mu_Z$, then $\mu_X = \mu_Z$.
(i) Case when $\mu_X > \mu_Z$: It follows from Theorem 1 and the almost-sure limit: $\lim_n \frac{B_n}{N_n^B} = \mu_Z$, that

Lemma 1 *If $\mu_X > \mu_Z > 0$, almost surely,*

$$N_n^B = \frac{K}{\mu_Z} n^\rho + o(n^\rho),$$

where $\rho = \frac{\mu_Z}{\mu_X} \in]0, 1[$ and K is the random variable defined by

$$K = (\mu_X)^\rho B(W + \frac{\mu_Y}{\mu_X - \mu_Z} B)^{-\rho}.$$

For a continuous time process $\mathcal{S}$ and a positive real t_0, we define $\Delta\mathcal{S}(t_0)$ by

$$\Delta\mathcal{S}(t_0) = \lim_{\varepsilon\to 0^+}\Big[\mathcal{S}(t_0+\varepsilon) - \mathcal{S}(t_0-\varepsilon)\Big].$$

It is the jump, at time t_0, of the process $(\mathcal{S}(t))_{t\geq 0}$.
Let $S_n = W_n + B_n$, we can write, for $n \geq 1$

$$\begin{aligned}\frac{S_n}{n} &= \frac{S_0}{n} + \frac{1}{n}\sum_{i=1}^{N_n^w}\Delta W(\tau_i^W) + \frac{1}{n}\sum_{i=1}^{N_n^B}\Big(\Delta B(\tau_i^B) + \Delta W(\tau_i^B)\Big)\\ &= \frac{S_0}{n} + \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n}\sum_{i=1}^{N_n^B}\Big(Z_i + Y_i - X_i\Big).\end{aligned}$$

Using Lemma 1, and the strong law of large number, we deduce that almost surely,

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \mu_X + o(1) \quad and \quad \frac{1}{n}\sum_{i=1}^{N_n^B}\Big(Z_i + Y_i - X_i\Big) = O(n^{\rho-1}),$$

then for n large, we can write almost surely

$$\frac{n}{S_n} = \frac{1}{\mu_X}(1 + +o(1)),$$

and, by [3] (Theorem [III.9.4]), we conclude:

$$\sum_{n\geq 1}\frac{1}{n}\Big(\frac{n}{S_n} - \frac{1}{\mu_X}\Big) \overset{a.s.}{=} C',$$

where C' is a finite random variable. It means that, as $n \to +\infty$

$$\sum_{j=1}^{n-1}\Big(\frac{1}{W_j + B_j}\Big) - \frac{1}{\mu_X}\ln n \overset{a.s.}{=} O(1).$$

Using the fact that T_n is an almost surely convergent martingale, and putting T_∞ as the corresponding limit, it follows that, almost surely

$$\tau_n = \sum_{j=1}^{n-1}\frac{1}{S_j} + T_\infty + o(1) = \frac{\ln n}{\mu_X} + o(\ln n).$$

(ii) Case when $\mu_X = \mu_Z$: From (3) and (4) we prove that

$$\lim_{t\to+\infty} \frac{e^{-\mu_X t} W(t)}{t} \overset{a.s}{=} \mu_Y B. \tag{14}$$

When replacing t by τ_n, the quotient of (14) by (4) gives

$$\lim_{n\to+\infty} \frac{W_n}{\tau_n B_n} \overset{a.s}{=} \mu_Y. \tag{15}$$

Then Equation (23), implies

$$\lim_{n\to+\infty} \frac{n}{\tau_n B_n} \overset{a.s}{=} \frac{\mu_Y}{\mu_X}. \tag{16}$$

Again, using (4), we conclude $\lim_{n\to+\infty} \frac{n\exp(-\mu_X \tau_n)}{\tau_n} \overset{a.s}{=} \frac{B\mu_Y}{\mu_X}$. This means that

$$\lim_{n\to+\infty} \frac{\tau_n}{\ln n} \overset{a.s}{=} \frac{1}{\mu_X}. \tag{17}$$

4.2 Proof of Theorem 2

Proof of claim 1)

1-a) By (1) and (2), we can write

$$\frac{W(t)}{(B(t))^{\frac{\mu_X}{\mu_Z}}} + \frac{\mu_Y}{\mu_X - \mu_Z}\big(\exp(-t(\mu_X - \mu_Z)) - 1\big)\left(e^{-t\mu_Z} B(t)\right)^{1-\frac{\mu_X}{\mu_Z}} \xrightarrow{a.s.} \frac{W}{(B)^{\frac{\mu_X}{\mu_Z}}}. \tag{18}$$

Replacing in (18) t by τ_n ; since $(W_n, B_n) = (W(\tau_n), B(\tau_n))$ almost surely and since τ_n tends almost surely to infinity as n tends to infinity (see [3]), we obtain

$$\begin{aligned} \frac{W_n}{(B_n)^{\frac{\mu_X}{\mu_Z}}} \quad + \quad & \frac{\mu_Y}{\mu_X - \mu_Z}\big(\exp(-\tau_n(\mu_X - \mu_Z)) - 1\big)\left(e^{-\tau_n\mu_Z} B_n\right)^{1-\frac{\mu_X}{\mu_Z}} \\ & \xrightarrow{a.s.} \frac{W}{(B)^{\frac{\mu_X}{\mu_Z}}}. \end{aligned} \tag{19}$$

But, it's known (see [3, 11]) that $\frac{W_n}{n} \overset{a.s.}{\to} \mu_X$, then

$$\frac{n\mu_X}{(B_n^{\frac{\mu_X}{\mu_Z}})} \xrightarrow{a.s.} \frac{\mu_Y}{\mu_X - \mu_Z} B^{\frac{\mu_Z - \mu_X}{\mu_Z}} + W B^{\frac{-\mu_X}{\mu_Z}},$$

this proves $1-a)$ second equation. The rest of proofs of $1-a)$ is immediate using the last equation and the fact that

$$\frac{W_n}{N_n^W} \xrightarrow{a.s.} \mu_X,\ \frac{B_n}{N_n^B} \xrightarrow{a.s.} \mu_Z \textit{ and } N_n^W + N_n^B = n.$$

1-b) We have $W_n = W_0 + \sum_{i=1}^{N_n^W} X_{\tau_i^W} + \sum_{j=1}^{N_n^B} Y_{\tau_i^B}$, then

$$\frac{W_n - n\mu_X}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{N_n^W}(X_{\tau_i^W} - \mu_X) + \frac{1}{\sqrt{n}}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_X) + \frac{W_0}{\sqrt{n}}.$$

Because $N_n^W = n + o(n)$ and by the central limit Theorem we deduce

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{N_n^W}(X_{\tau_i^W} - \mu_X) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_X^2).$$

On the other hand, from $(1-a)$,
if $\rho < \frac{1}{2}$,

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_X) &= \frac{N_n^B}{\sqrt{n}}\frac{1}{N_n^B}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_X) \\ &\xrightarrow{a.s.} 0 \times (\mu_Y - \mu_X) = 0.\end{aligned}$$

if $\rho = \frac{1}{2}$,

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_X) - \frac{K}{m_Z}(\mu_Y - \mu_X) &= \Big[\frac{1}{\sqrt{n}}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_Y) + N_n^B(\mu_Y - \mu_X)\Big] \\ &= \frac{N_n^B}{\sqrt{n}}\frac{1}{N_n^B}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_Y) + (\frac{N_n^B}{\sqrt{n}} - \frac{K}{m_Z})(\mu_Y - \mu_X),\end{aligned}$$

and we conclude using the fact that

$$(\frac{N_n^B}{\sqrt{n}} - \frac{K}{m_Z}) \overset{a.s.}{\to} 0, \quad \textit{and} \quad \frac{1}{N_n^B}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_Y) \overset{a.s.}{\to} 0.$$

1-c) if $\frac{1}{2} < \rho < 1$, We decompose W_n as in $1-b)$

$$\frac{W_n - n\mu_X}{n^\rho} = \frac{1}{n^\rho}\sum_{i=1}^{N_n^W}(X_{\tau_i^W} - \mu_X) + \frac{1}{n^\rho}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_Y) + \frac{N_n^B}{n^\rho}(\mu_Y - \mu_X) + \frac{W_0}{n^\rho}$$

The first term in the right hand side tends in L^2 to 0, in fact,

$$\mathbf{E}\Big(\frac{1}{n^\rho}\sum_{i=1}^{N_n^W}(X_{\tau_i^W} - \mu_X)\Big)^2 = \sigma_X^2\mathbf{E}(\frac{N_n^W}{n^{2\rho}}) \leq \sigma_X^2 n^{1-2\rho} \to 0. \tag{20}$$

On the other hand, by the strong law of large numbers and $(1-a)$, we conclude

$$\frac{1}{n^\rho}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_Y) = \frac{N_n^B}{n^\rho}\frac{1}{N_n^B}\sum_{j=1}^{N_n^B}(Y_{\tau_i^B} - \mu_Y) \overset{a.s.}{\to} 0 \ \textit{and} \ \frac{N_n^B}{n^\rho}(\mu_Y - \mu_X) \overset{a.s.}{\to} \frac{K}{\mu_Z}.$$

Proof of claim 2) It follows from equations (3) and (4), that

$$\mu_X\frac{W_n}{B_n} - \mu_Y\ln(B_n) \xrightarrow{a.s.} \mu_X\frac{W}{B} - \mu_Y\ln(B). \tag{21}$$

This leads to:

Lemma 2 *Under the conditions of Theorem 2(2.), almost surely,*

$$\lim_{n\to+\infty}\frac{W_n}{B_n} = +\infty,\ N_n^W = n + o(n) \ \textit{and}\ N_n^B = o(n).$$

Proof : Because of Equation (2), $B(t) \sim Be^{t\mu_Z}$ when t tends to $+\infty$, so that B_n tends to infinity when n tends to infinity. Thus Equation (21) implies that W_n/B_n tends almost surely to infinity as n tends to infinity. But, for $n \geq 1$

$$W_n = W_0 + \sum_{i=1}^{N_n^W} X_{\tau_i^W} + \sum_{j=1}^{N_n^B} Y_{\tau_j^B},\ \textit{and}\ B_n = B_0 + \sum_{j=1}^{N_n^B} Z_{\tau_j^B}, \tag{22}$$

so $\frac{B_n}{N_n^B} \xrightarrow{a.s.} \mu_Z$ and the first assertion of this Lemma, yields $\frac{W_n}{N_n^B} \xrightarrow{a.s.} +\infty$. This implies that

$$\frac{N_n^W}{N_n^B} \xrightarrow{a.s.} +\infty,$$

the result of Lemma 2 follows using the identity: $N_n^W + N_n^B = n$.
By the strong law of large numbers and Lemma 2, we conclude, almost surely

$$\lim_{n\to+\infty} \frac{W_n}{n} = \mu_X. \tag{23}$$

Asymptotic of B_n can be, easily, obtained using Equations (16) and (17) .

4.3 Proof of Theorem 3

From Theorem 1 we deduce that:

Corollary 1 *The process $(e^{-\mu_X t}(W(t) + \frac{\mu_Y}{\mu_X}))_{(t\geq 0)}$ is a positive martingale. Let*

$$\lim_{t\to\infty} e^{-\mu_X t}(W(t) + \frac{\mu_Y}{\mu_X}) = W + \frac{\mu_Y}{\mu_X}. \tag{24}$$

In particular

$$\mathbf{E}(W(t)) = \frac{\mu_Y + \mu_X}{\mu_X} e^{t\mu_X} - \frac{\mu_Y}{\mu_X}.$$

claim 1)**:** Let, for $t \geq 0$, $N(t) := \#\{k : \tau_k^B \leq t\}$ the number of splits of black balls up to time t, and let τ_n be the time of the n^{th} draw. Then

$$W(\tau_n) = W_n = W_0 + \sum_{j=1}^{n-N(\tau_n)} X_{\tau_j^W} + \sum_{i=1}^{N(\tau_n)} Y_{\tau_i^B}. \tag{25}$$

Since $(N(t))_{(t\geq 0)}$ is a Poisson process with intensity B_0, hence

$$\frac{N(t)}{t} \xrightarrow{a.s.} B_0, \; when \; t \to +\infty. \tag{26}$$

In particular, $e^{-\mu_X t}N(t) \xrightarrow{a.s.} 0$, and (24) and (25) yield, since $\tau_n \xrightarrow{a.s.} +\infty$,

$$\lim_n e^{-\mu_X \tau_n} W_n = W + \frac{\mu_Y}{\mu_X} \tag{27}$$

$$\lim_n \mu_X e^{-\mu_X \tau_n} n = W + \frac{\mu_Y}{\mu_X}. \tag{28}$$

Equation (27) can be written as

$$e^{\mu_X \tau_n} W_n = \Big(n\mu_X e^{-\mu_X \tau_n}\Big)\Big(\frac{W_n}{n\mu_X}\Big) \xrightarrow{a.s.} W + \frac{\mu_Y}{\mu_X}.$$

Consequently, by equation (28), we conclude that

$$\frac{W_n}{n\mu_X} \xrightarrow{a.s.} 1.$$

claim 2)**:** Using Equations (28) and (26), we deduce that

Lemma 3

$$\begin{aligned}\lim_n \frac{N(\tau_n)}{\ln n} &= \frac{B_0}{\mu_X} \ \ a.s., \ and \\ \tau_n &= \frac{\ln n}{\mu_X} + O(1).\end{aligned}$$

We have the following representation of W_n

$$W_n = W_0 + \sum_{k=1}^{n} X_k + \sum_{j=1}^{N(\tau_n)} (Y_{\tau_j^B} - X_{\tau_j^B}),$$

where $(X_i)_{1\leq i\leq n}$ (resp. $(Y_j)_{j\geq 1}$) are mutually independent and identically distributed as X (resp. Y) and $\tau_1^B, \dots, \tau_{N(\tau_n)}^B$ are the orders of the black ball splits up to time τ_n.

Let $\Delta(t) = \frac{1}{N(t)} \sum_{j=1}^{N(t)} (Y_{\tau_j^B} - X_{\tau_j^B})$ which by Lemma 3 and the strong law of large numbers, converges almost surely to: $\mu_Y - \mu_X$. Now, observe the identity

$$W_n - n\mu_X - W_0 = \sum_{k=1}^{n} (X_k - \mu_X) + N(\tau_n)\Delta(\tau_n).$$

By virtue of the central limit Theorem, Lemma 3 and Slutsky's Theorem, the result of 2) follows.

claim 3)**:** Since $(N(t))_{(t\geq 0)}$ is a Poisson process with intensity B_0 and $\tau_n \to +\infty$ when $n \to +\infty$, we have

$$\frac{N(\tau_n) - B_0\tau_n}{\sqrt{\tau_n}} \xrightarrow{\mathcal{L}} N(0, B_0). \tag{29}$$

Consider

$$\begin{aligned}
K_n &= \frac{1}{\sqrt{\ln n}}\Big(W_n - n\alpha - \frac{\mu_Y-\alpha}{\alpha}B_0\ln n\Big)\\
&= \frac{1}{\sqrt{\ln n}}\Big(N(\tau_n)(\mu_Y-\alpha) - \frac{\mu_Y-\alpha}{\alpha}B_0\ln n + \sum_{j=1}^{N(\tau_n)}(Y_{\tau_j^B}-\mu_Y)\Big)\\
&= (\mu_Y-\alpha)\Big(\frac{N(\tau_n)-\frac{B_0}{\alpha}\ln n}{\sqrt{\ln n}}\Big) + \frac{1}{\sqrt{\ln n}}\sum_{j=1}^{N(\tau_n)}(Y_{\tau_j^B}-\mu_Y).
\end{aligned}$$

Define, for $t\in(-1,1)$, $\phi_n(t)=\mathbf{E}(e^{-tK_n})$ (resp. $\phi(t)=\mathbf{E}(e^{-t(Y-\mu_Y)})$) the Laplace transform of K_n (resp of $Y-\mu_Y$). We have

$$\begin{aligned}
\phi_n(t) &= \mathbf{E}\Big[\exp\Big(-t(\mu_Y-\alpha)\Big(\frac{N(\tau_n)-\frac{B_0}{\alpha}\ln n}{\sqrt{\ln n}}\Big)\Big)\exp\Big(-\frac{t}{\sqrt{\ln n}}\sum_{j=1}^{N(\tau_n)}(Y_{\tau_j^B}-\mu_Y)\Big)\Big]\\
&= \mathbf{E}\Big[\exp\Big(-t(\mu_Y-\alpha)\Big(\frac{N(\tau_n)-\frac{B_0}{\alpha}\ln n}{\sqrt{\ln n}}\Big)\Big)\Big(\phi(\frac{t}{\sqrt{\ln n}})\Big)^{N(\tau_n)}\Big]\\
&= \mathbf{E}\Big[\exp\Big(-t(\mu_Y-\alpha)\Big(\frac{N(\tau_n)-\frac{B_0}{\alpha}\ln n}{\sqrt{\ln n}}\Big)\Big)\exp\Big(N(\tau_n)\ln(\phi(\frac{t}{\sqrt{\ln n}}))\Big)\Big]\\
&= \mathbf{E}\Big[\exp\Big(\frac{N(\tau_n)-\frac{B_0}{\alpha}\ln n}{\sqrt{\ln n}}\Big(\sqrt{\ln n}\ln(\phi(\frac{t}{\sqrt{\ln n}}))-t(\mu_Y-\alpha)\Big)\Big)\Big]\\
&\quad\times\exp\Big(\frac{B_0}{\alpha}\ln n\ln(\phi(\frac{t}{\sqrt{\ln n}}))\Big).
\end{aligned}$$

Since, for n large,

$$\begin{aligned}
\phi(\frac{t}{\sqrt{\ln n}}) &= \mathbf{E}\Big[1-\frac{t}{\sqrt{\ln n}}(Y-\mu_Y)+\frac{t^2}{2\ln n}(Y-\mu_Y)^2+o(\frac{1}{\ln n})\Big]\\
&= 1+\frac{t^2}{2\ln n}\sigma_Y^2+o(\frac{1}{\ln n})
\end{aligned}$$

we conclude that, as n large,

$$\frac{B_0}{\alpha}\ln n\ln(\phi(\frac{t}{\sqrt{\ln n}})) = \frac{B_0}{2\alpha}t^2\sigma_Y^2+o(1),$$

then

$$\exp\Big(\frac{B_0}{\alpha}\ln n\ln(\phi(\frac{t}{\sqrt{\ln n}})\Big)\overset{n\to\infty}{\longrightarrow}\exp\Big(\frac{B_0}{2\alpha}\sigma_Y^2t^2\Big)\Big).$$

On the other hand, using Equation (29), we have from Lemma 3,

$$\begin{aligned}
\frac{N(\tau_n)-\frac{B_0}{\alpha}\ln n}{\sqrt{\ln n}} &= \frac{N(\tau_n)-B_0\tau_n}{\sqrt{\ln n}}+\frac{B_0(\tau_n-\frac{\ln n}{\alpha})}{\sqrt{\ln n}}\\
&= \frac{N(\tau_n)-B_0\tau_n}{\sqrt{\tau_n}}\sqrt{\frac{\tau_n}{\ln n}}+\frac{B_0(\tau_n-\frac{\ln n}{\alpha})}{\sqrt{\ln n}}\\
&\xrightarrow{\mathcal{L}} N(0,\frac{B_0}{\alpha}),
\end{aligned}$$

and, when $n \to +\infty$, we have

$$\sqrt{\ln n}\ln(\phi(\frac{t}{\sqrt{\ln n}})) = \sqrt{\ln n}\Big(\frac{t^2}{2\ln n}\sigma_Y^2 + o(\frac{1}{\ln n})\Big) \longrightarrow 0.$$

Finally, we conclude that

$$\phi_n(t) \longrightarrow \exp\Big(\frac{B_0}{2\alpha}(\mu_Y-\alpha)^2t^2+\frac{B_0}{2\alpha}\sigma_Y^2t^2\Big)$$

4.4 Proof of Theorem 4

We study the asymptotic behavior of the number of black balls $B(t)$ and white balls $W(t)$. We see that $(B(t))_{(t\geq 0)}$ and $(W(t))_{(t\geq 0)}$ are independent Yule-type branching processes. Let $\mathcal{F}_t^W$ (resp. $\mathcal{F}_t^B$) the sigma fields generated by the process $(W(u), 0\leq u\leq t)$ (resp. $(B(u), 0\leq u\leq t)$). We deduce from Theorem 1 that:

$$\mathbf{E}(B(t)) = e^{\mu_Z t} \quad , \quad \mathbf{E}(W(t)) = e^{\mu_X t} \tag{30}$$

and

$$\mathbf{E}(W(t+s)|\mathcal{F}_t^W) = e^{\mu_X s}W(t), \tag{31}$$

$$\mathbf{E}(B(t+s)|\mathcal{F}_t^B) = e^{\mu_Z s}B(t), \tag{32}$$

$$e^{-\mu_X t}W(t) \xrightarrow{a.s} W, \quad e^{-\mu_Z t}B(t) \xrightarrow{a.s} B. \tag{33}$$

In order to prove claim (1) of Theorem 4, we need the following lemma

Lemma 4 *If* $\mu_X \times \mu_Z > 0$ *then :*

$$\frac{W_n}{\mu_X}+\frac{B_n}{\mu_Z} \stackrel{a.s}{=} n + o(n), \quad (n\to+\infty).$$

Proof: We have $W_n = W_0 + \sum_{k=1}^{N_n^W} X_{\tau_k^W}$ and $B_n = B_0 + \sum_{k=1}^{N_n^B} Z_{\tau_k^B}$.
By the strong law of large numbers, and the fact that almost surely,

$$\lim_{n\to\infty} N_n^W = +\infty, \ \lim_{n\to\infty} N_n^B = +\infty,$$

we deduce

$$\frac{W_n}{N_n^W} \xrightarrow{a.s.} \mu_X, \quad \frac{B_n}{N_n^B} \xrightarrow{a.s.} \mu_Z.$$

This implies, almost surely, that

$$\frac{W_n}{\mu_X} \simeq N_n^W + o(n), \quad \frac{B_n}{\mu_Z} \simeq N_n^B + o(n). \tag{34}$$

But $N_n^B + N_n^W = n$, then, by equation (34), we conclude

$$\frac{W_n}{\mu_X} + \frac{B_n}{\mu_Z} \stackrel{a.s}{=} n + o(n), \quad (n \to +\infty).$$

By (33), we conclude that:

$$\frac{W_n^\rho}{B_n} \xrightarrow{a.s} \frac{W^\rho}{B}. \tag{35}$$

Then from (35) and Lemma (4) we deduce $1-a$).
To prove $1-b$) observe from (34) and $1-a$) that

$$N_n = n + o(n), \qquad N_n^B = \frac{D}{\mu_Z} n^\rho + o(n^\rho). \tag{36}$$

For $n \geq 1$, we can decompose W_n as

$$\frac{W_n - n\mu_X}{\sqrt{n}} = \frac{W_0}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu_X) - \frac{1}{\sqrt{n}} \sum_{j=1}^{N_n^B} X_{\tau_j^B}.$$

By the central limit Theorem we have

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu_X) \xrightarrow{\mathcal{L}} N(0, \sigma_X^2).$$

In the other hand using (36) and the strong law of large numbers

$$if\ \rho < \frac{1}{2},\ \frac{1}{\sqrt{n}} \sum_{j=1}^{N_n^B} X_{\tau_j^B} \ = \ \frac{N_n^B}{\sqrt{n}} \frac{1}{N_n^B} \sum_{j=1}^{N_n^B} X_{\tau_j^B} \xrightarrow{a.s} 0 \times \mu_X = 0$$

$$if\ \rho = \frac{1}{2},\ \frac{D}{\mu_X}\mu_Z - \frac{1}{\sqrt{n}} \sum_{j=1}^{N_n^B} X_{\tau_j^B} \ = \ \frac{D}{\mu_X}\mu_Z - \frac{N_n^B}{\sqrt{n}} \frac{1}{N_n^B} \sum_{j=1}^{N_n^B} X_{\tau_j^B} \xrightarrow{a.s.} 0$$

We conclude $1-b)$ using Slutsky's Theorem.
In the case $1-c)$ observe that

$$\frac{W_n - n\mu_X}{n^\rho} = \frac{W_0}{n^\rho} + \frac{1}{n^\rho}\sum_{k=1}^{n}(X_k - \mu_X) - \frac{1}{n^\rho}\sum_{j=1}^{N_n^B} X_{\tau_j^B}.$$

from (20), $\frac{1}{n^\rho}\sum_{k=1}^{n}(X_k - \mu_X) \xrightarrow{L^2} 0$, and by (36), $\frac{1}{n^\rho}\sum_{j=1}^{N_n^B} X_{\tau_j^B} \xrightarrow{a.s.} -\frac{D\mu_X}{\mu_Z}$.
Proof of claim 2: Since $\tau_n \to +\infty$ when $n \to +\infty$, we deduce from (33), that

$$\frac{W_n}{B_n} \xrightarrow{a.s.} \frac{W}{B}. \tag{37}$$

Using the relation $N_n^W + N_n^B = n$ and that $N_n^B \to +\infty$, $N_n^W \to +\infty$, we deduce from (37) that, almost surely

$$\begin{aligned} N_n^B &= \frac{B}{B+W}n + o(n) \\ N_n^W &= \frac{W}{B+W}n + o(n), \end{aligned}$$

wich concludes the result of claim 2).

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