Multicriteria Decision Making

Goal Programming

Multicriteria Decision Problems

- Goal Programming
- Goal Programming: Formulation and Graphical Solution

Goal Programming

- Goal programming may be used to solve linear programs with multiple objectives, with each objective viewed as a "goal".
- In goal programming, d_i^+ and d_i^- , deviation variables, are the amounts a targeted goal i is overachieved or underachieved, respectively.
- The goals themselves are added to the constraint set with d_i^+ and d_i^- acting as the surplus and slack variables.
- One approach to goal programming is to satisfy goals in a priority sequence. Second-priority goals are pursued without reducing the first-priority goals, etc.

Goal Programming

- For each priority level, the objective function is to minimize the (weighted) sum of the goal deviations.
- Previous "optimal" achievements of goals are added to the constraint set so that they are not degraded while trying to achieve lesser priority goals.

Goal Programming Approach

Step 1: Decide the priority level of each goal.

Step 2: Decide the weight on each goal.

If a priority level has more than one goal, for each goal *i* decide the weight, w_i , to be placed on the deviation(s), d_i^+ and/or d_i^-, from the goal.

Step 3: Set up the initial linear program.

Min $w_1d_1^+ + w_2d_2^-$
s.t. Functional Constraints,
and Goal Constraints

Step 4: Solve the current linear program.

If there is a lower priority level, go to step 5. Otherwise, a final solution has been reached.

Goal Programming Approach

Step 5: Set up the new linear program.

Consider the next-lower priority level goals and formulate a new objective function based on these goals. Add a constraint requiring the achievement of the next-higher priority level goals to be maintained. The new linear program might be:

Min $w_3d_3^+ + w_4d_4^-$
s.t. Functional Constraints,
Goal Constraints, and
$w_1d_1^+ + w_2d_2^- = k$

Go to step 4. (Repeat steps 4 and 5 until

Example: Conceptual Products

Conceptual Products is a computer company that produces the CP400 and the CP500 computers. The computers use different mother boards produced in abundant supply by the company, but use the same cases and disk drives. The CP400 models use two floppy disk drives and no zip disk drives whereas the CP500 models use one floppy disk drive and one zip disk drive.

Example: Conceptual Products

The disk drives and cases are bought from vendors. There are 1000 floppy disk drives, 500 zip disk drives, and 600 cases available to Conceptual Products on a weekly basis. It takes one hour to manufacture a CP400 and its profit is $200 and it takes one and one-half hours to manufacture a CP500 and its profit is $500.

Example: Conceptual Products

The company has four goals which are given below:

Priority 1: Meet a state contract of 200 CP400 machines weekly. (Goal 1)

Priority 2: Make at least 500 total computers weekly. (Goal 2)

Priority 3: Make at least $250,000 weekly. (Goal 3)

Priority 4: Use no more than 400 man-hours per week. (Goal 4)

Example: Conceptual Products

- Variables

  x_1 = number of CP400 computers produced weekly

  x_2 = number of CP500 computers produced weekly

  d_i^- = amount the right hand side of goal i is deficient

  d_i^+ = amount the right hand side of goal i is exceeded

- Functional Constraints

  Availability of floppy disk drives: $2x_1 + x_2 \leq 1000$

  Availability of zip disk drives: $x_2 \leq 500$

  Availability of cases: $x_1 + x_2 \leq 600$

Example: Conceptual Products

- Goals

  (1) 200 CP400 computers weekly:

  $$x_1 + d_1^- - d_1^+ = 200$$

  (2) 500 total computers weekly:

  $$x_1 + x_2 + d_2^- - d_2^+ = 500$$

  (3) $250(in thousands) profit:

  $$.2x_1 + .5x_2 + d_3^- - d_3^+ = 250$$

  (4) 400 total man-hours weekly:

  $$x_1 + 1.5x_2 + d_4^- - d_4^+ = 400$$

  Non-negativity:

  $$x_1, x_2, d_i^-, d_i^+ \geq 0 \text{ for all } i$$

Example: Conceptual Products

- Objective Functions

  Priority 1: Minimize the amount the state contract is not met: Min d_1^-

  Priority 2: Minimize the number under 500 computers produced weekly: Min d_2^-

  Priority 3: Minimize the amount under $250,000 earned weekly: Min d_3^-

  Priority 4: Minimize the man-hours over 400 used weekly: Min d_4^+

Example: Conceptual Products

- Formulation Summary

$$
\begin{array}{llllllll}
\text{Min} & \multicolumn{7}{l}{P_1(d_1^-) + P_2(d_2^-) + P_3(d_3^-) + P_4(d_4^+)} \\
\text{s.t.} & 2x_1 & +x_2 & & & & & \le 1000 \\
 & & +x_2 & & & & & \le 500 \\
 & x_1 & +x_2 & & & & & \le 600 \\
 & x_1 & & +d_1^- - d_1^+ & & & & = 200 \\
 & x_1 & +x_2 & & +d_2^- - d_2^+ & & & = 500 \\
 & .2x_1 + & .5x_2 & & & +d_3^- - d_3^+ & & = 250 \\
 & x_1 + 1.5x_2 & & & & & +d_4^- - d_4^+ & = 400 \\
 & \multicolumn{7}{l}{x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+ \ge 0}
\end{array}
$$

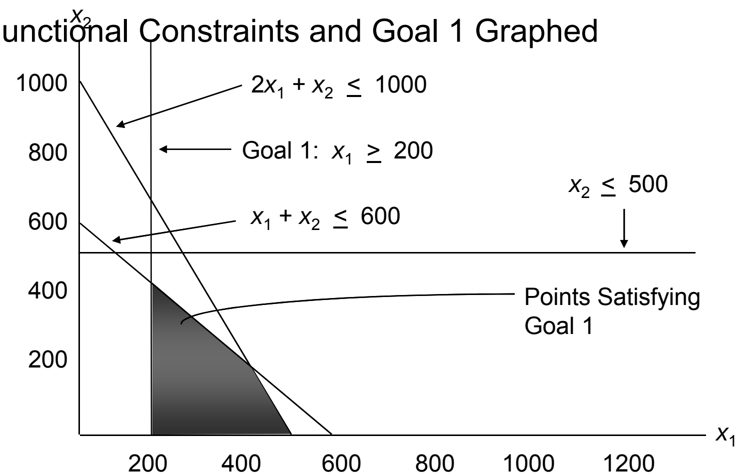
Example: Conceptual Products

- Graphical Solution, Iteration 1

  To solve graphically, first graph the functional constraints. Then graph the first goal: x_1 = 200. Note on the next slide that there is a set of points that exceed x_1 = 200 (where d_1^- = 0).

Example: Conceptual Products

- Functional Constraints and Goal 1 Graphed



Example: Conceptual Products

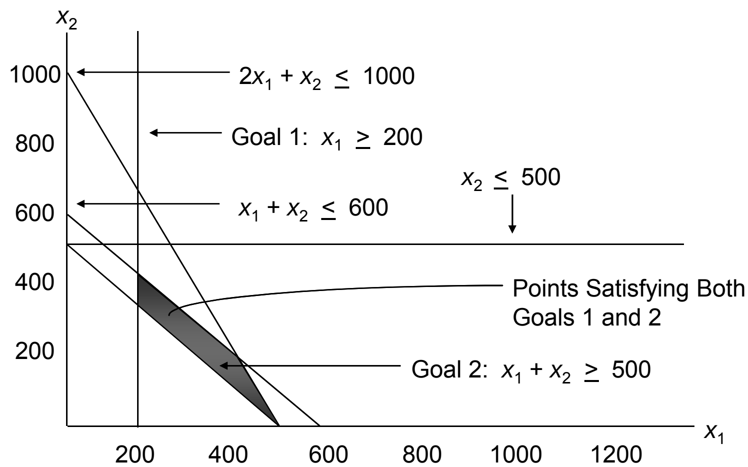
- Graphical Solution, Iteration 2

  Now add Goal 1 as $x_1 \geq 200$ and graph Goal 2:

  $x_1 + x_2 = 500$. Note on the next slide that there is still a set of points satisfying the first goal that also satisfies this second goal (where $d_2^- = 0$).

Example: Conceptual Products

- Goal 1 (Constraint) and Goal 2 Graphed



Example: Conceptual Products

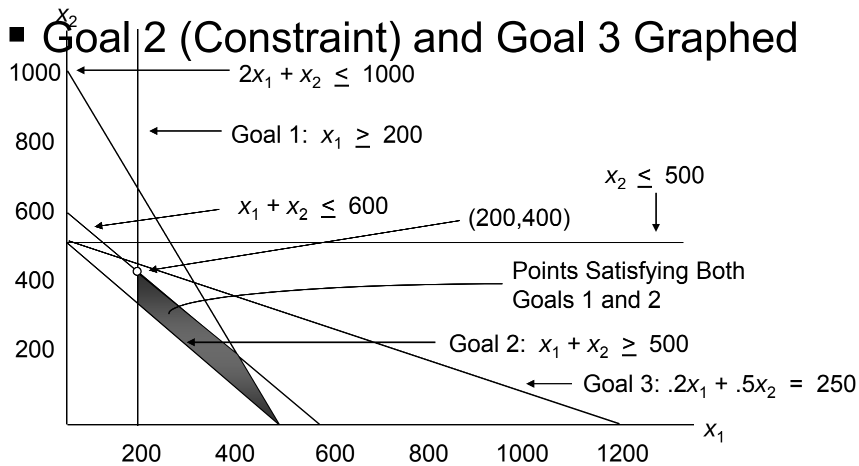
- Graphical Solution, Iteration 3

  Now add Goal 2 as $x_1 + x_2 \geq 500$ and Goal 3: $.2x_1 + .5x_2 = 250$. Note on the next slide that no points satisfy the previous functional constraints and goals <u>and</u> satisfy this constraint.

  Thus, to Min d_3^-, this minimum value is achieved when we Max $.2x_1 + .5x_2$. Note that this occurs at $x_1 = 200$ and $x_2 = 400$, so that $.2x_1 + .5x_2 = 240$ or $d_3^- = 10$.

Example: Conceptual Products

- Goal 2 (Constraint) and Goal 3 Graphed



Objective Function and Constraints

Maximize $Z = \$40\, x_1 + 50\, x_2$

Subject to

$$x_1 + 2x_2 \le 40 \text{ hr} \quad \text{(labor constraint)}$$
$$4x_1 + 3x_2 \le 120 \text{ lb} \quad \text{(clay constraint)}$$
$$x_1, x_2 \ge 0$$

Decision variables
***x*1 = number of bowls to produce**
***x*2 = number of mugs to produce**

Goals

Instead of having one objective, the pottery company has several objectives that are listed ***in order of importance***:

- To avoid layoffs, the company does not want to use fewer than 40 hours of labor per day;
- The company would like to achieve a satisfactory profit level of $1,600 per day;
- Because the clay must de stored in a special place so that it does not dry out, the company prefers not to keep more than 120 pounds on hand per day;
- Because high overhead costs results when the plant is kept open past normal hours, the company would like to minimize the amount of overtime.

Labor Goal

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40\,hr$$

X_1= 5 ; x_2=10 25 hours are used in production Labor is underutilized by 15 hrs	X_1= 10 ; x_2=20 50 hours are used in production The extra 10 hrs is overtime
$x_1 + 2x_2 + d_1^- - d_1^+ = 40\,hr$ $(5) + 2(10) + d_1^- - d_1^+ = 40\,hr$ $25 + d_1^- - d_1^+ = 40\,hr$ *If* $d_1^- = 15$ *and* $d_1^+ = 0$ *then* $25 + 15 - 0 = 40\,hr$	$x_1 + 2x_2 + d_1^- - d_1^+ = 40\,hr$ $(10) + 2(20) + d_1^- - d_1^+ = 40\,hr$ $50 + d_1^- - d_1^+ = 40\,hr$ *Thus* $d_1^- = 0$ *and* $d_1^+ = 10$ $(50 + 0 - 10 = 40\,hr)$

$$Minimize \quad P_1 d_1^- \;,\; P_4 d_1^+$$

Profit Goal

$$40x_1 + 50x_2 + d_2^- - d_2^+ = \$1{,}600$$

$$Minimize \quad P_1 d_1^-,\ P_2 d_2^-,\ P_4 d_1^+$$

Material Goal

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120\,lb$$

$$Minimize \quad P_1 d_1^-,\ P_2 d_2^-,\ P_3 d_3^+,\ P_4 d_1^+$$

The model

$$Minimize \quad P_1 d_1^-,\ P_2 d_2^-,\ P_3\ d_3^+, P_4 d_1^+$$

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40hr$$

$$40x_1 + 50x_2 + d_2^- - d_2^+ = \$1{,}600$$

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120\,lb$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Nonlinear Programming

Nonlinear profit analysis

- Breakeven analysis

$Z = v\,p - c_f - v\,c_v$

Where

v = Sales volume (i.e. demand)

p = price

c_f = fixed cost

c_v = variable cost

Nonlinear profit analysis

$$v = 1500 - 24.6\,p$$

- Breakeven analysis

$Z = (1500\text{-}24.6\,p)\,p - c_f - (1500\text{-}24.6\,p)\,c_v$

$\quad = 1500\,p - 24.6\,p^2 - c_f - 1500\,c_v + 24.6\,p\,c_v$

If $c_f = \$10000$ and $c_v = \$58$

$Z = 1696.8\,p - 24.6\,p^2 - 22000$

Nonlinear profit analysis (cont'd)

$$Z = 1696.8\,p - 24.6\,p^2 - 22000$$

$\partial Z / \partial p = 1696.8 - 24.6\,(2)\,p$

$\partial Z / \partial p = 0 \leftrightarrow 1696.8 - 49.2\,p = 0$

$p = \$34.49$

The optimal volumes of jeans is:

$v = 1500 - 24.6\,p = 1500 - 24.6\,(34.49)$

$= 651.6$ pairs of denim jeans

The maximum total profit is:

$Z = 1696.8\,(34.49) - 24.6\,(34.49)^2 - 22000$

$= \$7259.45$

Microsoft Excel - EXH10-3 [Read-Only]

C3 =C4*C5-C6-(C4)*C7

	A	B	C	D	E
1	**Western Clothing Company**				
2					
3		*Profit =*	7259		
4		*Demand =*	651.6		
5		*Price =*	34.49		
6		*Fixed cost =*	10000		
7		*Variable cost =*	8		
8					

$$\text{Max } (4-0.1X_1)X_1+(5-0.2X_2)X_2$$
$$\text{s.t. } X_1+2X_2 = 40$$

Microsoft Excel - EXH10-6 [Read-Only]

C11 =SUMPRODUCT(C5:C6,D5:D6)

	A	B	C	D	E	F	G
1	**Beaver Creek Pottery Company**						
2							
3				*Profit*			
4			*Production*	*per Unit*			
5		*Bowls =*	18.3	2.17			
6		*Mugs =*	10.8	2.83			
7							
8			*Used*	*Available*			
9		*Labor hours:*	40.00	40			
10							
11		*Total profit =*	70.42				
12							

Constrained Optimization

x1=1500 - 24.6 p1 (designer jeans)

x2=2700 – 63.8 p2 (straight-leg jeans)

Max Z=(p1-12) x1 + (p2-9) x2

s.t.

Cloth: 2 x1 + 2.7 x2 ≤ 6000 yards

Cutting: 3.6 x1 + 2.9 x2 ≤ 8500 minutes

Sewing: 7.2 x1 + 8.5 x2 ≤ 15000 minutes

X1,x2,p1,p2 ≥ 0

Microsoft Excel - EXH10-11 [Read-Only]

E7 =SUMPRODUCT(C5:C6,E5:E6)

Western Clothing Company

	Demand	Price	Profit
Designer jeans:	602.4	36.49	24.49
Straight-leg jeans:	1062.9	25.66	16.66
		Total=	32459.2

Resource constraints:

Resource	*Used*	*Available*
Cloth:	4074.63	6000
Cutting time:	5251.05	8500
Sewing time:	13371.9	15000

Facility Location Example

Compute load-distance

$$\text{Min} \sum_{i=1}^{n} t_i d_i$$

t_i = annual trips to town i

d_i = distance to town i

$$d = \sqrt{(x_i - x)^2 + (y_i - y)^2}$$

Load-Distance Example

Town	Coordinates X	Y	Annual Trips
Abbeville	**20**	**20**	**75**
Benton	**10**	**35**	**105**
Clayton	**25**	**9**	**135**
Dunning	**32**	**15**	**60**
Eden	**10**	**8**	**90**

Distribution Center Location

X	Y
21.72	**14.14**

Compute distance from each site to each supplier

Abbeville (A)

$$d_A = \sqrt{(x_A - x_1)^2 + (y_A - y_1)^2} = \sqrt{(20-21.72)^2 + (20-14.14)^2} = 6.11$$

$d_B = 23.93$ $\quad$ $d_D = 1.32$

$d_C = 6.09$ $\quad$ $d_E = 6.22$

Microsoft Excel - EXH10-12 [Read-Only]

E6 =SQRT((B6-C14)^2+(C6-C15)^2)

Clayton County Rescue Squad

Town	*Coordinates* x	y	*Annual Trips*	*Distance*
Abbeville	20	20	75	6.11
Benton	10	35	105	23.93
Clayton	25	9	135	6.09
Dunning	32	15	60	1.32
Eden	10	8	90	6.22

Rescue Squad Facility Location:

x =	21.72
y =	14.14

Total Annual Distance = 4432.53

Investment Portfolio Selection

$$S = x_1^2 s_1^2 + x_2^2 s_2^2 + \ldots + x_n^2 s_n^2 + \sum_{i \neq j} x_i x_j r_{ij} s_i s_j$$

x_i, x_j = the proportion of money invested in investments i or j

s_i^2 = the variance for investment i

r_{ij} = the covariance (or correlation) between returns on investments i and j

s_i, s_j = the standard deviation of returns for investments i and j

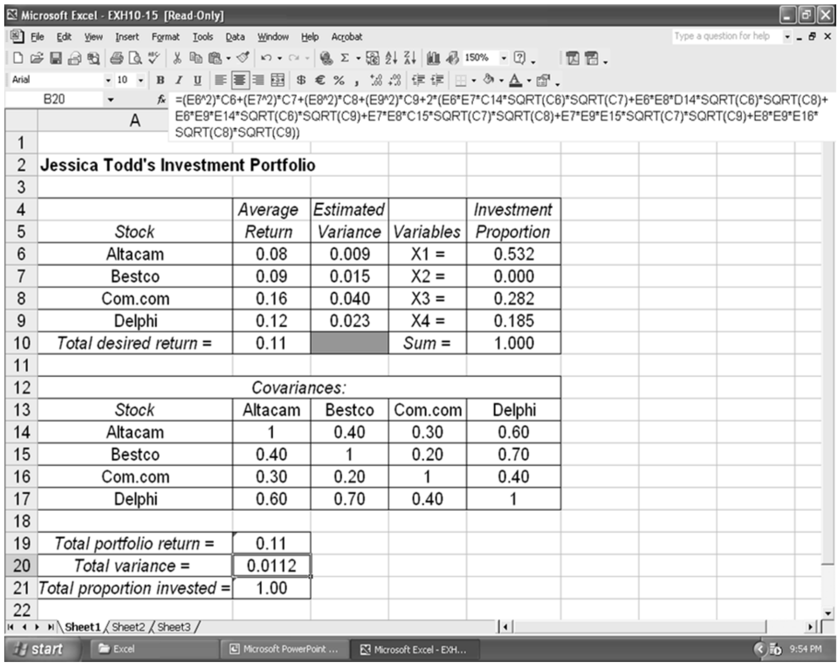
$$r_1 x_1 + r_2 x_2 + \ldots + r_n x_n \geq r_m$$

r_i = expected annual return in our investment i

x_i = proportion (or fraction) of money invested in investment i

r_m = the minimum desired annual return from the portfolio

$$x_1 + x_2 + \ldots + x_n = 1.0$$



Jessica Todd's Investment Portfolio

Stock	Average Return	Estimated Variance	Variables	Investment Proportion
Altacam	0.08	0.009	X1 =	0.532
Bestco	0.09	0.015	X2 =	0.000
Com.com	0.16	0.040	X3 =	0.282
Delphi	0.12	0.023	X4 =	0.185
Total desired return =	0.11		*Sum =*	1.000

Covariances:

Stock	Altacam	Bestco	Com.com	Delphi
Altacam	1	0.40	0.30	0.60
Bestco	0.40	1	0.20	0.70
Com.com	0.30	0.20	1	0.40
Delphi	0.60	0.70	0.40	1

Total portfolio return =	0.11
Total variance =	0.0112
Total proportion invested =	1.00