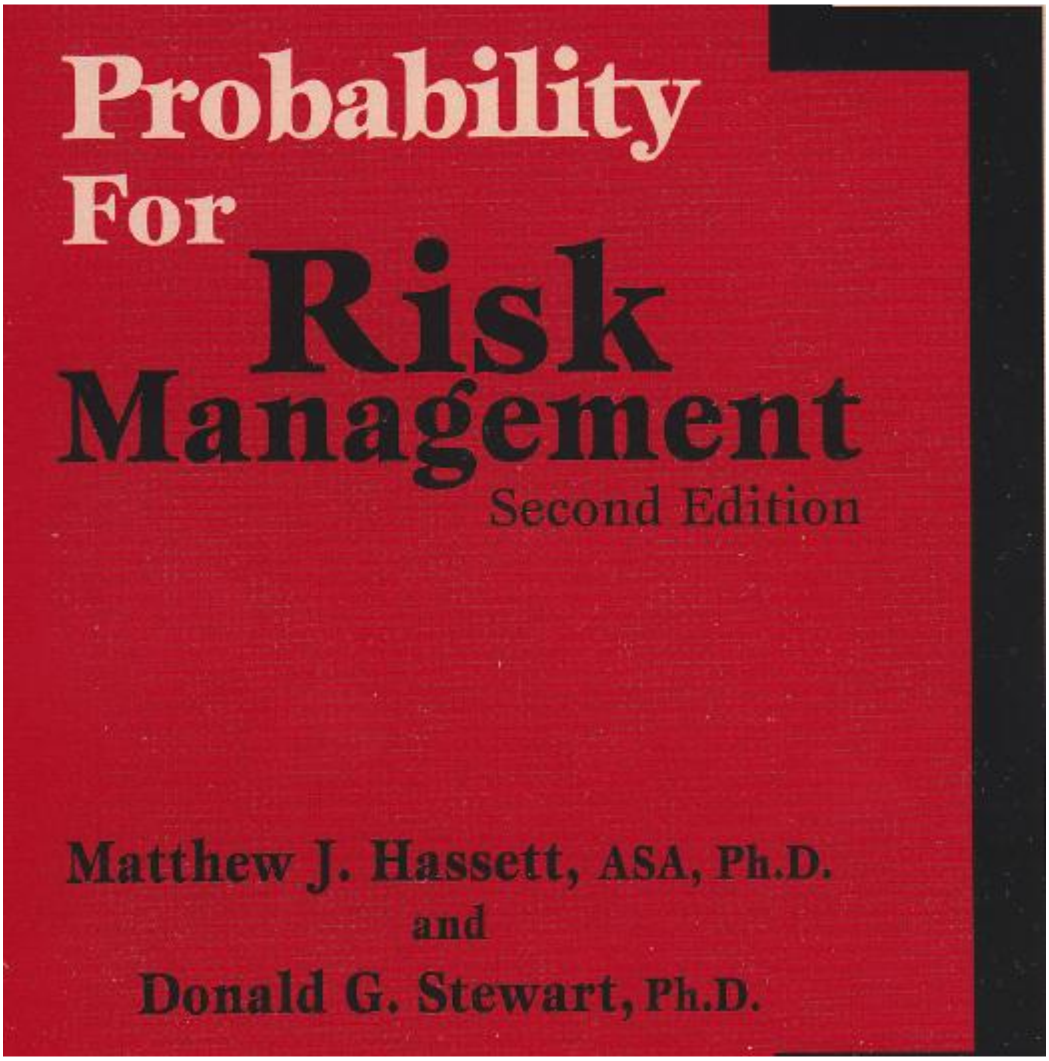


Chapter 1: Probability: A Tool for Risk Management 1

Chapter 2: Counting for Probability 7

2.6 Exercises

2.2 The Language of Probability; Sets, Sample Spaces and Events

2-1. From a standard deck of cards a single card is drawn. Let E be the event that the card is a red face card. List the outcomes in the event E.

2-2. An insurance company insures buildings against loss due to fire.
(a) What is the sample space of the amount of loss?
(b) What is the event that the amount of loss is strictly between \$1,000 and \$1,000,000 (i.e., the amount x is in the open interval (1,000, 1,000,000))?

2-3. An urn contains balls numbered from 1 to 25. A ball is selected and its number noted.
(a) What is the sample space for this experiment?
(b) If E is the event that the number is odd, what are the outcomes in E?

2-4. An experiment consists of rolling a pair of fair dice, one red and one green. An outcome is an ordered pair (r, g), where r is the number on the red die and g is the number on the green die. List all outcomes of this experiment.

2-5. Two dice are rolled. How many outcomes have a sum of (a) 7; (b) 8; (c) 11; (d) 7 *or* 11?

2-6. Suppose a family has 3 children. List all possible outcomes for the sequence of births by sex in this family.

2.3 Compound Events; Set Notation

2-7. Let S be the sample space for drawing a ball from an urn containing balls numbered from 1 to 25, and E be the event the number is odd. What are the outcomes in $\sim E$?

2-8. In the sample space for drawing a card from a standard deck, let A be the event the card is a face card and B be the event the card is a club. List all the outcomes in $A \cap B$.

2-9. Consider the insurance company that insures against loss due to fire. Let A be the event the loss is strictly between \$1,000 and \$100,000, and B be the event the loss is strictly between \$50,000 and \$500,000. What are the events in $A \cup B$ and $A \cap B$?

2-10. An experiment consists of tossing a coin and then rolling a die. An outcome is an ordered pair, such as $(H, 3)$. Let A be the event the coin shows heads and B be the event the number on the die is greater than 2. What is $A \cap B$?

2-11. In the experiment of tossing two dice, let E be the event the sum of the dice is 6 and F be the event both dice show the same number. List the outcomes in the events $E \cup F$ and $E \cap F$.

2-12. In the sample space for the family with three children in Exercise 2-6, let E be the event that the oldest child is a girl and F the event that the middle child is a boy. List the outcomes in E, F, $E \cup F$ and $E \cap F$.

2.4 Set Identities

2-13. Verify the two distributive laws by drawing the appropriate Venn diagrams.

2-14. Verify De Morgan's laws by drawing the appropriate Venn diagrams.

2-15. Let M be the set of students in a large university who are taking a mathematics class and E be the set taking an economics class.

(a) Give a verbal statement of the identity

$$\sim(M \cup E) = \sim M \cap \sim E.$$

(b) Give a verbal statement of the identity

$$\sim(M \cap E) = \sim M \cup \sim E.$$

2.5 Counting

2-16. An insurance agent sells two types of insurance, life and health. Of his clients, 38 have life policies, 29 have health policies and 21 have both. How many clients does he have?

2-17. A company has 134 employees. There are 84 who have been with the company more than 10 years and 65 of those are college graduates. There are 23 who do not have college degrees and have been with the company less than 10 years. How many employees are college graduates?

2-18. A stockbroker has 94 clients who own either stocks or bonds. If 67 own stocks and 52 own bonds, how many own both stocks and bonds?

2-19. In a survey of 185 university students, 91 were taking a history course, 75 were taking a biology course, and 37 were taking both. How many were taking a course in exactly one of these subjects?

2-20. A broker deals in stocks, bonds and commodities. In reviewing his clients he finds that 29 own stocks, 27 own bonds, 19 own commodities, 11 own stocks and bonds, 9 own stocks and commodities, 8 own bonds and commodities, 3 own all three, and 11 have no current investments. How many clients does he have?

2-21. An insurance agent sells life, health and auto insurance. During the year she met with 85 potential clients. Of these, 42 purchased life insurance, 40 health insurance, 24 auto insurance, 14 both life and health, 9 both life and auto, 11 both health and auto, and 2 purchased all three. How many of these potential clients purchased (a) no policies; (b) only health policies; (c) exactly one type of insurance; (d) life or health but not auto insurance?

2-22. If an experiment consists of tossing a coin and then rolling a die, how many outcomes are possible?

2-23. In purchasing a car, a woman has the choice of 4 body styles, 15 color combinations, and 6 accessory packages. In how many ways can she select her car?

2-24. A student needs a course in each of history, mathematics, foreign languages and economics to graduate. In looking at the class schedule he sees he can choose from 7 history classes, 8 mathematics classes, 4 foreign language classes and 7 economics classes. In how many ways can he select the four classes he needs to graduate?

2-25. An experiment has two stages. The first stage consists of drawing a card from a standard deck. If the card is red, the second stage consists of tossing a coin. If the card is black, the second stage consists of rolling a die. How many outcomes are possible?

2-26. Let X be the n-element set $\{x_1, x_2, \ldots, x_n\}$. Show that the number of subsets of X, including X and $\emptyset$, is 2^n. (Hint: For each subset A of X, define the sequence $(a_1, a_2, \ldots, a_n)$ such that $a_i = 1$ if $x_i \in A$ and 0 otherwise. Then count the number of sequences).

2-27. An arrangement of 4 letters from the set $\{A, B, C, D, E, F\}$ is called a (four-letter) word from that set. How many four-letter words are possible if repetitions are allowed? How many four-letter words are possible if repetitions are not allowed?

2-28. Suppose any 7-digit number whose first digit is neither 0 nor 1 can be used as a telephone number. How many phone numbers are possible if repetitions are allowed? How many are possible if repetitions are not allowed?

2-29. A row contains 12 chairs. In how many ways can 7 people be seated in these chairs?

2-30. At the beginning of the basketball season a sportswriter is asked to rank the top 4 teams of the 10 teams in the PAC-10 conference. How many different rankings are possible?

2-31. A club with 30 members has three officers: president, secretary and treasurer. In how many ways can these offices be filled?

2-32. The speaker's table at a banquet has 10 chairs in a row. Of the ten people to be seated at the table, 4 are left-handed and 6 are right-handed. To avoid elbowing each other while eating, the left-handed people are seated in the 4 chairs on the left. In how many ways can these 10 people be seated?

2-33. Eight people are to be seated in a row of eight chairs. In how many ways can these people be seated if two of them insist on sitting next to each other?

2-34. A club with 30 members wants to have a 3-person governing board. In how many ways can this board be chosen? (Compare with Exercise 2-31.)

2-35. How many 5-card (poker) hands are possible from a deck of 52 cards?

2-36. How many of those poker hands consist of (a) all hearts; (b) all cards in the same suit; (c) 2 aces, 2 kings and 1 jack?

2-37. In a class of 15 boys and 13 girls, the teacher wants a cast of 4 boys and 5 girls for a play. In how many ways can she select the cast?

2-38. The Power Ball lottery uses two sets of balls, a set of white balls numbered 1 to 55 and a set of red balls numbered 1 to 42. To play, you select 5 of the white balls and 1 red ball. In how many ways can you make your selection?

2-39. How many different ways are there to arrange the letters in the word MISSISSIPPI?

2-40. An insurance company has offices in New York, Chicago and Los Angeles. It hires 12 new actuaries and sends 5 to New York, 3 to Chicago, and 4 to Los Angeles. In how many ways can this be done?

2-41. A company has 9 analysts. It has a major project which has been divided into 3 subprojects, and it assigns 3 analysts to each task. In how ways can this be done?

2-42. Suppose that, in Exercise 2-41, the company divides the 9 analysts into 3 teams of 3 each, and each team works on the whole project. In how many ways can this be done?

2-43. Expand $(2s - t)^4$.

2-44. In the expansion of $(2u - 3v)^8$, what is the coefficient of the term involving u^5v^3?

2-45. Prove the Binomial Theorem. (Hint: How many ways can you get the term $x^{n-k}y^k$ from the product of n factors, each of which is $(x + y)$?)

2-46. Using the Binomial Theorem, give an alternate proof that the number of subsets of an n-element set is 2^n.

2.7 Sample Actuarial Examination Problem

2-47. An auto insurance company has 10,000 policyholders. Each policyholder is classified as

(i) young or old;
(ii) male or female; and
(iii) married or single.

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400young married persons. Finally, 600 of the policyholders are young married males.

How many of the company's policyholders are young, female, and single?

CHAPTER 2

2-1. The red suits are hearts (H) and diamonds (D), and the face cards kings (K), queens (Q) and jacks (J). Thus the outcomes are KH, QH, JH, KD, QD and JD.

2-2. (a) The loss can be any positive rational number.

(b) The loss is any rational number in (1,000, 1,000,000).

2-3. (a) S consists of the positive integers from 1 to 25.

(b) E consists of the odd integers from 1 to 25.

2-4. S consists of all ordered pairs (r,g) where r = 1,2,3,4,5, or 6 and g = 1,2,3,4,5, or 6.

2-5. Count the number of ordered pairs with the desired sum in the list in the answer For Exercise 2-4. For example, the only two pairs which sum to 11 are (5,6) and (6,5), so the answer to part (c) is 2.

2-6. S consist of all sequences xyz where each of x, y and z is either B or G.

2-7. The outcomes in ~E are the even (not odd) integers between 1 and 25.

2-8. The outcomes in $A \cap B$ are the club face cards.

2-9. $A \cup B$ is all rational numbers in (1,000, 500,000).

$A \cap B$ is all rational numbers in (50,000, 100,000).

2-10. $A \cap B$ requires an outcome where the coin shows a head and the number on the die is greater than 2. It consists of the ordered pairs (H,3), (H,4), (H,5) and (H,6).

2-11. E = {(1,5),(2,4),(3,3),(4,2),(5,1)}

F = {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)}

$E \cup F$ = {(1,5),(2,4),(3,3),(4,2),(5,1),(1,1),(2,2),(4,4),(5,5),(6,6)}

$E \cap F$ = {(3,3)}

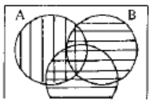
2-12. E = {GGG,GGB.GBG,GBB} F = {GBG,GBB,BBG,BBB}

$E \cup F$ = {GGG,GGB,GBG,GBB,BBG,BBB}

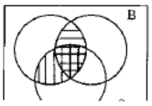
$E \cap F$ = {GBG,GBB}

2-13. To verify (2.1) draw a Venn diagram for $A \cap (B \cup C)$ and a second one showing $(A \cap B)$ and $(A \cap C)$ and observe that the first is the union of the second and the third.

$A \cap (B \cup C)$

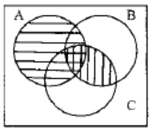


$(A \cap B) \cup (A \cap C)$

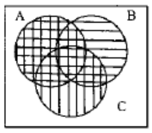


To verify (2.2) draw a Venn diagram for $A \cup (B \cap C)$ and a second one showing $(A \cup B)$ and $(A \cup C)$, and observe that first is the intersection of the second and the third.

$A \cup (B \cap C)$

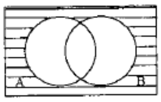


$(A \cup B) \cap (A \cup C)$

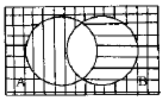


2-14. To verify (2.3) draw a Venn diagram for $\sim(A \cup B)$ and a second one showing $\sim A$ and $\sim B$, and verify that the first is the intersection of the second and the third.

$\sim(A \cup B)$

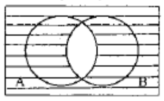


$\sim A \cap \sim B$

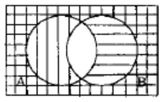


To verify (2.4) draw a Venn diagram for $\sim(A \cap B)$ and a second one showing $\sim A$ and $\sim B$, and verify that the first is the union of the second and the third.

$\sim(A \cap B)$



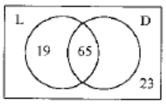
$\sim A \cup \sim B$



2-15. See verbal statements in Answers to the Exercises.

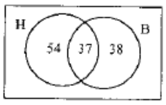
2-16. Let H be the set of those with health insurance and L be the set of those with life insurance. Then $(L \cup H) = 38 + 29 - 21 = 46$.

2-17. Let L be the set of those with the company more than 10 years and D be the set of those with college degrees. The given data can be used to fill in the Venn diagram below. Then $n(L \cup D) = 134 - 23 = 111$. Then $n(D) = n(L \cup D) - 19 = 111 - 19 = 92$.

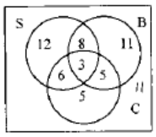


2-18 Let S be the set of those who own stocks and B be the set of those who own bonds. So $n(S \cap B) = 67 + 52 - 94 = 25$.

2-19. The given data can be used to fill in the Venn diagram below. The number of students taking exactly one of these courses is $54 + 38 = 92$.



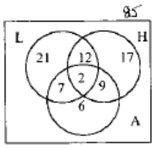
2-20. Using the given information we get the following Venn diagram.



The total number of clients is 61.

2-21. Using the given information we get the following Venn diagram.

85



(a) The number with no policies is $85 - 74 = 11$.
(b) The number with only health policies is 17.
(c) The number with exactly one policy is $21 + 17 + 6 = 44$.
(d) The number with life or health but not auto is
$21 + 12 + 17 = 50$.

2-22. Number of outcomes is $2 \cdot 6 = 12$.

2-23. Number of ways to select car is $4 \cdot 15 \cdot 6 = 360$.

2-24. Number of ways to select classes is $7 \cdot 8 \cdot 4 \cdot 7 = 1568$.

2-25. Number of outcomes is $26 \cdot 2 + 26 \cdot 6 = 208$.

2-26. Each subset corresponds with exactly one sequence. For example the subset $\{x_1, x_2\}$ corresponds with the sequence $(1,1,0,0,\ldots,0)$. There are 2^n sequences, so there are 2^n subsets.

2-27. If repetitions are allowed, the number of words is $6^4 = 1296$.

If repetitions are not allowed, the number is $P(6,4) = 360$.

2-28. If repetitions are allowed there are $8 \cdot 10^6$ possible numbers.

If repetitions are not allowed there are
$8 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 483{,}840$ numbers.

2-29. The number of seating arrangements (in order) is $P(12,7) = 3{,}991{,}280$.

2-30. The number of rankings (in order) is $P(10,4) = 5040$.

2-31. The number of ways to fill the offices is $P(30,3) = 24{,}360$.

2-32. The number of seating arrangements is $P(4,4)P(6,6) = 17{,}280$. (Arrange the left-handers in the first 4 seats, and then arrange the right-handers in the remaining 6 seats.)

2-33. First choose a pair of adjacent seats, which can be done in 7 ways, and the two people can be seated in those seats in 2 ways. Then arrange the 6 remaining people in remaining 6 chairs. This can be done in 6! ways, so the number of seating arrangements is $7 \cdot 2 \cdot 6! = 10{,}080$.

2-34. The number of possible committees is $C(30,3) = 4060$. (Order is irrelevant.)

2-35. The number of hands is $C(52,5) = 2{,}598{,}960$. (Order is irrelevant.)

2-36. (a) Number of hands with all hearts is $C(13,5) = 1287$.

(b) Number of hands with all same suit is $4C(13,5) = 5148$. (Choose one of the 4 suits and then choose 5 cards.)

(c) Number of hands with (AAKKJ) is $C(4,2)C(4,2) \cdot 4 = 144$. (Choose 2 aces and then 2 kings and then 1 jack.)

2-37. Number of ways to pick cast is $C(15,4)C(13,5) = 1{,}756{,}755$. (Choose 4 boys and then 5 girls.)

2-38. Number of ways to select balls is $C(29,5) \cdot 42 = 80{,}089{,}128$.

2-39. Number of distinguishable arrangements of MISSISSIPPI is $11!/(4!4!2!) = 34{,}650$. (There are 11 slots to fill with the 11 letters. First select 4 slots for the 4 I's, and then 4 slots for the 4 S's, and 2 slots for the 2 P's, leaving one slot left for the M.)

2-40. Number of ways to assign the actuaries is $12!/(5!3!4!) = 27{,}720$, since this constitutes a partition of the group.

2-41. Number of ways to assign the analysts is $9!/(3!)^3 = 1680$, since this constitutes a partition of the group.

2-42. In this case the 3 teams are not distinguishable by task. If the 9 analysts are divided into 3 three-man teams, A, B and C, and have distinguishable tasks, these teams can be assigned to the 3 tasks in $3! = 6$ ways. But if the tasks are all the same, those 6 partitions constitute a single division into teams. Hence the divisions into teams is the number of partitions divided by 6 which is $1680/6 = 280$.

2-43. $(2s - t)^4 = (2s)^4 + 4(2s)^3(-t) + 6(2s)^2(-t)^2 + 4(2s)(-t)^3 + (-t)^4$

2-44. Term is $C(8,3)(2u)^5(-3v)^3$, and coefficient is $-48{,}384$.

2-45. An $x^{n-k}y^k$ is obtained by selecting k of the $(x+y)$ factors from which to take y and taking x from the remaining n–k factors. This can be done in $C(n,k)$ ways.

2-46. The number of subsets of size k from a set of size n is $C(n,k)$. Then $2^n = (1 + 1)^n = \Sigma C(n,k)$ = the total number of subsets.

Answers to the Exercises

CHAPTER 2

2-1. KH, QH, JH, KD, QD, JD

2-2. (a) $S = \{x|x > 0 \text{ and } x \text{ rational}\}$
(b) $E = \{x|\ 1{,}000 < x < 1{,}000{,}000 \text{ and } x \text{ rational}\}$

2-3. (a) $S = \{1, 2, 3, \ldots, 25\}$ (b) $E = \{1, 3, 5, \ldots, 25\}$

2-4. $(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)$

2-5. (a) 6 (b) 5 (c) 2 (d) 8

2-6. BBB, BBG, BGB, BGG, GBB, GBG, GGB,GGG

2-7. $\sim E = \{2, 4, 6, \ldots, 24\}$

2-8. KC, QC, JC

2-9. $A \cup B = \{x|1{,}000 < x < 500{,}000 \text{ and } x \text{ rational}\}$,
$A \cap B = \{x|50{,}000 < x < 100{,}000 \text{ and } x \text{ rational}\}$

2-10. (H, 3), (H, 4), (H, 5), (H, 6)

2-11. $E \cup F$
$= \{(1,5),(2,4),(3,3),(4,2),(5,1),(1,1),(2,2),(4,4),(5,5),(6,6)\}$
$E \cap F = \{(3,3)\}$

2-12. $E = \{\text{GGG}, \text{GGB}, \text{GBG}, \text{GBB}\}$, $F = \{\text{GBG}, \text{GBB}, \text{BBG}, \text{BBB}\}$,
$E \cup F = \{\text{GGG}, \text{GGB}, \text{GBG}, \text{GBB}, \text{BBG}, \text{BBB}\}$,
$E \cap F = \{\text{GBG}, \text{GBB}\}$

2-15. (a) "You are not taking either a mathematics course or an economics course" is equivalent to "you are not taking a mathematics course *and* you are not taking an economics course."
(b) "You are not taking both a mathematics course and an economics course" is equivalent to "you are either not taking a mathematics course or you are not taking an economics course."

2-16. 46

2-17. 92

2-18. 25

2-19. 92

2-20. 61

2-21. (a) 11 (b) 17 (c) 44 (d) 50

2-22. 12

2-23. 360

2-24. 1568

2-25. 208

2-27. 1296; 360

2-28. 8,000,000; 483,840

2-29. 3,991,680

2-30. 5040

2-31. 24,360

2-32. 17,280

2-33. 10,080

2-34. 4,060

2-35. 2,598,960

2-36. (a) 1,287 (b) 5,148 (c) 144

2-37. 1,756,755

2-38. 146,107,962

2-39. 34,650

2-40. 27,720

2-41. 1,680

2-42. 280

2-43. $16s^4 - 32s^3t + 24s^2t^2 - 8st^3 + t^4$

2-44. $-48,384$

2-47. 880

Chapter 3
Elements of Probability

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3.6 Exercises

3.1 Probability by Counting for Equally Likely Outcomes

3-1. You toss a fair coin 3 times. What is the probability that you get 2 heads and 1 tail? (Note: All possible outcomes for this experiment were given in a tree in Section 2.5.3.)

3-2. If a fair coin is tossed 3 times what is the probability of getting at least 1 head?

3-3 An urn contains 3 red balls, 7 green balls and 6 blue balls. If a ball is selected at random from the urn, what is the probability that it is (a) red; (b) not green?

3-4. A consulting company has 68 employees. Of these 21 have degrees in mathematics, 33 have degrees in economics and 7 have degrees in both. What is the probability that an employee chosen at random has a degree in either mathematics or economics?

3-5. If a pair of dice is rolled, what is the probability that the sum of the two dice is (a) 7; (b) 11; (c) less than 5?

3-6. An insurance agent has 78 clients. Of these 45 have life insurance, 32 have auto insurance, and 16 have both types. What is the probability that a client chosen at random has neither life nor auto insurance?

3-7. An urn contains 4 red balls and 6 green balls. Three balls are selected at random. What is the probability (a) all 3 are red; (b) 1 is red and 2 are green; (c) all 3 are the same color?

3-8. A computer company has a shipment of 40 computer components of which 5 are defective. If 4 components are chosen at random to be tested, what is the probability that (a) all are good; (b) 2 are good and 2 are defective?

3-9. Ten people, 5 men and 5 women, are to be seated in a row of ten chairs. What is the probability that the men and women end up in alternate chairs?

3-10. 8 people were all born in January. What is the probability that at least 2 of them have the same birthday?

3-11. What is the probability that at least 2 of a group of 4 people were born on the same day of the week?

3-12. 4 balls are picked at random from an urn containing 5 red balls and 6 blue balls. What is the probability that you get balls of both colors?

3-13. A 5-card poker hand is dealt from a standard deck of cards. What is the probability that you get a full house (3 of one kind plus a different pair, such as KKK55) ?

3-14. If a poker hand is dealt, what is the probability that you get 2 pairs (e.g., QQ993)?

3-15. The odds for an event E are defined as the ratio $P(E)$ to $P(\sim E)$. Odds are generally written as the ratio of two integers, such as 5:4, which is read "5 to 4". The odds against E are given by the reverse ratio (i.e., 4:5). If a pair of dice are rolled, what are (a) the odds for a 7; (b) the odds against an 11?

3-16. If the odds for E are known, say $r{:}s$, then $P(E) = r/(r + s)$. If the odds against F are $a{:}b$, what is the $P(F)$?

3.2 Probability When Outcomes Are Not Equally Likely

3-17. Prove $P(\sim E) = 1 - P(E)$.

3-18. Prove $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ using the axioms in Section 3.2.2. Hint: First show that

$$(A \cup B) = (A \cap \sim B) \cup (A \cap B) \cup (\sim A \cap B).$$

3-19. A four-year college has the following enrollment by class: 27.8% freshman, 26.3% sophomore, 24.4% junior and 21.5% senior. What is the probability that a student chosen at random is a junior or senior.

3-20. An auto insurance company finds that in the past 10 years 22% of its policyholders have filed liability claims, 37% have filed comprehensive claims, and 13% have filed both types of claims. What is the probability that a policyholder chosen at random has not filed a claim of either kind?

3-21. A teacher's grade distribution for the year is as follows: A, 13.1%; B, 27.8%; C, 31.2%; D, 8.9%; E, 9.4%; and W, 9.6%. What is the probability that a student of this teacher got (a) a grade C or better; (b) a grade of D or E?

3-22. In a survey of college students it was discovered that 37% had received flu shots, 58% had a skin test for tuberculosis, and 21% had received neither. What is the probability that a student received both?

3.3 Conditional Probability

3-23. In Exercise 3-21 what is the probability that a randomly selected student got an A, given that she got a grade of C or better?

3-24. In the first quarter of a year, a company's records showed that 63.5% of its employees missed no work, 23.7% missed one day of work, 8.1% missed two days, and 4.7% missed three days. What is the probability that an employee who missed work missed only one day?

3-25. An insurance company classifies its claims as low if they are under $10,000, and high otherwise. During the year 79.2% of its policyholders filed no claims, 16.9% filed low claims, and 3.9% filed high claims. If a policyholder filed a claim, what is the probability that it was a low claim?

3-26. Two cards are drawn from a standard deck without replacement. What is the probability that (a) both are hearts; (b) neither is a heart; (c) exactly one is a heart?

3-27. For the experiment of tossing a single fair coin 3 times, what is the probability of getting exactly 2 heads, given that you get at least one head?

3-28. For the experiment in Exercise 3-27 what is the probability of getting exactly 2 heads, given that the first toss is a head?

3-29. Three cards are drawn from a standard deck. What is the probability that all three are hearts, given that at least two of them are hearts?

3.4 Independence

3-30. Let X be the experiment of drawing a single card from a deck. Let A be the event the card is a spade or a heart, B be the event it is a spade or a diamond, and C be the event it is a spade or a club. Show that each of the pairs (A, B), (A, C) and (B, C) is independent. Show that $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$.

3-31. Two cards are drawn from a standard deck with replacement. Let A1 be the event the first card is an ace and A2 be the event the second card is an ace. Show that A1 and A2 are independent.

3-32. Let S be the sample space for rolling a single die. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4\}$, and $C = \{3, 4, 5\}$. Which of the pairs (A, B), (A, C) and (B, C) is independent?

3-33. A company needs some of its employees for a task that requires that they not be color blind. In testing them it finds that 7 of the 130 men are color blind and 2 of the 170 women are color blind. Are the events male and color blind independent or dependent?

3-34. A student is taking a history course and an English course. He decides that the probability of passing the history course is .75 and the probability of passing the English course is .84. If these events are independent, what is the probability that (a) he passes both courses; (b) he passes exactly one of them?

3-35. A company has three identical machines operating independently of each other. The probability of any one machine breaking down during the next year is .05. What is the probability that during the next year there will be no breakdowns?

3-36. A machine has two parts that could fail and have to be replaced. The probabilities of failure of parts A and B are .17 and .12, respectively. If failures of these parts are independent of each other, what is the probability that at least one of them will fail?

3-37. For the experiment of tossing a single fair coin 3 times, let E be the event the first toss is a head and F be the event 2 heads and 1 tail are tossed. Are E and F independent?

3.5 Bayes' Theorem

3-38. A manufacturing company has a fabrication plant and an assembly line. The fabrication plant has 60% of the employees and the assembly line 40%. During the past year 35% of the workers in the fabrication plant sustained injuries and 20% of the assembly line workers had injuries.
- (a) What percentage of all workers had injuries in this period?
- (b) If an employee had an injury, what is the probability that he worked on the assembly line?

3-39. Two jars contain coins. Jar I contains 5 pennies, 4 nickels and 6 dimes. Jar II contains 6 pennies, 4 nickels and 2 dimes. A jar is selected at random and a coin is selected from that jar. If the coin is a nickel, what is the probability that it came from Jar II?

3-40. An insurance company divides its policyholders into low-risk and high-risk classes. For the year, of those in the low-risk class, 80% had no claims, 15% had one claim, and 5% had 2 claims. Of those in the high-risk class, 50% had no claims, 30% had one claim, and 20% had two claims. Of the policyholders, 60% were in the low-risk class and 40% in the high-risk class.
- (a) If a policyholder had no claims in the year, what is the probability that he is in the low-risk class?
- (b) If a policyholder had two claims in the year, what is the probability that he is in the high-risk class?

3-41. A manufacturer has three machines producing light bulbs. Machine A produces 40% of the light bulbs with 1% of them defective. Machine B produces 35% of them with 2% being defective. Machine C produces 25% with 4% being defective. If a light bulb is tested and found to be defective, what is the probability that it was produced by machine A?

3-42. A skin test for a disease is less expensive but less accurate than an X-ray. In a country 20% of the adult population has this disease. For a person with the disease, the skin test is positive 95% of the time. If a person does not have the disease, it will be positive 30% of the time.

(a) What is the probability that a person who tests positive does not have the disease?

(b) What is the probability that a person who tests negative has the disease?

3-43. A card is drawn from a deck, not replaced, and a second card is drawn. What is the probability that the second card is a heart?

3-44. A company classifies injuries to its workers as minor if the worker does not have to take time off and severe if the worker has to take time off. The company has two plants, A and B. In plant A 60% of the workers had no injuries, 30% had minor injuries, and 10% had severe injuries. In plant B 50% had no injuries, 35% minor injuries, and 15% severe injuries. 70% of all workers work in plant A and 30% in plant B. What is the probability that a worker with a severe injury worked in plant A?

3-45. In Exercise 3-44, what is the probability that a worker who had an injury worked in plant B and had a minor injury?

3.7 Sample Actuarial Examination Problems

3-46. The probability that a visit to a primary care physicians (PCP) office results in neither lab work nor referral to a specialist is 35%. Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work.

Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

3-47. You are given $P(A \cup B) = 0.7$ and $P(A \cup B') = 0.9$.
Determine $P[A]$.

3-48. An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.
(ii) 64% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.

What is the probability that a randomly selected customer insures exactly one car, and that car is not a sports car?

3-49. Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist.

Determine the probability that a randomly chosen member of this group visits a physical therapist.

3-50. A survey of a group's viewing habits over the last year revealed the following information:

(i) 28% watched gymnastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gymnastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gymnastics and soccer
(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

3-51. An actuary studying the insurance preferences of automobile owners makes the following conclusions:

(i) An automobile owner is twice as likely to purchase collision coverage as disability coverage.
(ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.
(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.

What is the probability that an automobile owner purchases neither collision nor disability coverage?

3-52. An insurance company pays hospital claims. The number of claims that include emergency room or operating room charges is 85% of the total number of claims. The number of claims that do not include emergency room charges is 25% of the total number of claims. The occurrence of emergency room charges is independent of the occurrence of operating room charges on hospital claims.
Calculate the probability that a claim submitted to the insurance company includes operating room charges.

3-53. The number of injury claims per month is modeled by a random variable N with $P[N=n]=\frac{1}{(n+1)(n+2)}$, where $n \geq 0$.
Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

3-54. A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease.
Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease.

Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

3-55. An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44.

Calculate the number of blue balls in the second urn.

3-56. An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is 1/3.

What is the probability that a woman has none of the three risk factors, given that she does not have risk factor A?

3-57. An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are 1/4, 1/3, and 5/12, respectively.

Determine the probability that a randomly chosen employee will choose no supplementary coverage.

3-58. An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of those policies next year. Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy.

Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

3-59. A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease.

Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

3-60. An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policyholder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year. A policyholder dies in the next year.

What is the probability that the deceased policyholder was ultra-preferred?

3-61. Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;
(ii) 30% of the emergency room patients were serious;
(iii) the rest of the emergency room patients were stable;
(iv) 40% of the critical patients died;
(vi) 10% of the serious patients died; and
(vii) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

3-62. An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of Driver	**Percentage of all drivers**	**Probability of at least one collision**
Teen	8%	0.15
Young Adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

3-63. The probability that a randomly chosen male has a circulation problem is 0.25. Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem.

What is the conditional probability that a male has a circulation problem, given that he a smoker?

3-64. A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period.

Calculate the probability that the participant was a heavy smoker.

CHAPTER 3

3-1. There are 8 total outcomes, 3 of which give 2 heads and 1 tail, HHT, HTH or THH. Hence P(2 heads and 1 tail) = 3/8.

3-2. P(at least 1 head) = 1 – P(no heads) = 1 – 1/8 = 7/8

3-3. (a) P(ball is red) = 3/16
(b) P(ball is not green) = P(ball is red or blue) = (6 + 3)/16 = 9/16

3-4. Number of employees with degree in either mathematics or economics is 21+33–7 = 47. The probability that the person chosen has one of these is 47/68.

3-5. There are 36 outcomes, of which 6 give a 7, 2 an 11, and 6 give a number less that 5.
(a) P(7) = 6/36
(b) P(11) = 2/36
(c) P(less that 5) = 6/36

3-6. Number of clients with either life or auto insurance is 45+32 – 16 = 61. The probability a client has neither is 17/78.

3-7. (a) P(all 3 red) = C(4,3)/C(10,3) = 4/120 = 1/30
(b) P(1 red and 2 green) = C(4,1)C(6,2)/120 = 60/120 = 1/2
(c) P(all red or all green) = [C(4,3)+C(6,3)]/120 = 24/120 = 1/5

3-8. (a) P(all 4 good) = C(35,4)/C(40,4) = 52360/91390 = .5279

(b) P(2 good and 2 defective) = C(35,2)C(5,2)/C(40,4)
= 5950/91390 = .0651

3-9. Total seating arrangements is 10! = 3,628,800. There are two choices, M or W, for the first chair. This determines the 5 chairs for the 5 men and the 5 chairs for the 5 women. The number of seating arrangements alternating men and women is 2(5!)(5!) = 28,800. The probability is .0079.

3-10. P(at least 2 born on same date)
= 1 – P(all born on different days)
= 1 – $P(31,8)/31^8$ = .6271

3-11. P(at least 2 born on same day of week)
= 1 – P(all born on different days)
= 1 – $P(7,4)/7^4$ = .6501

3-12. P(balls of both colors) = 1 – P(all red or all blue)
= 1 – [C(5,4) + C(6,4)]/C(11,4)
= 1 – 20/330 = 31/33

3-13. Number of hands is 2,598,960 (Exercise 2-35). Number of ways to pick the 2 ranks is P(13,2). Number of full houses is P(13,2)C(4,3)C(4,2) = 3744, so P(full house) = .0014.

3-14. Number of ways to pick 2 suits for pairs is C(13,2). (Order doesn't matter here.) Number of hands is C(13,2)C(4,2)C(4,2)·44 = 123,552, so P(two pair) = .0475.

3-15. (a) P(7) = 1/6, P(~7) = 5/6. Odds for a 7 are 1:5.
(b) P(11) = 1/18, P(~11) = 17/18. Odds against an 11 are 17:1.

3-16. If the odds against F are a:b, the odds for F are b:a, and P(F) = b/(b+a).

3-17. $E \cup \sim E = S$ and $E \cap \sim E = \varnothing$, so P(E) + P(~E) = P(S) = 1.

3-18. $A = (A \cap B) \cup (A \cap \sim B)$ and $B = (B \cap A) \cup (B \cap \sim A)$, so $(A \cup B) = (A \cap \sim B) \cup (A \cap B) \cup (B \cap \sim A)$.

$P(A \cup B) = P(A \cap \sim B) + P(A \cap B) + P(B \cap \sim A)$
$= P((A \cap \sim B) + P(A \cap B) + P(B \cap \sim A) + [P(A \cap B) - P(A \cap B)]$
$= P(A) + P(B) - P(A \cap B)$

3-19. P(junior or senior) = .244 + .215 = .459

3-20. P(no claim) = 1 – P(liability or comprehensive)
= 1 – (.22 + .37 – .13) = .54

3-21. (a) P(C or better) = .131 +.278 + .312 = .721

(b) P(D or E) = .089 + .094 = .183

3-22. P(flu shot or tuberculosis test) = 1 – .21 = .79
= P(flu shot) + P(tuberculosis test) – P(both)
= .37 + .58 – P(both). P(both) = .95 – .79 = .16

3-23. P(A | C or better) = .131/.721 = .1817

3-24. P(missed 1 day | missed work) = .237/.365 = .6493

3-25. P(low claim | claim filed) = .169/.208 = .8125

3-26. (a) P(both hearts) = P(1st heart)P(2nd heart | 1st heart)
= (1/4)(12/51) = .0588
(b) P(neither a heart)
= P(1st not a heart)P(2nd not a heart | 1st not a heart)
= (3/4)(38/51) = .5588
(c) P(exactly one heart) = 1 – .0588 – .5588 = .3824

3-27. P(exactly 2 heads | at least 1 head) = (3/8)/(7/8) = 3/7

3-28. P(exactly 2 heads | 1st is a head)
= P(exactly 2 heads and 1st is a head)/P(1st is a head)
= (2/8)/(4/8) = 1/2

3-29. Let E be the event exactly 2 cards are hearts and F be the event all three are hearts. E∪F is the event at least 2 are hearts. Then n(E) = C(13,2)(39) = 3042 and n(F) = C(13,3) = 286, so P(F | E∪F) = 286/(286+3042) = .0859.

3-30. Let E be the event the card is a spade. P(A) = P(B) = P(C) = 1/2. P(A∩B) = P(A∩C) = P(B∩C) = P(S) = 1/4 = P(A)P(B) = P(A)P(C) = P(B)P(C), so each pair is independent.

P(A∩B∩C) = P(S) = 1/4 ≠ P(A)P(B)P(C)

3-31. P(A1∩A2) = $4^2/52^2$, and P(A2) = 4/52 = P(A1)
P(A2 | A1) = P(A1∩A2)/P(A1) = 4/52 = P(A2), so events are independent.

3-32. P(A) = 2/3, P(B) = 1/2 and P(C) = 1/2.

P(A∩B) = 1/2 ≠ P(A)P(B), so pair is dependent.

P(A∩C) = 1/3 = P(A)P(C), so pair is independent.

P(B∩C) = 1/3 ≠ P(B)P(C), so pair is dependent.

3-33. P(color blind) = 9/300, and P(color blind | male) = 7/130.
Male and color blind are dependent.

3-34. (a) P(pass both classes) = (.75)(.84) = .63

(b) P(fail both classes) = (.25)(.16) = .04
P(pass exactly one class) = 1 – (.63+.04) = .33

3-35. P(none break down) = $.95^3$ = .8574

3-36. P(at least one fails) = 1 – P(neither fails) = 1– (.83)(.88) = .2696

3-37. P(E) = 1/2 and P(F) = 3/8. P(E∩F) = 1/4. (E∩F = {HHT,HTH})
P(E∩F) ≠ P(E)P(F), so events are dependent.

3-38. (a) P(injury) = (.6)(.35) + (.4)(.2) = .29
(b) P(assembly line worker | injury) = .08/.29 = .2759

3-39. P(nickel) = (1/2)(4/15) + (1/2)(1/3) = 3/10
P(jar II | nickel) = (1/6)/(3/10) = 5/9

3-40. P(0 claims) = (.6)(.8) + (.4) (.5) = .68
P(2 claims) = (.6)(.05) + (.4)(.02) = .11

(a) P(low risk | 0 claims) = .48/.68 = .7059

(b) P(high risk | 2 claims) = .08/.11 = .7273

3-41. P(defective) = (.4)(.01) + (.35)(.02) + (.25)(.04) = .021
P(machine A | defective) = .004/.021 = .1905

3-42. Define the events; D: has disease, ~D: doesn't have disease, Y: tests positive, and N: tests negative.

P(Y) = (.2)(.95) + (.8)(.3) = .43, and
P(N) = (.2)(.05) + (.8)(.7) = .57

(a) P(~D | Y) = .24/.43 = .5581

(b) P(D | N) = .01/.57 = .0175

3-43. Let A be event first card is a heart, and B that second is a heart.
P(A∩B) = (1/4)(12/51), and P(~A∩B) = (3/4)(13/51).

P(second card is a heart) = (12 +39)/(4·51) = 1/4

3-44. P(severe injury) = (.7)(.10) + (.3)(.15) = .115

P(plant A | severe injury) = .07/.115 = .6087

3-45. P(injury) = 1 – P(no injury) = 1 – [(.7)(.6) + (.3)(.5)] = .43

P(plant B and minor injury | injury) = (.3)(.35)/.43 = .2442

CHAPTER 3

3-1. 3/8

3-2. 7/8

3-3. (a) 3/16 (b) 9/16

3-4. 47/68 ≈ .6912

3-5. (a) 1/6 (b) 1/18 (c) 1/6

3-6. $17/78 \approx .2179$

3-7. (a) 1/30 (b) 1/2 (c) 1/5

3-8. (a) .5729 (b) .0651

3-9. .0079

3-10. .6271

3-11. .6501

3-12. $31/33 \approx .9394$

3-13. .0014

3-14. .0475

3-15. (a) 1:5 (b) 17:1

3-16. $\dfrac{b}{a+b}$

3-19. .459

3-20. .54

3-21. (a) .721 (b) .183

3-22. .16

3-23. .1817

3-24. .6493

3-25. .8125

3-26. (a) .0588 (b) .5588 (c) .3824

3-27. 3/7

3-28. 1/2

3-29. .0859

3-32. (A, C)

3-33. Dependent

3-34. (a) .63 (b) .33

3-35. .8574

3-36. .2696

3-37. No

3-38. (a) 29% (b) .2759

3-39. 5/9

3-40. (a) .7059 (b) .7273

3-41. .1905

3-42. (a) .5581 (b) .0175

3-43. 1/4

3-44. .6087

3-45. .2442

3-46. .05

3-47. .60

3-48. .256

3-49. .48

3-50. .52

3-51. .33

3-52. .40

3-53. 2/5

3-54. .173

3-55. 4

3-56. .467

3-57. 1/2

3-58. .53

3-59. .657

3-60. .0141

3-61. .2922

3-62. .21955

3-63. .40

3-64. .42

Chapter 4
Discrete Random Variables

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4.6 Exercises

4.2 The Probability Function of a Discrete Random Variable

4-1. Let X be the random variable for the number of heads obtained when three fair coins are tossed. What is the probability function for X?

4-2. Ten cards are face down in a row on a table. Exactly one of them is an ace. You turn the cards over one at a time, moving from left to right. Let X be the random variable for the number of cards turned *before* the ace is turned over. What is the probability function for X?

4-3. A fair die is rolled repeatedly. Let X be the random variable for the number of times the die is rolled *before* a six appears. What are the probability function and the cumulative distribution function for X?

4-4. Let X be the random variable for the sum obtained by rolling two fair dice. What are the $p(x)$ and $F(x)$ functions for X?

4.3 Measuring Central Tendency; Expected Value

4-5. For the X defined in Exercise 4-4, what is $E(X)$?

4-6. The GPA (grade point average) random variable X assigns to the letter grades A, B, C, D and E the numerical values 4, 3, 2, 1 and 0. Find the expected value of X for a student selected at random from a class in which there were 15 A grades, 33 B grades, 51 C grades, 6 D grades, and 3 E grades. (This expected value can be thought of as the class average GPA for the course.)

4-7. A construction company whose workers are used on high-risk projects insures its workers against injury or death on the job. One unit of insurance for an employee pays \$1,000 for an injury and \$10,000 for death. Studies have shown that in a year 7.3% of the workers suffer an injury and 0.41% are killed. What is the expected unit claim amount (pure premium) for this insurance? If the company has 10,000 employees and exactly 7.3% are injured and exactly 0.41% are killed, what is the average cost per unit of the insurance claims?

4-8. Suppose that in the above problem the administrative costs are \$50 per person insured. The company purchases 10 units of insurance for each worker. Let X be the total of expected claim amount and administrative costs for each worker. Find $E(X)$.

4-9. Verify Equation (4.4b).

4-10. Let X be the random variable for the number of times a fair die is tossed before a six appears (Exercise 4-3). Find $E(X)$.

4-11. The mode of a probability function does not have to be unique. Find the mode of the probability function in Exercise 4-1, for the random variable for the number of heads obtained when three fair coins are tossed.

4.4 Variance and Standard Deviation

4-12. If X is the random variable for the sum obtained by rolling two fair dice (Exercise 4-4), what is $V(X)$?

4-13. For the insurance policy that pays \$1,000 for an injury and \$10,000 for death (Exercise 4-7), what is the standard deviation for the claim amount on 5 units of insurance? (Note: Some employees receive \$0 of claim payment. This value of the random variable must be included in your calculation.)

4-14. Verify Equation (4.5b). (Hint: It is sufficient to show that $V(X+b) = V(X)$. If $Y = X + b$ and $E(X) = \mu_X$, what is $Y - \mu_Y$?)

4-15. Let X be the random variable for the sum obtained by rolling two fair dice (Exercise 4-4).

(a) Using Chebychev's Theorem, what is a lower bound for the probability that the value of X is within 2 standard deviations of the mean of X?

(b) What is the exact probability that this sum is within this range?

4.5 Population and Sample Statistics

4-16. An auto insurance company has 15,000 policyholders with comprehensive automobile coverage. In the past year 11,425 filed no claims, 3,100 filed one claim, 385 filed two claims, and 90 filed three claims. What are the mean and the standard deviation for the number of claims filed by a policyholder?

4-17. A marketing company polled 50 people at a mall about the number of movies they had seen in the previous month. The results of this poll are as follows:

Number of movies	0	1	2	3	4	5	6	7	8
Number of viewers	3	5	6	9	11	7	5	3	1

What are the sample mean and sample standard deviation for the number of movies seen by an individual in a month?

4.7 Sample Actuarial Examination Problems

4-18. A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim Size	Probability
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

What percentage of the claims are within one standard deviation of the mean claim size?

4-19. A recent study indicates that the annual cost of maintaining and repairing a car in a town in Ontario averages 200 with a variance of 260.

If a tax of 20% is introduced on all items associated with the maintenance and repair of cars (i.e., everything is made 20% more expensive), what will be the variance of the annual cost of maintaining and repairing a car?

4-20. A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists.

Each ticket costs 50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay 100 (ticket cost + 50 penalty) to the tourist.

What is the expected revenue of the tour operator?

CHAPTER 4

4-1. Using the tree on page 26 we get the following table.

Number of heads (x)	0	1	2	3
Number of outcomes	1	3	3	1
p(x)	1/8	3/8	3/8	1/8

4-2. $P(X = 0) = P(\text{first card is the ace}) = 1/10$,
$P(X = 1) = P(\text{second card is the ace} \mid \text{first was not the ace})$
$= 9/10 \cdot 1/9 = 1/10$,
$P(X = 2) = P(\text{third card is the ace} \mid \text{neither the first nor the second was the ace}) = 9/10 \cdot 8/9 \cdot 1/8 = 1/10$, etc.

4-3. $P(X = x) = (5/6)^x(1/6), \ x = 0,1,2,\ldots$
$F(x) = 1/6 + 1/6 \cdot 5/6 + \ldots + (1/6)(5/6)^x$
$= (1/6)(1 + 5/6 + \ldots + (5/6)^x) = (1/6)[1 - (5/6)^{x+1}]/(1 - 5/6)$
$= 1 - (5/6)^{x+1}, x = 0, 1, 2, \ldots$

4-4.

x	Number of Outcomes	p(x)	F(x)
2	1	1/36	1/36
3	2	2/36	3/36
4	3	3/36	6/36
5	4	4/36	10/36
6	5	5/36	15/36
7	6	6/36	21/36
8	5	5/36	26/36
9	4	4/36	30/36
10	3	3/36	33/36
11	2	2/36	35/36
12	1	1/36	36/36

4-5. $E(X) = \Sigma xp(x) = (2 + 3\cdot2 + 4\cdot3 + 5\cdot4 + 6\cdot5 + 7\cdot6 + 8\cdot5 + 9\cdot4 + 10\cdot3 + 11\cdot2 + 12)/36 = 7$

4-6. $E(X) = (4\cdot15 + 3\cdot33 + 2\cdot51 + 6\cdot1 + 3\cdot0)/108 = 2.47$

4-7. $E(X) = 0(.9929) + \$1000(.073) + \$10,000(.0041) = \$114$
Average cost per claim $= (730(\$1000) + 41(\$10,000))/10,000 = \$114$

4-8. If Y is the claim amount for one unit, then $X = 10Y + 50$.
From Exercise 4-7 we have $E(Y) = \$114$.
$E(X) = E(10Y + 50) = 10E(Y) + 50 = \$1140 + 50 = \$1190$

4-9. $E(aX + b) = \Sigma(ax+b)p_Y(ax+b)$
$= a\Sigma xp_X(x) + b\Sigma p_X(x)$
$= aE(X) + b$

4-10.
$$E(X) = \sum_{x=0}^{\infty} x(5/6)^x(1/6)$$
$$= (5/6)(1/6)\sum_{x=1}^{\infty} x(5/6)^{x-1}$$
$$= (5/6)(1/6)/(1-5/6)^2$$
$$= (5/6)/(1/6) = 5$$

4-11. Three outcomes yield 1 head and three outcomes yield 2 heads. Thus either 1 or 2 is a mode.

4-12. $V(X) = \Sigma(x-7)^2p(x) = (25 + 2(16) + 3(9) + 4(4) + 5(1) + 6(0) + 5(1) + 4(4) + 3(9) + 2(16) + 25)/36 = 210/36$

4-13. $V(X) = (10,000 - 114)^2(.0041) + (1000 - 114)^2(.073) + (0 - 114)^2(.9229) = 470.004$
For 5 units, $\sigma_{5X} = 5\sigma_X = 3427.84$

4-14. If $Y = X + b$, $E(Y) = \mu_Y = E(X) + b = \mu_X + b$.
$Y - \mu_Y = X + b - (\mu_X + b) = X - \mu_X$
$V(Y) = \Sigma(y - \mu_Y)^2 = \Sigma(x - \mu_X)^2 = V(X)$

4-15. (a) Lower bound is $1 - (1/2)^2 = .75$.
(b) Two standard deviations equals $2(5.8333)^{1/2} = 4.8305$.
$P(7 - 4.8305 < X < 7 + 4.8305) = P(3 \le X \le 11) = 34/36 = .9444$

4-16. $\mu = (0 \cdot 11,425 + 1 \cdot 3100 + 2 \cdot 385 + 3 \cdot 90)/15,000 = .276$
$\sigma^2 = [11,425(0-.276)^2 + 3100(1-.276)^2 + 385(2-.276)^2 + 90(3-.276)^2]/15,000 = .0287157$
$\sigma = .53587$

4-17. $\bar{x} = (0\cdot3 + 1\cdot5 + 2\cdot6 + 3\cdot9 + 4\cdot11 + 5\cdot7 + 6\cdot5 + 7\cdot3 + 8\cdot1)/50 = 3.64$
$s = [\Sigma f\cdot(n - 3.64)^2/49]^{1/2} = 1.9667$

CHAPTER 4

4-1.

Number of heads (x)	0	1	2	3
$p(x)$	1/8	3/8	3/8	1/8

4-2. $p(x) = 1/10 \quad x = 0, 1, \ldots, 9$

4-3. $p(x) = (1/6)(5/6)^x \quad x = 0, 1, 2, \ldots$
$F(x) = 1 - (5/6)^{x+1} \quad x = 0, 1, 2, \ldots$

4-4.

x	$p(x)$	$F(x)$
2	1/36	1/36
3	1/18	1/12
4	1/12	1/6
5	1/9	5/18
6	5/36	5/12
7	1/6	7/12
8	5/36	13/18
9	1/9	5/6
10	1/12	11/12
11	1/18	35/36
12	1/36	1

4-5. 7

4-6. $267/108 \approx 2.47$

4-7. \$114; \$114

4-8. \$1190

4-10. 5

4-11. Modes are 1 and 2

4-12. $210/36 \approx 5.8333$

4-13. 3,427.84

4-15. (a) .75 (b) .9444

4-16. $\mu = .276$; $\sigma = .53587$

4-17. $\overline{x} = 3.64$; $s = 1.9667$

4-18. 45%

4-19. 374.4

4-20. 984.58

Chapter 5
Commonly Used Discrete Distributions

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5.7 Exercises

5.1 The Binomial Distribution

5-1. A student takes a 10 question true-false test. He has not attended class nor studied the material, and so he guesses on every question. What is the probability that he gets (a) exactly 5 questions correct; (b) he gets 8 or more correct?

5-2. A single fair die is rolled 10 times. What is the probability of getting (a) exactly 2 sixes; (b) at least 2 sixes?

5-3. An insurance agent has 12 policyholders who are considered high risk. The probability that one of these clients will file a major claim in the next year is .023. What is the probability that exactly 3 of them will file major claims in the next year?

5-4. A company produces light bulbs of which 2% are defective.
- (a) If 50 bulbs are selected for testing, what is the probability that exactly 2 are defective?
- (b) If a distributor gets a shipment of 1,000 bulbs, what are the mean and the variance of the number of defective bulbs?

5-5. In the game of craps (dice table) the simplest bet is the pass line. The probability of winning such a bet is .493 and the payoff is even money, i.e., if you win you receive $1 more for each dollar that you bet. A gambler makes a series of 100 $10 bets on the pass line. What is his expected gain or loss at the end of this sequence of bets?

5-6. In a large population 10% of the people have type B+ blood. At a blood donation center 20 people donate blood. What is the probability that (a) exactly 4 of these have B+ blood; (b) at most 3 have B+ blood?

5-7. In the population of Exercise 5-6, 50,000 pints of blood are donated. What is the expected number of pints of B+ blood? What is the variance of the number of pints of B+ blood?

5-8. An experiment consists of picking a card at random from a standard deck and replacing it. If this experiment is performed 12 times, what is the probability that you get (a) exactly 2 aces; (b) exactly 3 hearts; (c) more than 1 heart?

5-9. Suppose that 5% of the individuals in a large population have a certain disease. If 15 individuals are selected at random, what is the probability that no more than 3 have the disease?

5-10. For a binomial random variable X with $n = 2$ and $P(S) = p$, show that (a) $E(X) = 2p$; (b) $V(X) = 2p(1 - p)$.

5.2 The Hypergeometric Distribution

5-11. There are 10 cards lying face down on a table, and 2 of them are aces. If 5 of these cards are selected at random, what is the probability that 2 of them are aces?

5-12. In a hospital ward there are 16 patients, 4 of whom have AIDS. A doctor is assigned to 6 of these patients at random. What is the probability that he gets 2 of the AIDS patients?

5-13. A baseball team has 16 non-pitchers on its roster. Of these, 6 bat left-handed and 10 right-handed. The manager, having already selected the pitcher for the game, randomly selects 8 players for the remaining positions.

(a) What is the probability that he selects 4 left-handed batters and 4 right-handed batters?

(b) What is the expected number of left-handed batters chosen?

5-14. The United States Senate has 100 members. Suppose there are 54 Republicans and 46 Democrats.

(a) If a committee of 15 is selected at random, what is the expected number of Republicans on this committee?

(b) What is the variance of the number of Republicans?

5-15. A bridge hand consists of 13 cards. If X is the random variable for the number of spades in a bridge hand, what are $E(X)$ and $V(X)$?

5.3 The Poisson Distribution

5-16. An auto insurance company has determined that the average number of claims against the comprehensive coverage of a policy is 0.6 per year. What is the probability that a policyholder will file (a) 1 claim in a year; (b) more than 1 claim in a year?

5-17. A city has an intersection where accidents have occurred at an average rate of 1.5 per year. What is the probability that in a year there will be (a) 0; (b) 1; (c) 2 accidents in a year?

5-18. Policyholders of an insurance company file claims at an average rate of 0.38 per year. If the company pays \$5,000 for each claim, what is the mean claim amount for a policyholder in a year?

5-19. An insurance company has 5,000 policyholders who have had policies for at least 10 years. Over this period there have been a total of 12,200 claims on these policies. Assuming a Poisson distribution for these claims, answer each of the following.

(a) What is λ, the average number of claims per policy per year?

(b) What is the probability that a policyholder will file less than 2 claims in a year?

(c) If all claims are for \$1,000, what is the mean claim amount for a policyholder in a year?

5-20. Claims filed in a year by a policyholder of an insurance company have a Poisson distribution with $\lambda = .40$. The number of claims filed by two different policyholders are independent events.

(a) If two policyholders are selected at random, what is the probability that each of them will file one claim during the year?

(b) What is the probability that at least one of them will file no claims?

5-21. Show that a Poisson distribution with parameter $\lambda = k$ (an integer) has two modes, $k - 1$ and k.

5-22. Show that $V(X) = \lambda$ for a Poisson random variable X with parameter λ. Hint: Show $V(X) = E(X^2) + E(-2\lambda X + \lambda^2)$ and $E(X^2) = \lambda^2 + \lambda$.

5.4 The Geometric Distribution

5-23. If you roll a pair of fair dice, the probability of getting an 11 is 1/18. (See Exercise 4-4.) If you roll the dice repeatedly, what is the probability that the first 11 occurs on the eighth roll?

5-24. An experiment consists of drawing a card at random from a standard deck and replacing it. If this experiment is done repeatedly, what is the probability that (a) the first heart appears on the fifth draw; (b) the first ace appears on the tenth draw?

5-25. For the experiment in Exercise 5-24, let X be the random variable for the number of unsuccessful draws before the first ace is drawn. Find $E(X)$ and $V(X)$.

5-26. At a medical clinic, patients are given X-rays to test for tuberculosis.
   (a) If 15% of these patients have the disease, what is the probability that on a given day the first patient to have the disease will be the fifth one tested?
   (b) What is the probability that the first with the disease will be the tenth one tested?

5.5 The Negative Binomial Distribution

5-27. Consider the experiment of drawing from a deck of cards with replacement (Exercise 5-24).
   (a) What is the probability that the third heart appears on the tenth draw?
   (b) What is the mean number of non-hearts drawn before the fifth heart is drawn?

5-28. A single fair die is rolled repeatedly.

(a) What is the probability that the fourth six appears on the twentieth roll?

(b) What is the mean number of total rolls needed to get 4 sixes?

5-29. For the experiment in Exercise 5-28, let X be the random variable for the number of non-sixes rolled before the fifth six is rolled. What are $E(X)$ and $V(X)$?

5-30. A telemarketer makes successful calls with probability .20. What is the probability that her fifth sale will be on her sixteenth call?

5-31. If each sale made by the person in Exercise 5-30 is for \$250, what is the mean number of total calls she will have to make to reach \$2,000 in total sales?

5-32. Consider the clinic in Exercise 5-26, where 15% of the patients have tuberculosis.

(a) What is the probability that the fifteenth patient tested will be the third with tuberculosis?

(b) What is the mean number of patients without tuberculosis tested before the sixth patient with tuberculosis is tested?

5.6 The Discrete Uniform Distribution

5-33. Verify the results of Example 5.25 by direct calculation using the definitions of $E(X)$ and $V(X)$.

5-34. A contestent on a game show selects a ball from an urn containing 25 balls numbered from 1 to 25. His prize is \$1,000 times the number of the ball selected. If X is the random variable for the amount he wins, find the mean and standard deviation of X.

5-35. Derive the formulas for $E(X)$ and $V(X)$ for the discrete uniform distribution. (Recall that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ and $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.)

5.8 Sample Actuarial Examination Problems

5-36. A company prices its hurricane insurance using the following assumptions:

(i) In any calendar year, there can be at most one hurricane.
(ii) In any calendar year, the probability of a hurricane is 0.05.
(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.

Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period.

5-37. A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants).

What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?

5-38. A hospital receives 1/5 of its flu vaccine shipments from Company X and the remainder of its shipments from other companies. Each shipment contains a very large number of vaccine vials.

For Company X's shipments, 10% of the vials are ineffective. For every other company, 2% of the vials are ineffective. The hospital tests 30 randomly selected vials from a shipment and finds that one vial is ineffective.

What is the probability that this shipment came from Company X?

5-39. An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims. If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

5-40. A company buys a policy to insure its revenue in the event of major snowstorms that shut down business. The policy pays nothing for the first such snowstorm of the year and 10,000 for each one thereafter, until the end of the year. The number of major snowstorms per year that shut down business is assumed to have a Poisson distribution with mean 1.5.

What is the expected amount paid to the company under this policy during a one-year period?

5-41. In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying assumption that for all integers $n \geq 0$, $p_{n+1} = \frac{1}{5} p_n$, where p_n represents the probability that the policyholder files n claims during the period.

Under this assumption, what is the probability that a policyholder files more than one claim during the period?

CHAPTER 5

5-1. (a) $P(X = 5) = C(10,5)(1/2)^5(1/2)^5 = .2461$ ($p = q = ½$)

(b) $P(X \geq 8) = C(10,8)(1/2)^{10} + C(10,9)(1/2)^{10} + C(10,10)(1/2)^{10} = .05467$

5-2. (a) $P(X = 2) = C(10,2)(1/6)^2(5/6)^8 = .2907$

(b) $P(X \geq 2) = 1 - P(X < 2) = 1 - [(5/6)^{10} + 10(1/6)(5/6)^9] = .5155$

5-3. P(three file claims) $= C(12,3)(.023)^3(.977)^9 = .00217$

5-4. (a) P(exactly 2 defective) $= C(50,2)(.02)^2(.98)^{48} = .1858$

(b) $\mu = 1000(.02) = 20$; $\sigma^2 = 1000(.02)(.98) = 19.6$

5-5. Let X be the random variable for the number of wins, and Y = 100 – X be the random variable for the number of losses. The expected amount of gain = E(10X) = 10(100)(.493) = 493 and, the expected amount of loss = E(10Y) = 10(100)(.507) = 507. The expected end result is 493 – 507 = –14 (a loss of $14).

5-6. (a) P(4 have B+ blood) $= C(20,4)(.10)^4(.90)^{16} = .0898$

(b) P(at most 3 have B+ blood) $= (.90)^{20} + C(20,1)(.10)(.90)^{19} + C(20,2)(.10)^2(.90)^{18} + C(20,3)(.10)^3(.90)^{17} = .8670$

5-7. Let X be the number of pints of B+ blood donated.
$E(X) = 50{,}000(.10) = 5000$; $V(X) = 50{,}000(.10)(.90) = 4500$

5-8. (a) P(2 aces) $= C(12,2)(1/13)^2(12/13)^{10} = .1754$

(b) P(3 hearts) $= C(12,3)(1/4)^3(3/4)^9 = .2581$

(c) P(more than 1 heart) $= 1 - [(3/4)^{12} + 12(1/4)(3/4)^{11}] = .8416$

5-9. P(no more than 3) $= (.95)^{15} + C(15,1)(.05)(.95)^{14} + C(15,2)(.05)^2(.95)^{13} + C(15.3)(.05)^3(.95)^{12} = .9945$

5-10. For a binomial random variable with n = 2 and P(S) = p we have

k	0	1	2
P(k)	$(1-p)^2$	$2p(1-p)$	p^2

(a) $E(X) = 0(1-p)^2 + 1(2p)(1-p) + 2p^2 = 2p$

(b) $V(X) = (0-2p)^2(1-p)^2 + (1-2p)^2(2p)(1-p) + (2-2p)^2p^2 = 2p(1-p)[2p(1-p) + (1-2p)^2 + 2p(1-p)] = 2p(1-p)$

5-11. P(2 aces) $= C(2,2)C(8,3)/C(10,5) = 2/9$

5-12. P(2 with AIDS) $= C(4,2)C(12,4)/C(16,6) = .3709$

5-13. (a) P(4 of each) = C(10,4)C(6,4)/C(16,8) = .2448

(b) If X is number of left-handed batters, E(X) = 8(6/16) = 3

5-14. Let X be the number of Republicans on committee.
(a) E(X) = 15(54/100) = 8.1
V(X) = 15(54/100)(46/100)(85/99) = 3.199

5-15. E(X) = 13(1/4) = 3.25; V(X) = 13(1/4)(3/4)(39/51) = 1.864

5-16. (a) $P(X = 1) = .6e^{-.6} = .3293$

(c) $P(X > 1) = 1 - e^{-.6} - .6e^{-.6} = .1219$

5-17. (a) $P(X = 0) = e^{-1.5} = .2231$

(b) $P(X = 1) = 1.5e^{-1.5} = .3347$

(d) $P(X = 2) = 1.5^2e^{-1.5}/2 = .2510$

5-18. If X is the number of claims per year, 5000X is the claim amount. E(5000X) = 5000E(X) = 5000(.38) = 1900

5-19. (a) The rate of claims per policy per year is λ = 12,200/50,000 = .244.
(b) $P(X = 0 \text{ or } 1) = e^{-.244}(1 + .244) = .9747$

(c) E(1000X) = 1000E(X) = 1000(.244) = 244

5-20. Let X and Y be the number of claims filed by two policyholders.
(a) $P(X = 1 \text{ and } Y = 1) = (0.4e^{-.4})^2 = .0719$
P(X = 0 or Y = 0) = P(X = 0) + P(Y = 0) − P(X = 0 and Y = 0)
$= 2e^{-.4} - e^{-.8} = .8913.$

5-21. $P(X = k-1) = \frac{k^{k-1}e^{-k}}{(k-1)!} = \frac{k}{k}\cdot\frac{k^{k-1}e^{-k}}{(k-1)!} = \frac{k^ke^{-k}}{k!} = P(X = k)$

For $n \geq k$, $P(X = n) = \frac{k^ne^{-k}}{n!} > \frac{k}{n+1}\cdot\frac{k^ne^{-k}}{n!}$

$$= \frac{k^{n+1}e^{-k}}{(n+1)!} = P(X = n+1)$$

Hence for n > k, the probabilities get smaller as n gets larger.

For $n < k-1$, $P(X = n) = \frac{k^ne^{-k}}{n!} < \frac{k}{n+1}\cdot\frac{k^ne^{-k}}{n!}$

$$= \frac{k^{n+1}e^{-k}}{(n+1)!} = P(X = n + 1)$$

Hence for n < k − 1, the probabilities get larger as n gets larger. Therefore the largest probability occurs when n = k − 1 or k.

5-22. In Section 5.3.4 it was shown that $E(X) = \lambda$ for the Poisson random variable.

$$V(X) = E[(X-\lambda)^2] = E(X^2 - 2\lambda X + \lambda^2)$$

$$= \sum_{k=0}^{\infty} (k^2 - 2\lambda k + \lambda^2)\frac{\lambda^ke^{-\lambda}}{k!}$$

$$= \sum_{k=0}^{\infty} k^2\frac{\lambda^ke^{-\lambda}}{k!} - 2\lambda\sum_{k=0}^{\infty} k\frac{\lambda^ke^{-\lambda}}{k!} + \lambda^2\sum_{k=0}^{\infty}\frac{\lambda^ke^{-\lambda}}{k!}$$

The second term is $-2\lambda E(X) = -2\lambda^2$, and the last term is $\lambda^2\Sigma p(x) = \lambda^2(1)$.

$$\sum_{k=0}^{\infty} k^2\frac{\lambda^ke^{-\lambda}}{k!} = \sum_{k=1}^{\infty} k\frac{\lambda^ke^{-\lambda}}{(k-1)!}$$

If we let n = k − 1, then k = n + 1 and this sum becomes

$$\sum_{n=0}^{\infty}(n+1)\frac{\lambda^{n+1}e^{-\lambda}}{n!} = \lambda\left(\sum_{n=0}^{\infty} n\frac{\lambda^ne^{-\lambda}}{n!} + \sum_{n=0}^{\infty}\frac{\lambda^ne^{-\lambda}}{n!}\right)$$

$$= \lambda[E(X) + 1] = \lambda^2 + \lambda.$$

Hence $V(X) = \lambda^2 + \lambda - 2\lambda^2 + \lambda^2 = \lambda$.

5-23. If the first 11 occurs on the eighth roll, there are 7 failures first, and p = 1/18 and q = 17/18.

$P(X = 7) = (11/18)^7(1/18) = .0372$

5-24. (a) The number of initial failures is 4, and p = ¼, q = ¾.
$P(X = 4) = (3/4)^4(1/4) = .0791$

(b) The number of initial failures is 9, and p = 1/13, q = 12/13.
$P(X = 9) = (12/13)^9(1/13) = .0374$

5-25. Probability of an ace is p = 1/13, so q = 12/13.

E(X) = q/p = (12/13)/(1/13) = 12

$V(X) = q/p^2 = (12/13)/(1/13)^2 = 156$

5-26. (a) If the fifth patient is the first one with the disease, then there are 4 initial failures.

$P(X = 4) = (.85)^4(.15) = .0783$

(b) For the second part there are 9 initial failures.

$P(X = 9) = (.85)^9(.15) = .0347$

5-27. (a) If the third heart appears on the tenth draw, there are 7 failures.

$P(X = 7) = C(7{+}3{-}1,3{-}1)q^7p^3 = C(9,2)(3/4)^7(1/4)^3 = .0751$

(b) If X is the number of non-hearts drawn before the fifth heart, then

E(X) = rq/p = 5(3/4)/(1/4) = 15.

5-28. (a) If the fourth 6 appears on the twentieth roll, there are 16 initial failures.

$P(X = 16) = C(19,3)(5/6)^{16}(1/6)^4 = .0404$

(b) If X is the number of failures before the fourth 6, then

E(X) = rq/p = 4(5/6)/(1/6) = 20.

The expected total number of rolls is 20 + 4 = 24.

5-29. Let X be the number of failures before the fifth 6.

E(X) = rq/p = 5(5/6)/(1/6) = 25

$V(X) = rq/p^2 = 5(5/6)/(1/6)^2 = 150$

5-30. If the fifth success occurs on the sixteenth call, there are 11 initial failures.

$P(X = 11) = C(15,4)(.8)^{11}(.2)^5 = .0375$

5-31. Number of successes needed is 2000/250 = 8. If X is the number of failures before the eighth success, then

E(X) = 8(.8)/(.2) = 32.

The total expected number of calls is 32 + 8 = 40.

5-32. (a) There would be 12 failures before the third success.

$$P(X = 12) = C(14,2)(.85)^{12}(.15)^3 = .0437$$

(b) If X is the number patients without the disease tested before the sixth with, then

$$E(X) = 6(.85)/(.15) = 34.$$

5-33. The outcome X is a discrete uniform random variable with $p(x) = 1/6$, $x = 1,2,3,4,5,6$.

$$E(X) = (1 + 2 + 3 + 4 + 5 + 6)/6 = 21/6 = 3.5$$

$$\begin{aligned} V(X) &= [(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 \\ &\quad +(6-3.5)^2]/6 = (6.25 + 2.25 + .25 + .25 + 2.25 + 6.25)/6 \\ &= 17.5/6 = 35/12 \end{aligned}$$

5-34. Let Y be the number on the ball chosen. Y is a discrete uniform random variable, with $n = 25$ and $p(n) = 1/25$, $n = 1,2,\ldots,25$, and $X = 1000Y$.

$$E(X) = 1000E(Y) = 1000(26)/2 = \$13{,}000$$

$$V(X) = 1000^2V(Y) = 1000^2(25^2 - 1)/12 = 1000^2(52) = \sigma^2$$

$$\sigma = \$7{,}211.10$$

5-35. Let X be the discrete uniform random variable with $p(x) = 1/n$ for $x = 1,2,3,\ldots,n$.

$$E(X) = \Sigma xp(x) = \frac{1 + 2 + \cdots + n}{n} = \frac{1}{n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2} = \mu$$

$$\begin{aligned} V(X) &= E[(X-\mu)^2] = E(X^2 - 2\mu X + \mu^2) = \Sigma(k^2 - 2\mu k + \mu^2)p(k) \\ &= \Sigma k^2p(k) - 2\mu\Sigma kp(k) + \mu^2\Sigma p(k) \end{aligned}$$

The second term is $-2\mu(X) = -2\mu^2$, and the last term is μ^2. The sum of these two is $-\mu^2 = \dfrac{(n+1)^2}{4}$.

$$\Sigma k^2p(k) = \frac{1^2 + 2^2 + \cdots + n^2}{n} = \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{2n^2 + 3n + 1}{6}$$

$$V(X) = \frac{2n^2 + 3n + 1}{6} - \frac{n^2 + 2n + 1}{4} = \frac{8n^2 + 12n + 4 - 6n^2 - 12n - 6}{24} = \frac{n^2 - 1}{12}$$

CHAPTER 5

5-1. (a) 0.2461 (b) 0.05469

5-2. (a) 0.2907 (b) 0.5155

5-3. 0.00217

5-4. (a) 0.1858 (b) $\mu = 20$; $\sigma^2 = 19.6$

5-5. Loss of $14

5-6. (a) .0898 (b) .8670

5-7. 5,000; 4,500

5-8. (a) .1754 (b) .2581 (c) .8416

5-9. .9945

5-11. $2/9 \approx .2222$

5-12. .3709

5-13. (a) .2448 (b) 3

5-14. (a) 8.1 (b) 3.199

5-15. 3.25, 1.864

5-16. (a) .3293 (b) .1219

5-17. (a) .2231 (b) .3347 (c) .2510

5-18. 1,900

5-19. (a) .244 (b) .9747 (c) 244

5-20. (a) .0719 (b) .8913

5-23. .0372

5-24. (a) .0791 (b) .0374

5-25. $E(X) = 12$; $V(X) = 156$

5-26. (a) .0783 (b) .0347

5-27. (a) .0751 (b) 15

5-28. (a) .0404 (b) 24 (20 failures and 4 successes)

5-29. $E(X) = 25$; $V(X) = 150$

5-30. .0375

5-31. 40 (32 failures and 8 successes)

5-32. (a) .0437 (b) 34

5-34. $\mu = \$13{,}000$; $\sigma = \$7{,}211.10$

5-36. .92452

5-37. .469

5-38. .0955

5-39 2

5-40 7,231

5-41. .04

Chapter 6
Applications for Discrete Random Variables

6.5 Exercises

6.1 Functions of Random Variables and Their Expectations

6-1. In a year, a policyholder with an insurance company has no claims with probability .69, 1 claim with probability .23, 2 claims with probability .07, and 3 claims with probability .01. If X is the random variable for the number of claims, find (a) $E(500X + 50)$; (b) $E(X^2)$; (c) $E(X^3)$.

6-2. Let X be the random variable for the sum obtained by rolling a pair of fair dice (see Exercise 4-4). Find $V(X)$ by using the alternate formula $V(X) = E(X^2) - E(X)^2$.

6-3. Rework Example 6.2 using the logorithmic utility function $u(w) = ln(w + 1)$. What are $E[u(W_1)]$ and $E[u(W_2)]$ for this utility function?

6-4. Overflow problems occur when you exceed the precision of the computer or calculator you are using. Consider the distribution whose values of x are 1,000,000,000.1, 1,000,000,000 and 999,999,999.9, each with probability 1/3. The variance for this distribution is .00666. If you try to compute the variance using Equation (6.2), the value you get will depend on the precision of your computer or calculator and may not be correct. Use your calculator to find $E(X^2)$ and $E(X)$. Then use Equation (6.2) and determine whether or not you found the correct value of $V(X)$.

6.2 Moments and the Moment Generating Function

6-5. Show that the moment generating function for the binomial distribution is $(q + pe^t)^n$. Hint: Expand $(q + p)^n$ using the binomial theorem and use it to get the moment generating function.

6-6. Use the moment generating function for the Poisson distribution to verify that $E(X) = V(X) = \lambda$.

6-7. Use the moment generating function for the geometric distribution to obtain its mean and variance.

6-8. Use the moment generating function for the negative binomial distribution to obtain its mean and variance.

6-9. Let X be a discrete random variable with $p(x) = \frac{1}{n}$ for $x = 1, \ldots, n$. (X is a discrete uniform random variable.)

(a) Show that the moment generating function for X is $M_X(t) = \frac{1}{n}\sum_{x=1}^{n} e^{xt}$.

(b) Find $E(X)$ and $V(X)$.

6-10. Let X be a random variable whose probability function is given below.

x	0	1	2	3
$p(x)$	.42	.30	.17	.11

Find $M_X(t)$ and use its derivatives to find $E(X)$ and $E(X^2)$.

6-11. Prove $M_{aX+b}(t) = e^{tb} \cdot M_X(at)$.

6-12. If X is a binomial random variable with $p = .60$ and $n = 8$, and if $Y = 3X + 4$, what is $M_Y(t)$?

6-13. If $M_X(t) = [.70/(1 - .3e^t)]^5$, what is the distribution of X.

6.4 Simulation of Discrete Distributions

6-14. Using the linear congruence $y = 9x + 11 \pmod{16}$, with seed $x_1 = 6$, find $x_2, x_3, \ldots, x_{16}$.

For Exercises 6-15 and 6-16, use the following sequence of random numbers from [0, 1).

1. .5619	6. .9983	11. .7855	16. .3729
2. .4500	7. .0225	12. .9955	17. .1326
3. .3566	8. .8026	13. .6558	18. .9246
4. .5844	9. .3516	14. .1280	19. .6867
5. .8638	10. .4584	15. .3908	20. .9638

6-15. Random numbers from [0, 1) are used to simulate a binomial distribution with $n = 20$ and $p = .40$. If the random number x is less than .40 on a trial, then a success has occurred. Count the number of successes in the 20 trials.

6-16. Random numbers from [0, 1) are used to simulate repeated trials of the experiment of tossing 5 fair coins. The first five numbers represent the first trial, the second five numbers the second, and so on. If the random number x is less than .50, the coin is a head. How many heads appear on each of the first four repetitions of this experiment?

6.6 Sample Actuarial Examination Problems

6-17. A baseball team has scheduled its opening game for April 1. If it rains on April 1, the game is postponed and will be played on the next day that it does not rain. The team purchases insurance against rain. The policy will pay 1000 for each day, up to 2 days, that the opening game is postponed.

The insurance company determines that the number of consecutive days of rain beginning on April 1 is a Poisson random variable with mean 0.6.

What is the standard deviation of the amount the insurance company will have to pay?

6-18. Let X_1, X_2, X_3 be a random sample from a discrete distribution with probability function

$$p(x) = \begin{cases} \frac{1}{3} & \text{for } x = 0 \\ \frac{2}{3} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the moment generating function, $M(t)$, of $Y = X_1X_2X_3$.

CHAPTER 6

6-1.

x	0	1	2	3
p(x)	.69	.23	.07	.01

(a) $E(X) = 0(.69) + 1(.23) + 2(.07) + 3(.01) = .40$
$E(500X + 50) = 500E(X) + 50 = 250$

(b) $E(X^2) = 0(.69) + 1(.23) + 4(.07) + 9(.01) = .60$

(c) $E(X^3) = 0(.69) + 1(.23) + 8(.07) + 27(.01) = 1.06$

6-2. The following table relates to the rolling of a pair of dice.

X	p(x)	xp(x)	$x^2p(x)$
2	1/36	2/36	4/36
3	2/36	6/36	18/36
4	3/36	12/36	48/36
5	4/36	20/36	100/36
6	5/36	30/36	180/36
7	6/36	42/36	294/36
8	5/36	40/36	320/36
9	4/36	36/36	324/36
10	3/36	30/36	300/36
11	2/36	22/36	242/36
12	1/36	12/36	144/36

$E(X) = \Sigma xp(x) = 252/36 = 7$

$E(X^2) = \Sigma x^2p(x) = 1974/36 = 54.8333$

$V(X) = E(X^2) - E(X)^2 = 54.8333 - 49 = 5.8333$

6-3. Using the utility function $u(x) = \ln(x+1)$ we have the following:

Method 1

Wealth w	0	10,000
u(w) = ln(w+1)	0	ln(10,001)
p(w)	.10	.90

Method 2

Wealth w	0	9,025
u(w) = ln(w+1)	0	ln(9,026)
p(w)	.02	.98

$E[u(W_1)] = .10(0) + .90\ln(10{,}001) = 8.289$

$E[u(W_2)] = .02(0) + .98\ln(9{,}026) = 8.926$

Method 2 gives a higher expected utility.

6-4. This is primarily an exercise for the reader to observe the precision of his or her particular computer or calculator.

$$E(X) = 1{,}000{,}000{,}000 = \mu$$
$$V(X) = E[(X - \mu)^2] = (1/3)(.01 + 0 + .01) = .006\bar{6}$$

The value you get using $V(X) = E(X^2) - E(X)^2$ will vary from calculator to calculator.

6-5. Let X be the binomial random variable with n trials and $P(S) = p$.

$M_X(t) = E(e^{tX}) = \Sigma e^{tx}p(x) = \Sigma e^{tx}C(n,x)p^xq^{n-x} = \Sigma C(n,x)(e^tp)^xq^{n-x}$

$(q + p)^n = \Sigma C(n,x)p^x q^{n-x}$

If we replace p with pe^t, we get $M_X(t) = (q + pe^t)^n$.

6-6. For the Poisson random variable with rate λ, $M_X(t) = e^{\lambda(e^t-1)}$.

$$M'_X(t) = \lambda e^t e^{\lambda(e^t-1)}, \quad M'_X(0) = \lambda(1)(1) = \lambda = E(X)$$

$$M''_X(t) = e^{\lambda(e^t-1)}(\lambda e^t + (\lambda e^t)^2), \quad M''_X(0) = \lambda + \lambda^2 = E(X^2)$$

$$V(X) = E(X^2) - E(X)^2 = \lambda$$

6-7. For the geometric random variable with $P(S) = p$,

$$M_X(t) = \frac{p}{1-qe^t}, \text{ and } M'_X(t) = \frac{-p(-qe^t)}{(1-qe^t)^2}.$$

$$E(X) = M'_X(0) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

$$M''_X(t) = \frac{pqe^t(1-qe^t) + 2pq^2e^{2t}}{(1-qe^t)^3}$$

$$E(X^2) = M''_X(0) = \frac{pq(1-q) + 2pq^2}{(1-q)^3} = \frac{p^2q + 2pq^2}{p^3}$$

$$= \frac{pq + 2q^2}{p^2}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{pq + 2q^2}{p^2} - \frac{q^2}{p^2} = \frac{pq + q^2}{p^2}$$

$$= \frac{q(p+q)}{p^2} = \frac{q}{p^2}$$

6-8. For the negative binomial random variable with $P(S) = p$ and X the number of failures before r successes,

$$M_X(t) = \frac{p^r}{(1-qe^t)^r}.$$

$$M'_X(t) = \frac{rp^r qe^t}{(1-qe^t)^{r+1}}$$

$$E(X) = M'_X(0) = \frac{rp^r q}{(1-q)^{r+1}} = \frac{rp^r q}{p^{r+1}} = \frac{rq}{p}$$

$$M''_X(t) = \frac{rp^r qe^t(1-qe^t) + r(r+1)p^r q^2 e^{2t}}{(1-qe^t)^{r+2}}$$

$$E(X^2) = M''_X(0) = \frac{rp^r q(1-q) + r(r+1)p^r q^2}{(1-q)^{r+2}}$$

$$= \frac{rpq + r(r+1)q^2}{p^2}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{rpq + r(r+1)q^2}{p^2} - \frac{r^2q^2}{p^2}$$

$$= \frac{rpq + rq^2}{p^2} = \frac{rq(p+q)}{p^2} = \frac{rq}{p^2}$$

6-9. For the discrete uniform random variable, $p(x) = 1/n$, for $x = 1, 2, \ldots, n$.

(a) $$M_X(t) = \sum_{x=1}^{n} e^{xt} p(t) = \frac{1}{n}\sum_{x=1}^{n} e^{xt} = \frac{e^t(1-e^{nt})}{n(1-e^t)}, \quad t \neq 0$$

(b) $$M'_X(t) = \frac{1}{n}\sum_{x=1}^{n} xe^{xt}$$

$$E(X) = M'_X(0) = \frac{1}{n}\sum_{x=1}^{n} x = \frac{1}{n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2}$$

$$M''_X(t) = \sum_{x=1}^{n} x^2 e^{xt}$$

$$E(X^2) = M''_X(0) = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n^2-1}{12}$$

6-10. $M_X(t) = \Sigma e^{xt}p(x) = .42 + .30e^t + .17e^{2t} + .11e^{3t}$

$$M'_X(t) = .30e^t + .17(2)e^{2t} + .11(3)e^{3t}$$
$$E(X) = M'_X(0) = .30 + .34 + .33 = .97$$

$$M''_X(t) = .30e^t + .17(4)e^{2t} + .11(9)e^{3t}$$
$$E(X^2) = M''_X(0) = .30 + .68 + .99 = 1.97$$

6-11. $M_X(at) = E(e^{atX}) = \Sigma e^{atx}p(x)$

$$M_{aX+b}(t) = E(e^{(aX+b)t}) = \Sigma e^{axt+bt}p(x) = e^{bt}\Sigma e^{axt}p(x)$$

$$= e^{bt}M_X(at)$$

6-12. For the binomial random variable with n trials and $P(S) = p$, $M_X(t) = (q+pe^t)^n$.

If $n = 8$ and $p = .6$, then $M_X(t) = (.4 + .6e^t)^8$

If $Y = 3X + 4$, then $M_Y(t) = e^{4t}(.4 + .6e^{3t})^8$

6-13. The moment generating function for the negative binomial distribution is $M_X(t) = \left(\frac{p}{1-qe^t}\right)^r$.

Therefore $\left(\frac{.70}{1-.3e^t}\right)^5$ is the moment generating of the negative binomial distribution with $r = 5$ and $p = .70$.

6-14. Successive applications of the linear congruence $y = 9x + 11 \pmod{16}$ yields the following table.

k	x_k	$9x_k + 11$	$9x_k + 11$ (mod16)
1	6	65	1
2	1	20	4
3	4	47	15
4	15	146	2
5	2	29	13
6	13	128	0
7	0	11	11
8	11	110	14
9	14	137	9
10	9	92	12
11	12	119	7
12	7	74	10
13	10	101	5
14	5	56	8
15	8	83	3
16	3	38	6

The following table is for the simulations in Exercises 6-15 and 6-16. In Exercise 6-15, if the random number $x < .40$, the result was a success. Otherwise it was a failure. In Exercise 6-16, if the random number $x < .50$, the result was a head. Otherwise it was a tail.

Trial	Random Number	S or F	H or T
1	.5619	F	T
2	.4500	F	H
3	.3566	S	H
4	.5844	F	T
5	.8638	F	T
6	.9983	F	T
7	.0225	S	H
8	.8026	F	T
9	.3516	S	H
10	.4584	F	H
11	.7855	F	T
12	.9955	F	T
13	.6558	F	T
14	.1280	S	H
15	.3908	S	H
16	.3729	S	H
17	.1326	S	H
18	.9246	F	T
19	.6867	F	T
20	.9638	F	T

6-15. In these 20 trials there are 7 successes.

6-16. The first trial (first set of 5 numbers) yields 2 heads, the second trial yields 3 heads, the third trial yields 2 heads, and the fourth trial yields 2 heads.

CHAPTER 6

6-1. (a) 250 (b) 0.6 (c) 1.06

6-2. 5.8333

6-3. $E[u(W_1)] = 8.289$; $E[u(W_2)] = 8.926$

6-9. (b) $E(X) = (n+1)/2$; $V(X) = (n^2-1)/12$

6-10. $M_X(t) = .42 + .30e^t + .17e^{2t} + .11e^{3t}$;
$E(X) = .97$; $E(X^2) = 1.97$

6-12. $e^{4t}(.4 + .6e^{3t})^8$

6-13. Negative binomial with $p = .7$ and $r = 5$

6-14. 1, 4, 15, 2, 13, 0, 11, 14, 9, 12, 7, 10, 5, 8, 3

6-15. 7

6-16. 2, 3, 2, 2

6-17. 698.9

6-18. $\frac{19}{27} + \frac{8}{27}e^t$

Chapter 7
Continuous Random Variables

7.4 Exercises

7.1 Defining a Continuous Random Variable

7-1. Let $f(x) = 1.5x + .25$, for $0 \leq x \leq 1$, and $f(x) = 0$ elsewhere.
(a) Show that $f(x)$ is a probability density function.
(b) What is the cumulative distribution function?
(c) Find $P(0 \leq X \leq \frac{1}{2})$ and $P(\frac{1}{4} \leq X \leq \frac{3}{4})$.

7-2. Let $f(x) = a(e^{-2x} - e^{-3x})$, for $x \geq 0$, and $f(x) = 0$ elsewhere.
(a) Find a so that $f(x)$ is a probability density function.
(b) What is $P(X \leq 1)$?

7-3 Let

$$f(x) = \begin{cases} 25x & 0 \leq x \leq .20 \\ 1.5625(1-x) & .20 < x \leq 1. \\ 0 & \text{elsewhere} \end{cases}$$

Find $P(.10 \leq X \leq .60)$.

7-4. Let $f(x) = a/(1 + x^2)$, for $x \geq 0$, and $f(x) = 0$ elsewhere.
(a) Find a so that $f(x)$ is a probability density function.
(b) What is $P(X \leq 1)$?

7.2 The Mode, the Median, and Percentiles

7-5. For the density function in Exercise 7-1, find $x_{.25}$, $x_{.50}$ and $x_{.75}$.

7-6. Let $f(x) = e^x$, for $0 \leq x \leq ln2$, and $f(x) = 0$ elsewhere.
(a) Find $x_{.50}$ and $x_{.90}$.
(b) What is the mode of this distribution?

7-7. For the density function in Exercise 7-3, find the median and $x_{.80}$.

7.3 The Mean and Variance of a Continuous Random Variable

7-8. If X is the random variable whose density function is defined in Exercise 7-1, what are $E(X)$ and $V(X)$?

7-9. If X is the random variable whose density function is defined in Exercise 7-3, what is $E(X)$?

7-10. For the random variable in Example 7.1 whose density function is $f(x)=.75(1-x^2)$, for $-1 \le x \le 1$, and $f(x)=0$ elsewhere, show that both the mean and the median are equal to 0.

7-11. Let X be a random variable whose density function is $\frac{2}{\pi(1+x^2)}$, for $x \ge 0$, and 0 elsewhere (Exercise 7-4). Show that $E(X)$ does not exist.

7.5 Sample Actuarial Examination Problems

7-12. The lifetime of a machine part has a continuous distribution on the interval $(0,40)$ with probability density function f, where $f(x)$ is proportional to $(10+x)^{-2}$.

Calculate the probability that the lifetime of the machine part is less than 6.

7-13. An insurer's annual weather-related loss, X, is a random variable with density function

$$f(x)=\begin{cases} \dfrac{2.5(200)^{2.5}}{x^{3.5}} & \text{for } x>200 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the difference between the 30th and 70th percentiles of X.

7-14. An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1+x)^{-4}$ where $0 < x < \infty$.

Determine the company's expected monthly claims.

7-15. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \dfrac{|x|}{10} & \text{for} \quad -2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the expected value of X.

7-16. The loss due to a fire in a commercial building is modeled by a random variable X with density function

$$f(x) = \begin{cases} .005(20-x) & \text{for} \quad 0 < x < 20 \\ 0 & \text{otherwise} \end{cases}$$

Given that a fire loss exceeds 8, what is the probability that it exceeds 16?

7-17. An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} 3x^{-4} & \text{for} \quad x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, what is the probability that it is insured for less than 2?

CHAPTER 7

7-1. The function $f(x) = 1.5x + .25$, for $0 \le x \le 1$, and 0 elsewhere.

(a) Clearly $f(x) \ge 0$ for all x, so we only need to show that the area under the curve is 1.

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 (1.5x + .25)dx = (.75x^2 + .25x)\Big|_0^1$$
$$= .75 + .25 = 1$$

Hence f(x) is a probability density function.

(b) $F(x) = P(X \le x) = \int_0^x (1.5t + .25)dt = (.75x^2 + .25x)$ for $0 \le x \le 1$. $F(x) = 0$ for $x < 0$, and $F(x) = 1$ for $x > 1$.

(c) $P(0 \le X \le 1/2) = F(.5) - F(0) = [.75(.5)^2 + .25(.5)] - [0 + 0]$
$= .3125$

$P(1/4 \le X \le 3/4) = F(.75) - F(.25)$
$= [.75(.75)^2 + .25(.75)] - [.75(.25)^2 + .25(.25)]$
$= .609375 - .109375 = .50$

7-2. (a) We need to find a so that the area under the curve is 1.

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} a(e^{-2x} - e^{-3x})dx = a[-(1/2)e^{-2x} + (1/3)e^{-3x}]\Big|_0^{\infty}$$
$$= a(0) - a(-1/2 + 1/3) = a/6$$

For this to be a probability density function $a/6 = 1$, and $a = 6$.

(b) $P(X \le 1) = 6\int_0^1 (e^{-2x} - e^{-3x})dx = (-3e^{-2x} + 2e^{-3x})\Big|_0^1$
$= (-3e^{-2} + 2e^{-3}) - (-3 + 2) = -.3064 + 1 = .6936$

7-3. $P(.10 \le X \le .60) = \int_{.1}^{.6} f(x)dx = \int_{.1}^{.2} 25x dx + \int_{.2}^{.6} 1.5625(1 - x)dx$
$= 12.5x^2\Big|_{.1}^{.2} - .78125(1 - x)^2\Big|_{.2}^{.6}$
$= .375 + .375 = .75$

7-4. (a) $\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} \frac{a}{1 + x^2} = a\tan^{-1}(x)\Big|_0^{\infty} = a(\pi/2)$

For this to be a probability density function, $a(\pi/2)$ must equal 1, so $a = 2/\pi$.

(b) $P(X \le 1) = (2/\pi)\int_0^1 \frac{1}{1 + x^2}dx = (2/\pi)[\tan^{-1}(1) - \tan^{-1}(0)]$
$= (2/\pi)(\pi/4 - 0) = 1/2$

7-5. For the density function in Exercise 7-1, $F(x) = .75x^2 + .25x$. To find x_p we need to solve $F(x) = p$.

For $p = .25$ we have $.75x^2 + .25x = .25$, or $3x^2 + x = 1$. The positive solution of this equation is $x = .4343 = x_{.25}$.

For $p = .50$ we have $.75x^2 + .25x = .5$, or $3x^2 + x = 2$. The positive solution of this equation is $x = 2/3 = x_{.50}$.

For $p = .75$ we have $.75x^2 + .25x = .75$, or $3x^2 + x = 3$. The positive solution of the equation is $x = .8471 = x_{.75}$.

7-6. If $f(x) = e^x$ for $0 \le x \le \ln 2$, then

$F(x) = \int_0^x e^t dt = e^x - 1$.

(a) Solving $F(x) = p$ when $p = .50$ we get

$e^x - 1 = .50$, or $e^x = 1.5$.

The solution of this equation is $x = \ln 1.5 = .4055 = x_{.50}$.

Solving $F(x) = p$ when $p = .90$ we get

$e^x - 1 = .90$, or $e^x = 1.9$.

The solution of this equation is $x = \ln 1.9 = .6419 = x_{.90}$.

(b) The mode of this distribution is the value of x for which the density function, f(x), is a maximum. Since e^x is increasing, the maximum occurs at the right hand endpoint, $x = \ln 2$.

7-7. For the density function in Exercise 7-3, if $0 \le x \le .2$, then

$F(x) = \int_0^x 25t dt = 12.5x^2$, and $F(.2) = .50$.

Hence the median is .20.

For $.2 < x \le 1$, $F(x) = .5 + 1.5625\int_{.2}^{x}(1 - t)dt$
$= .5 + 1.5625(t - t^2/2)\Big|_{.2}^{x}$
$= .5 + 1.5625(x - x^2/2 - .18)$
$= .5 + 1.5625x - .78125x^2 - .28125.$

Solving $F(x) = .8$ we get

$.78125x^2 - 1.5625x + .58125 = 0.$

The solution to this equation in $(.2,1)$ is $x = .4940 = x_{.80}$.

7-8. For the density function in Exercise 7-1:

$E(X) = \int_0^1 x(1.5x + .25)dx = \int_0^1 (1.5x^2 + .25x)dx$
$= (.5x^3 + .125x^2)\Big|_0^1 = .625$

$E(X^2) = \int_0^1 x^2(1.5x + .25)dx = \int_0^1 (1.5x^3 + .25x^2)dx$
$= [(1.5/4)x^4 + (.25/3)x^3]\Big|_0^1 = .45833$

$V(X) = E(X^2) - E(X)^2 = .45833 - .625^2 = .0677$

7-9. $E(X) = \int_0^{.2} x(25x)dx + \int_{.2}^{1} 1.5625x(1-x)dx$

$$= \int_0^{.2} 25x^2 dx + 1.5625\int_{.2}^{1}(x - x^2)dx$$

$$= (25/3)x^3\Big|_0^{.2} + 1.5625(x^2/2 - x^3/3)\Big|_{.2}^{1}$$

$$= .066\bar{6} + .233\bar{3} = .30$$

7-10. For the density function $f(x) = .75(1 - x^2)$ if $-1 \le x \le 1$ and 0 elsewhere,

$$E(X) = .75\int_{-1}^{1} x(1 - x^2)dx = .75(x^2/2 - x^4/4)\Big|_{-1}^{1} = 0.$$

$$F(x) = \int_{-1}^{x} .75(1 - t^2)dt = .75(t - t^3/3)\Big|_{-1}^{x}$$
$$= .75x - .25x^3 + .50$$

To find the median we have to solve $F(x) = .50$.

$$.75x - .25x^3 + .50 = .50$$
$$.25(3x - x^3) = 0$$

The only solution in [-1, 1] is 0, so the median is 0, and the mean and the median are the same.

7-11. For the density function $f(x) = \dfrac{2}{\pi(1 + x^2)}$ for $x \ge 0$,

$$E(X) = \frac{2}{\pi}\int_0^{\infty}\frac{x}{1 + x^2}dx = (1/\pi)\ln(1 + x^2)\Big|_0^{\infty}.$$

This does not have a finite value, so E(X) does not exist.

CHAPTER 7

7-1. (b) $F(X) = 0$ for $x < 0$, $.75x^2 + .25x$ for $0 \leq x \leq 1$, and 1 for $x > 1$ (c) $P(0 \leq X \leq 1/2) = .3125$; $P(1/4 \leq X \leq 3/4) = .50$

7-2. (a) 6 (b) .6936

7-3. .75

7-4. (a) $2/\pi$ (b) 1/2

7-5. .4343; 2/3; .8471

7-6. (a) .4055; .6419 (b) $ln2$

7-7. .20; .4940

7-8. .625; .0677

7-9. 0.3

7-12. .46875

7-13. 93.06

7-14. 1/2

7-15. 28/15

7-16. 1/9

7-17. .57813

Chapter 8
Commonly Used Continuous Distributions

8.10 Exercises

8.1 The Uniform Distribution

8-1. Derive Equation (8.5b).

8-2. If T is the random variable in Example 8.3 whose distribution is uniform on $[0, 100]$, find $E(T)$ and $V(T)$.

8-3. In a hospital the time of birth of a baby within an hour interval (e.g. between 5:00 and 6:00 in the morning) is uniformly distributed over that hour. What is the probability that a baby is born between 5:15 and 5:25, given that it was born between 5:00 and 6:00?

8-4. On a large construction site the lengths of pieces of lumber are rounded off to the nearest centimeter. Let X be the rounding error random variable (the actual length of a piece of lumber minus the rounded-off value). Suppose that X is uniformly distributed over $[-.50, .50]$. Find (a) $P(-.10 \leq X \leq .20)$; (b) $V(X)$.

8-5. A professor gives a test to a large class. The time limit for the test is 50 minutes, and the first student to finish is done in 35 minutes. The professor assumes that the random variable T for the time it takes a student to finish the test is uniformly distributed over $[35, 50]$.
   (a) Find $E(T)$ and $V(T)$.
   (b) At what time T will 60 percent of the students be finished?

8-6. Let T be a random variable whose distribution is uniform on $[a, b]$ and $a \le c \le d \le b$. Suppose you are given that the value of T falls in the interval $[c, d]$. Let Y be the conditional random variable for those values of T that are in $[c, d]$. Show that the distribution of Y is uniform over $[c, d]$.

8-7. Suppose you consider the subset of the population in Example 8.3 who survive to age 40. If T is the random variable for the age at time of death of these survivors, T has a uniform distribution over $[40, 100]$.

(a) Find $E(T)$ and $V(T)$.

(b) What is $P(T > 57)$ for this group? (Compare this with the result in Example 8.3.)

8-8. For the population in Example 8.3 where the time until death random variable T is uniform over $[0, 100]$, consider a couple whose ages are 45 and 50. Assume that their deaths are independent events.

(a) What is the probability that they both live at least 20 more years?

(b) What is the probability that both die in the next 20 years?

8.2 The Exponential Distribution

8-9. Tests on a certain machine part have determined that the mean time until failure of this part is 500 hours. Assume that the time T until failure of this part is exponentially distributed.

(a) What is the probability that one of these parts will fail within 300 hours?

(b) What is the probability that one of these parts will still be working after 900 hours?

8-10. If T has an exponential distribution with parameter λ, what is the median of T?

8-11. For a certain population the time until death random variable T has an exponential distribution with mean 60 years.

(a) What is the probability that a member of this population will die by age 50?

(b) What is the probability that a member of this population will live to be 100?

8-12. If T is uniformly distributed over $[a, b]$, what is its failure rate?

8-13. Researchers at a medical facility have discovered a virus whose mean incubation period (time from being infected until symptoms appear) is 38 days. Assume the incubation period has an exponential distribution

(a) What is the probability that a patient who has just been infected will show symptoms in 25 days?

(b) What is the probability that a patient who has just been infected will not show symptoms for at least 30 days?

8-14. If T has an exponential distribution, show that $P[T \leq E(T)]$ is $F[E(T)] = 1 - e^{-1} \approx .632$.

8-15. A city engineer has studied the frequency of accidents at two busy intersections. He has determined that the time T in months between accidents at each intersection has an exponential distribution. The parameters for these two distributions are 2 and 2.5. Assume that the occurrence of accidents at these intersections is independent.

(a) What is the probability that there are no accidents at either intersection in the next month?

(b) What is the probability that there will be no accidents for at least one of these intersections in the next month?

8-16. If T has an exponential distribution with parameter .15, what are the 25^{th} and 75^{th} percentiles for T?

8-17. Using Equation (8.8) and integration by parts, derive the identity $\Gamma(n) = (n-1) \cdot \Gamma(n-1)$.

8-18. Let T be a random variable whose distribution is exponential with parameter λ. Show that $P(T \geq a + b | T \geq a) = P(T \geq b)$.

8-19. Consider the population in Exercise 8-11.

(a) What is the probability that a member of this population who lives to age 40 will die by age 50?

(b) What is the probability that a person who lives to age 40 will then live to age 100?

8.3 The Gamma Distribution

8-20. Using Equation (8.10) and the result in Exercise 8.17, show that the mean of the gamma distribution with parameters α and β is α/β.

8-21. Use Equation (8.10) and Exercise 8.17 to show if X has a gamma distribution with parameters α and β, then $E(X^2) = \alpha(\alpha + 1)/\beta^2$ and hence $V(X) = \alpha/\beta^2$.

8-22. At a dangerous intersection accidents occur at a rate of 2.5 per month, and the time between accidents is exponentially distributed. Let T be the random variable for the waiting time from the beginning of observation until the third accident. Find $E(T)$ and $V(T)$.

8-23. Suppose a company hires new people at a rate of 8 per year and the time between new hires is exponentially distributed. What are the mean and variance of the time until the company hires its 12^{th} new employee?

8-24. A gamma distribution has a mean of 18 and a variance of 27. What are α and β for this distribution?

8-25. A gamma distribution has parameters $\alpha = 2$ and $\beta = 3$. Find (a) $F(x)$; (b) $P(0 \le X \le 3)$; (c) $P(1 \le X \le 2)$.

8-26. The length of stay X in a hospital for a certain disease has a gamma distribution with parameters $\alpha = 2$ and $\beta = 1/3$. The cost of treatment in the hospital is $C = 500X + 50X^2$. What is the expected cost of a hospital treatment for this disease?

8.4 The Normal Distribution

8-27. Using the z-table in Appendix A, find the following probabilities:

(a) $P(-1.15 \le Z \le 1.56)$ (b) $P(0.15 \le Z \le 2.13)$
(c) $P(|Z| \le 1.0)$ (d) $P(|Z| \ge 1.65)$.

8-28. Using the z-tables in Appendix A, find the value of z that satisfies the following probabilities:

(a) $P(Z \leq z) = .8238$ (b) $P(Z \leq z) = .0287$
(c) $P(Z \geq z) = .9115$ (d) $P(Z \geq z) = .1660$
(e) $P(|Z| \geq z) = .10$ (f) $P(|Z| \leq z) = .95$

8-29. Let z be the standard normal random variable. If $z > 0$ and $F_Z(z) = \alpha$, what are $F_Z(-z)$ and $P(-z \leq Z \leq z)$?

8-30. If X is a normal random variable with a mean of 17.1 and a standard deviation of 3.2, what is $P(14 \leq X \leq 25)$?

8-31. An insurance company has 5000 policies and assumes these policies are all independent. Each policy is governed by the same distribution with a mean of \$495 and a variance of \$30,000. What is the probability that the total claims for the year will be less than \$2,500,000?

8-32. A company manufactures engines. Specifications require that the length of a certain rod in this engine be between 7.48 cm. and 7.52 cm. The lengths of the rods produced by their supplier have a normal distribution with a mean of 7.505 cm. and a standard deviation of .01 cm.

(a) What is the probability that one of these rods meets these specifications?
(b) If a worker selects 4 of these rods at random, what is the probability that at least 3 of them meet these specifications?

8-33. The lifetimes of light bulbs produced by a company are normally distributed with mean 1500 hours and standard deviation 125 hours.

(a) What is the probability that a bulb will last at least 1400 hours?
(b) If 3 new bulbs are installed at the same time, what is the probability that they will all still be burning after 1400 hours?

8-34. If a number is selected at random from the interval $[0, 1]$, its value has a uniform distribution over that interval. Let S be the random variable for the sum of 50 numbers selected at random from $[0, 1]$. What is $P(24 \le S \le 27)$?

8-35. Let X have a normal distribution with mean 25 and unknown standard deviation. If $P(X \le 29.9) = .9192$, what is σ?

8.5 The Lognormal Distribution

8-36. If $Y = e^X$, where X is a normal random variable with $\mu = 5$ and $\sigma = .40$, what are $E(Y)$ and $V(Y)$?

8-37. If Y is lognormal and X, the normally distributed exponent, has parameters $\mu = 5.2$ and $\sigma = .80$, what is $P(100 \le Y \le 500)$?

8-38. The claim severity random variable for an insurance company is lognormal, and the normally distributed exponent has mean 6.8 and standard deviation 0.6. What is the probability that a claim is greater than \$1750?

8-39. If Y is a lognormal random variable, and the normally distributed exponent has parameters μ and σ, what is the median of Y?

8-40. For the stock in Example 8.24, whose value in one year is $Y = 100e^X$ where X is normal with parameters $\mu = .10$ and $\sigma = .03$, what is the probability that the value of the stock in one year will be (a) greater than 112.50; (b) less than 107.50.

8-41. If $Y = e^X$ is a lognormal random variable with $E(Y) = 2{,}500$ and $V(Y) = 1{,}000{,}000$, what are the parameters μ and σ for X?

8.6 The Pareto Distribution

8-42. Let X be the Pareto random variable with parameters α and β, $\alpha > 2$ and $x \geq \beta > 0$.

(a) Verify that $F(x) = 1 - (\beta/x)^\alpha$.
(b) Verify that $E(X) = \alpha\beta/(\alpha - 1)$.
(c) Verify that $E(X^2) = \alpha\beta^2/(\alpha - 2)$, and use this result to obtain $V(X)$.

8-43. For the Pareto random variable with $\alpha = 3.5$ and $\beta = 4$, find (a) $E(X)$; (b) $V(X)$; (c) the median of X; (d) $P(6 \leq X \leq 12)$.

8-44. A comprehensive insurance policy on commercial trucks has a deductible of \$500. The random variable for the loss amount (before deductible) on claims filed has a Pareto distribution with a failure rate of $3.5/x$ (x measured in hundreds of dollars). Find (a) the mean loss amount; (b) the expected value of the amount paid on a single claim; and (c) the variance of the amount of a single loss.

8.7 The Weibull Distribution

8-45. It can be shown (although beyond the scope of this text) that $\Gamma(1/2) = \pi^{1/2}$. Using this and the result of Exercise 8-17, find (a) $\Gamma(3/2)$; (b) $\Gamma(5/2)$; (c) $\Gamma(7/2)$. (Can you see a pattern?)

8-46. Let X be the Weibull random variable with $\alpha = 3$ and $\beta = 3.5$. Find (a) $P(X \leq 0.4)$; (b) $P(X > 0.8)$.

8-47. What is the failure rate for the random variable in Exercise 8-46?

8-48. For the Weibull random variable X with $\alpha = 2$ and $\beta = 3.5$, find (a) $E(X)$; (b) $V(X)$; (c) $P(.25 \leq X \leq .75)$.

8-49. Using Equation (8.10), verify that the mean of a Weibull distribution is $\Gamma(1 + 1/\alpha)/\beta^{1/\alpha}$. (Hint: Transform the integral using the substitution $u = x^\alpha$.)

8.8 The Beta Distribution

8-50. Find the density function for the beta distribution with $\alpha = 4$ and $\beta = 1.5$. (Hint: Use the results of Exercise 8.17.)

8-51. Find the value of k so that $f(x) = kx^4(1-x)^2$ for $0 \le x \le 1$ is a beta density function.

8-52. A meter measuring the volume of a liquid put into a bottle has an accuracy of $\pm$ 1 cm^3. The absolute value of the error has a beta distribution with $\alpha = 3$ and $\beta = 2$. What are the mean and variance for this error?

8-53. In Exercise 8-52, what is the probability that the error is no more than $0.5cm^3$?

8-54. A company markets a new product and surveys customers on their satisfaction with this product. The fraction of customers who are dissatisfied has a beta distribution with $\alpha = 2$ and $\beta = 4$. What is the probability that no more than 30 percent of the customers are dissatisfied?

8-55. Using Equation (8.33), verify that the mean of the beta distribution is $\alpha/(\alpha+\beta)$.

8.11 Sample Actuarial Examination Problems

8-56. The time to failure of a component in an electronic device has an exponential distribution with a median of four hours.

Calculate the probability that the component will work without failing for at least five hours.

8-57. The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with 6 years and 3 years, respectively.

What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

8-58. The lifetime of a printer costing 200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year.

If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

8-59. The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year.

What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?

8-60. An insurance policy reimburses dental expense, X, up to a maximum benefit of 250. The probability density function for X is:

$$f(x) = \begin{cases} ce^{-0.004x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

Calculate the median benefit for this policy.

8-61. You are given the following information about N, the annual number of claims for a randomly selected insured:

$$P(N=0) = \frac{1}{2} \qquad P(N=1) = \frac{1}{3} \qquad P(N>1) = \frac{1}{6}$$

Let S denote the total annual claim amount for an insured. When $N=1$, S is exponentially distributed with mean 5. When $N>1$, S is exponentially distributed with mean 8.

Determine $P(4 < S < 8)$.

8-62. An insurance company issues 1250 vision care insurance policies. The number of claims filed by a policyholder under a vision care insurance policy during one year is a Poisson random variable with mean 2. Assume the numbers of claims filed by distinct policyholders are independent of one another.

What is the approximate probability that there is a total of between 2450 and 2600 claims during a one-year period?

8-63. The total claim amount for a health insurance policy follows a distribution with density function

$$f(x) = \frac{1}{1000} e^{-\frac{x}{1000}} \quad \text{for} \quad x \geq 0.$$

The premium for the policy is set at 100 over the expected total claim amount.

If 100 policies are sold, what is the approximate probability that the insurance company will have claims exceeding the premiums collected?

8-64. A city has just added 100 new female recruits to its police force. The city will provide a pension to each new hire who remains with the force until retirement. In addition, if the new hire is married at the time of her retirement, a second pension will be provided for her husband. A consulting actuary makes the following assumptions:

(i) Each new recruit has a 0.4 probability of remaining with the police force until retirement.

(ii) Given that a new recruit reaches retirement with the police force, the probability that she is not married at the time of retirement is 0.25.

(iii) The number of pensions that the city will provide on behalf of each new hire is independent of the number of pensions it will provide on behalf of any other new hire.

Determine the probability that the city will provide at most 90 pensions to the 100 new hires and their husbands.

8-65. In an analysis of healthcare data, ages have been rounded to the nearest multiple of 5 years. The difference between the true age and the rounded age is assumed to be uniformly distributed on the interval from –2.5 years to 2.5 years. The healthcare data are based on a random sample of 48 people.

What is the approximate probability that the mean of the rounded ages is within 0.25 years of the mean of the true ages?

8-66. A charity receives 2025 contributions. Contributions are assumed to be independent and identically distributed with mean 3125 and standard deviation 250.

Calculate the approximate 90^{th} percentile for the distribution of the total contributions received.

CHAPTER 8

8-1. For the uniform distribution on [a, b], $f(x) = 1/(b - a)$.

$$E(X^2) = \int_a^b \frac{x^2}{b-a}dx = \frac{b^3 - a^3}{3(b-a)} = (1/3)(b^2 + ab + a^2)$$

$$\begin{aligned} V(X) &= E(X^2) - E(X)^2 \\ &= (1/3)(b^2 + ab + a^2) - (1/4)(b^2 + 2ab + a^2) \\ &= (b^2 - 2ab + a^2)/12 = (b - a)^2/12 \end{aligned}$$

8-2. If T is uniformly distributed on [0,100]:

$$E(T) = (100 + 0)/2 = 50$$

$$V(T) = (100 - 0)^2/12 = 833.33$$

8-3. Let T be the time (in minutes from 5:00) that the baby is born; then T is uniformly distributed on [0,60].

$$F(t) = P(T \le t) = t/60$$

$$P(15 \le T \le 25) = F(25) - F(15) = 25/60 - 15/60 = 1/6$$

8-4. Since X is uniform on [–.50, .50],

$$F(x) = \frac{x - (-.50)}{.50 - (-.50)} = x + .50\,.$$

(a) $P(-.10 \le X \le .20) = F(.20) - F(-.10) = .70 - .40 = .30$

(b) $V(X) = [.50 - (-.50)]^2/12 = 1/12$

8-5. (a) T is uniform on (35,50), so

$$E(T) = (50 + 35)/2 = 42.5$$

$$V(T) = (50 - 35)^2/12 = 15^2/12 = 18.75$$

(b) The time at which 60 percent will be finished, t_{60}, is the solution of $F(t) = .60$.

$F(t) = (t - 35)/15$, so we solve
$(t - 35)/15 = .60$
$t = 44$.

8-6. For y in [c,d] and T uniform on [a,b]:

$$P(Y \le y) = P(T \le y \mid c \le T \le b) = \frac{P(c \le T \le y)}{P(c \le T \le d)} = \frac{(y-c)/(b-a)}{(d-c)/(b-a)} = \frac{y-c}{d-c}$$

Hence Y has a uniform distribution over [c,d].

8-7. (a) Since T is uniform on [40,100] we have

$$E(T) = (100 + 40)/2 = 70$$

$$V(T) = (100 - 40)^2/12 = 300$$

(b) $P(T > 57) = 1 - P(T \le 57) = 1 - 17/60 = .7167$

8-8. Let X and Y be the random variables of the ages at time of death for the 45-year-old and the 50-year-old, respectively. Then X is uniform on [45,100] and Y is uniform on [50,100].

The probability that the 45-year-old dies in the next 20 years is

$$P(X \le 65) = (65 - 45)/(100 - 45) = 20/55 = 4/11.$$

The probability that the 50-year-old dies in the next 20 years is

$$P(Y \le 70) = 20/50 = 2/5.$$

(a) $P(X > 65 \text{ and } Y > 70) = (1 - 4/11)(1 - 2/5) = .3818$

(b) $P(X \le 65 \text{ and } Y \le 70) = (4/11)(2/5) = .1455$

8-9. T has an exponential distribution with mean $1/\lambda = 500$, or $\lambda = .002$.

$$P(T \le t) = F(t) = 1 - e^{-.002t}$$
$$P(T > t) = S(t) = e^{-.002t}$$

(a) $P(T \le 300) = 1 - e^{-.002(300)} = 1 - e^{-.6} = .4512$

(b) $P(T > 900) = e^{-.002(900)} = e^{-1.8} = .1653$

8-10. To find the median of the exponential distribution we solve

$F(t) = 1 - e^{-\lambda t} = .50$, or $e^{-\lambda t} = .50$.
Then $-\lambda t = \ln(.5) = -\ln 2$, and the median is $(1/\lambda)\ln 2$.

8-11. For the exponential distribution with mean 60, $\lambda = 1/60$.

(a) $P(T \le 50) = 1 - e^{-50/60} = .5654$

(b) $P(T > 100) = e^{-100/60} = .1889$

8-12. For the uniform distribution on [a,b],

$F(t) = 1/(b - a)$ and $S(t) = (b - t)/(b - a)$.

The failure rate is

$$\lambda(t) = \frac{f(t)}{S(t)} = \frac{1}{b-a} \div \frac{b-t}{b-a} = \frac{1}{b-t}.$$

8-13. Let T be the random variable of the incubation period. T is exponential with $\lambda = 1/38$.

(a) $P(T \le 25) = F(25) = 1 - e^{-25/38} = .4821$

(b) $P(T > 30) = S(30) = e^{-30/38} = .4541$

8-14. Let T have an exponential distribution with parameter λ. Then $E(T) = 1/\lambda$.

$$F(t) = 1 - e^{-\lambda t}$$
$$F[E(T)] = F(1/\lambda) = 1 - e^{-\lambda(1/\lambda)} = 1 - e^{-1} \approx .632$$

8-15. Let X and Y be the random variables for the time between accidents at these intersections. Each is exponentially distributed with parameters 2 and 2.5, respectively. Let S_1 and S_2 be their respective survival functions. No accident in a month at an intersection means X (or Y) > 1.

(a) $P(X > 1 \text{ and } Y > 1) = S_1(1)S_2(1) = (e^{-2})(e^{-2.5}) = .0111$

(b) $P(X > 1 \text{ or } Y > 1) = S_1(1) + S_2(1) - S_1(1)S_2(1)$
$= e^{-2} + e^{-2.5} - (e^{-2})(e^{-2.5}) = .2063$

8-16. If T has an exponential distribution with $\lambda = .15$, then $F(t) = 1 - e^{-.15t}$. To find t_p, solve $F(t) = p$.

If $p = .25$, $1 - e^{-.15t} = .25$
$e^{-.15t} = .75$
$-.15t = \ln .75$
$t_{.25} = 1.9179$

If $p = .75$, $1 - e^{-.15t} = .75$
$e^{-.15t} = .25$
$-.15t = \ln .25$
$t_{.75} = 9.2420$

8-17. $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ (See Equation 8.8)

let $u = x^{n-1}$ $\quad dv = e^{-x}dx$
$du = (n-1)x^{n-2}dx$ $\quad v = -e^{-x}$

$$\Gamma(n) = -e^{-x}x^{n-1}\Big|_0^\infty + (n-1)\int_0^\infty x^{n-2}e^{-x}dx$$

If $n > 1$, then $-e^{-x}x^{n-1}\Big|_0^\infty = 0$. (See Equation 8.6)

$$\Gamma(n) = (n-1)\int_0^\infty x^{(n-1)-1}e^{-x}dx = (n-1)\Gamma(n-1)$$

8-18. If T is an exponential random variable with parameter λ, then

$$P(T \ge a + b \mid T \ge a) = \frac{P(T \ge a + b)}{P(T \ge a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b} = P(T \ge b).$$

8-19. For the population in Exercise 8-11, T (the time until death) was exponential with $\lambda = 1/60$.

(a) $P(T \le 50 \mid T \ge 40) = 1 - P(T \ge 50 \mid T \ge 40)$
$= 1 - P(T \ge 10) = 1 - e^{-10/60} = .1535$

(b) $P(T \ge 100 \mid T \ge 40) = P(T \ge 60) = e^{-60/60} = .3679$

8-20. For the gamma distribution $f(x) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

and $\int_0^\infty x^n e^{-ax} dx = \dfrac{\Gamma(n+1)}{a^{n+1}}$.

$$E(X) = \int_0^\infty x f(x) dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\beta x} dx$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\alpha\Gamma(\alpha)}{\beta\Gamma(\alpha)} = \alpha/\beta$$

8-21. $E(X^2) = \int_0^\infty x^2 f(x) dx = \dfrac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\beta x} dx$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} = \frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)\beta^2}$$
$= (\alpha+1)\alpha/\beta^2$

$V(X) = E(X^2) - E(X)^2 = \alpha(\alpha+1)/\beta^2 - \alpha^2/\beta^2 = \alpha/\beta^2$

8-22 If the accidents occur at the rate of 2.5 per month, the waiting time between accidents is exponential with $\beta = 2.5$. The waiting time from the beginning of observation until the third accident, T, has a gamma distribution with $\alpha = 3$ and $\beta = 2.5$.

$E(T) = \alpha/\beta = 3/2.5 = 1.2$
$V(T) = \alpha/\beta^2 = 3/(2.5)^2 = .48$

8-27. (a) $F_Z(1.56) - F_Z(-1.15) = .9406 - .1251 = .8155$

(b) $F_Z(2.13) - F_Z(0.15) = .9834 - .5596 = .4238$

(c) $F_Z(1.0) - F_Z(-1.0) = .8413 - .1587 = .6826$

(d) $1 - F_Z(1.65) + F_Z(-1.65) = 1 - .9505 + .0495 = .0990$

8-28. (a) $F_Z(z) = .8238$, so $z = 0.93$

(b) $F_Z(z) = .0287$, so $z = -1.90$

(c) $1 - F_Z(z) = .9115$, so $F_Z(z) = .0885$ and $z = -1.35$

(d) $1 - F_Z(z) = .1660$, so $F_Z(z) = .8340$ and $z = 0.97$

(e) By symmetry of the standard normal density function, $P(Z \ge z) = P(Z \le -z)$.

$P(|Z| \ge z) = P(Z \ge z) + P(Z \le -z) = 2P(Z \ge z)$
$= 2(1 - F_Z(z)) = .10$

$F_Z(z) = .9500$, so $z = 1.645$
(Note that .9500 falls between two z-values, and we haven't done any interpolation. But this is a very important z-value to know.)

(f) $P(|Z| \le z) = 1 - P(|Z| \ge z) = 1 - 2(1 - F_Z(z))$
$= 2F_Z(z) - 1 = .95$
$F_Z(z) = .9750$, so $z = 1.96$

8-29. Let $z > 0$ and $F_Z(z) = \alpha = P(Z \le z)$. By symmetry of standard density normal density function:

$F_Z(-z) = P(Z \le -z) = P(Z \ge z) = 1 - P(Z \le z) = 1 - \alpha$

$P(-z \le Z \le z) = P(Z \le z) - P(Z \le -z) = \alpha - (1 - \alpha) = 2\alpha - 1$

8-23. The waiting time T until the 12th new employee is hired has a gamma distribution with $\alpha = 12$ and $\beta = 8$.

$E(T) = \alpha/\beta = 12/8 = 1.5$
$V(T) = \alpha/\beta^2 = 12/64 = .1875$

8-24. Let T be a gamma distribution with a mean of 18 and a variance of 27.

$$\frac{E(T)}{V(T)} = \frac{\alpha/\beta}{\alpha/\beta^2} = \beta = 18/27 = 2/3$$

$\alpha = \beta E(T) = (2/3)18 = 12$

8-25. Let X be a gamma distribution with $\alpha = 2$ and $\beta = 3$.
Then $f(x) = \dfrac{3^2}{\Gamma(2)} x^{2-1} e^{-3x}$.

(a) $F(x) = 9\int_0^x te^{-3t} dt$ (integrating by parts)
$= 9(-te^{-3t}/3 - e^{-3t}/9)\Big|_0^x = 1 - e^{-3x}(3x+1)$

(b) $P(0 \le X \le 3) = F(3) = 1 - e^{-9}(10) = .9988$

(c) $P(1 \le X \le 2) = F(2) - F(1) = (1 - 7e^{-6}) - (1 - 4e^{-3})$
$= .1818$

8-26. X has a gamma distribution with $\alpha = 2$ and $\beta = 1/3$.

$E(X) = 6$, and $E(X^2) = \alpha(\alpha+1)/\beta^2 = 54$
$E(C) = E(500X + 50X^2)$
$= \int_0^\infty (500x + 50x^2) f(x) dx$
$= 500\int_0^\infty x f(x) dx + 50\int_0^\infty x^2 f(x) dx$
$= 500E(X) + 50E(X^2) = 3270$

In subsequent problems involving probabilities of the normal random variable, the z-values will be rounded to 2 decimal places to make use of the z-table in Appendix A. If you are using the TI-83 or other calculator, this will not be necessary. The solutions here will only be those using the z-table.

8-30. X is a normal random variable with $\mu = 17.1$ and $\sigma = 3.2$.

$$P(14 \le X \le 25) = \left(\frac{14-\mu}{\sigma} \le Z \le \frac{25-\mu}{\sigma}\right)$$
$$= \left(\frac{14-17.7}{3.2} \le Z \le \frac{25-17.1}{3.2}\right)$$
$= (-.97 \le Z \le 2.47) = .9932 - .1660$
$= .8272$

8.31 Let S be the total claims on the 5000 policies. Then S has a normal distribution with $\mu = 5000(495)$ and $\sigma^2 = 5000(30{,}000)$ and $\sigma = 12{,}247.44$.

$$P(S \le 2{,}500{,}000) = P\left(Z \le \frac{2{,}500{,}000 - 2{,}475{,}000}{12{,}247.44}\right)$$
$= P(Z \le 2.04) = .9793$

8-32. Let X be the length of the rod. X is normally distributed with mean of 7.505 and standard deviation of .01.

(a) $P(7.48 \le X \le 7.52) = P\left(\dfrac{7.48 - 7.505}{.01} \le Z \le \dfrac{7.52 - 7.505}{.01}\right)$
$= P(-2.5 \le Z \le 1.5) = .9270$

(b) If 4 rods, let Y be the number that meet the specifications.

$P(Y \ge 3) = 4(.927)^3(.073) + (.927)^4 = .9711$

8-33. Let X be the lifetime of a light bulb, and $\mu = 1500$ and $\sigma = 125$.

(a) $P(X \geq 1400) = P\left(Z \geq \frac{1400-1500}{125}\right)$
$= P(Z \geq -.8) = 1 - P(Z \leq -.8) = .7881$

(b) P(all three burning after 1400 hrs) $= (.7881)^3 = .4895$

8-34. If X is a number picked from [0,1], x is uniformly distributed with mean .5 and variance 1/12. Let S be the sum of 50 such numbers. Then S is approximately normal with $\mu = 50(.5) = 25$ and $\sigma = \sqrt{50/12} = 2.0412$.

$P(24 \leq S \leq 27) = P\left(\frac{24-25}{2.0412} \leq Z \leq \frac{27-25}{2.0412}\right)$
$= P(-.49 \leq Z \leq .98) = .5244$

8-35. $P(X \leq 29.9) = .9192$
$P(Z \leq (29.9 - 25)/\sigma) = .9192 = F_Z(1.4)$
$4.9/\sigma = 1.4,\ \sigma = 3.5$

8-36. Let $Y = e^X$, where X is normal with $\mu = 5$ and $\sigma = .40$.

$E(Y) = e^{\mu + \frac{\sigma^2}{2}} = e^{5.08} = 160.77$

$V(Y) = E(Y)^2(e^{\sigma^2} - 1) = e^{10.16}(e^{.16} - 1) = 4{,}484.96$

8-37. Let $Y = e^X$, where X is normal with $\mu = 5.2$ and $\sigma = .80$.

$P(100 \leq Y \leq 500) = P(\ln 100 \leq X \leq \ln 500)$
$= P\left(\frac{\ln 100 - 5.2}{.8} \leq X \leq \frac{\ln 500 - 5.2}{.8}\right)$
$= P(-.74 \leq X \leq 1.27) = .6684$

8-38. Let $Y = e^X$, where X is normal with $\mu = 6.8$ and $\sigma = .60$.

$P(Y \geq 1750) = 1 - P(X \leq \ln 1750)$
$= 1 - P(X \leq \ln 1750)$
$= 1 - P(Z \leq (\ln 1750 - 6.8)/.6)$
$= 1 - P(Z \leq 1.11) = .1335$

8-39. To find the median of Y we solve the equation
$P(Y \leq y) = .50$
$P(X \leq \ln y) = .50$
$P(Z \leq (\ln y - \mu)/\sigma) = .50$
$(\ln y - \mu)/\sigma = 0$
$\ln y = \mu$.
Then $y = e^\mu$ is the median.

8-40. $Y = 100e^X$, where X is normal with $\mu = .10$ an $\sigma = .03$.

(a) $P(100Y \geq 112.50) = 1 - P(100Y \leq 112.5)$
$= 1 - P(Y \leq 1.125)$
$= 1 - P(X \leq \ln 1.125)$
$= 1 - P(Z \leq (\ln 1.125 - .10)/.03)$
$= 1 - P(Z \leq .59) = .2776$

(b) $P(100Y \leq 107.5) = P(Z \leq (\ln 1.075 - .1)/.03)$
$= P(Z \leq -.92) = .1788$

8-41. Let Y be lognormal with $E(Y) = 2{,}500$ and $V(Y) = 1{,}000{,}000$. Note that V(Y) can be written as

$V(Y) = E(Y)^2(e^{\sigma^2} - 1)$.
$e^{\sigma^2} - 1 = 1{,}000{,}000/(2{,}500)^2 = .16$
$\sigma^2 = \ln 1.16$
$\sigma = .3853$

$\ln(E(Y)) = \mu + \sigma^2/2 = \ln 2500$
$\mu = \ln 2500 - (.3853)^2/2 = 7.7498$

8-42. Rewrite the Pareto density function as $f(x) = \alpha\beta^\alpha x^{-\alpha-1}$.

(a) $F(x) = \alpha\beta^\alpha \int_\beta^x t^{-\alpha-1}dt = \alpha\beta^\alpha \frac{t^{-\alpha}}{(-\alpha)}\Big|_\beta^x$
$= -\beta^\alpha(x^{-\alpha} - \beta^{-\alpha}) = 1 - (\beta/x)^\alpha$

(b) $E(X) = \int_\beta^\infty xf(x)dx = \alpha\beta^\alpha \int_\alpha^\infty x^{-\alpha}dx$
$= \frac{\alpha\beta^\alpha}{-\alpha+1}x^{-\alpha+1}\Big|_\beta^\infty = \frac{\alpha\beta^\alpha}{-(\alpha-1)}(0 - \beta^{-\alpha+1})$
$= \alpha\beta/(\alpha - 1),\ (\alpha > 1)$

(c) $E(X^2) = \int_\beta^\infty x^2 f(x)dx = \alpha\beta^\alpha \int_\beta^\infty x^{-\alpha+1}dx$
$= \frac{\alpha\beta^\alpha}{-\alpha+2}x^{-\alpha+2}\Big|_\beta^\infty = \frac{\alpha\beta^\alpha}{-(\alpha-2)}(0 - \beta^{-\alpha+2})$
$= \alpha\beta^2/(\alpha - 2),\ (\alpha > 2)$

$V(X) = E(X^2) - E(X)^2 = \frac{\alpha\beta^2}{\alpha-2} - \left(\frac{\alpha\beta}{\alpha-1}\right)^2$

8-43. Let X be a Pareto random variable with $\alpha = 3.5$ and $\beta = 4$.

(a) $E(X) = \alpha\beta/(\alpha - 1) = 4(3.5)/2.5 = 5.6$

(b) $V(X) = \alpha\beta^2/(\alpha - 2) - [\alpha\beta/(\alpha - 1)]^2 = 3.5(16)/1.5 - 5.6^2$
$= 5.9733$

(c) To find the median solve $F(x) = 1 - (4/x)^{3.5} = .50$.
$(4/x)^{3.5} = .50$
$x = 4/(5)^{1/3.5} = 4.8761$

(d) $P(6 \leq X \leq 12) = F(12) - F(6)$
$= [1 - (4/12)^{3.5}] - [1 - (4/6)^{3.5}]$
$= .97862 - .75808 = .22054$

8-44. Because of the deductible, only claims for losses of more than 5 (hundreds of dollars) are filed. Hence $\beta = 5$. Since the failure rate is 3.5/x, $\alpha = 3.5$.

(a) Mean loss amount is $E(X) = (3.5)(5)/2.5 = 7$. ($700)

(b) Expected amount of a single claim is loss – deductible = $200.

(c) $V(100X) = 100^2[(3.5)(25)/1.5 - 7^2] = 93{,}333.33$

8-45. We are given $\Gamma(1/2) = \pi^{1/2}$, and we know $\Gamma(x + 1) = x\Gamma(x)$.

(a) $\Gamma(3/2) = (1/2)\Gamma(1/2) = (1/2)\pi^{1/2}$

(b) $\Gamma(5/2) = (3/2)\Gamma(3/2) = (3/4)\pi^{1/2}$

(c) $\Gamma(7/2) = (5/2)\Gamma(5/2) = (15/8)\pi^{1/2}$

8-46. For the Weibull distribution $F(x) = 1 - e^{-\beta x^\alpha}$.
If $\alpha = 3$ and $\beta = 3.5$, then $F(x) = 1 - e^{-3.5x^3}$.

(a) $P(X \leq 0.4) = F(0.4) = 1 - e^{-3.5(.4)^3} = .2007$

(b) $P(X > 0.8) = 1 - F(0.8) = e^{-3.5(.8)^3} = .1666$

8-47. The failure rate for the Weibull distribution is $\lambda(x) = \alpha\beta x^{\alpha-1}$.
If $\alpha = 3$ and $\beta = 3.5$, then $\lambda(x) = 10.5x^2$.

8-48. Let X be a Weibull distribution with $\alpha = 2$ and $\beta = 3.5$.

(a) $$E(X) = \frac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}} = \frac{\Gamma(3/2)}{\sqrt{3.5}} = \frac{(1/2)\pi^{1/2}}{\sqrt{3.5}} = .4737$$

(b) $$V(X) = \frac{1}{\beta^{2/\alpha}}[\Gamma(1+2/\alpha) - \Gamma(1+1/\alpha)^2]$$
$$= (1/3.5)[\Gamma(2) - \Gamma(3/2)^2] = (1/3.5)(1 - \pi/4)$$
$$= .0613$$

(c) $$P(.25 \le X \le .75) = (1 - e^{-3.5(.75)^2}) - (1 - e^{-3.5(.25)^2})$$
$$= (1 - .13963) - (1 - .80352)$$
$$= .66389$$

8-49. The Weibull density function is $f(x) = \alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$.

$$E(X) = \int_0^\infty x f(x)dx = \alpha\beta\int_0^\infty x^\alpha e^{-\beta x^\alpha}dx$$

Let $u = x^\alpha$ $\quad x = u^{1/\alpha}$ $\quad dx = (1/\alpha)u^{1/\alpha-1}du$

$$x^\alpha e^{-\beta x^\alpha}dx = ue^{-\beta u}(1/\alpha)u^{1/\alpha-1}du = (1/\alpha)u^{1/\alpha}e^{-\beta u}du$$

$$E(X) = \alpha\beta\int_0^\infty (1/\alpha)u^{1/\alpha}e^{-\beta u}du = \beta\frac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha+1}} = \frac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$$

8-50. Let X be a beta distribution with $\alpha = 4$ and $\beta = 1.5$.

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(5.5)}{\Gamma(4)\Gamma(1.5)}x^3(1-x)^{.5}$$
$$= \frac{(9/2)(7/2)(5/2)(3/2)\Gamma(3/2)}{(3!)\Gamma(3/2)}x^3(1-x)^{.5}$$
$$= 315x^3(1-x)^{1/2}/32$$

8-51. We need to find k so that $\int_0^1 kx^4(1-x)^2dx = 1$.

$$\int_0^1(x^4 - 2x^5 + x^6)dx = (x^5/5 - 2x^6/6 + x^7/7)\Big|_0^1$$
$$= 1/5 - 1/3 + 1/7 = 1/105$$

Hence k = 105.

8-52. For the beta distribution with $\alpha = 3$ and $\beta = 2$:

$$E(X) = \alpha/(\alpha+\beta) = 3/6 = 0.6$$

$$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{6}{25(6)} = 0.04$$

8-53. $$P(X \le 0.5) = \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)}\int_0^{.5}x^2(1-x)dx = 12\int_0^{.5}(x^2 - x^3)dx$$
$$= 12(x^3/3 - x^4/4)\Big|_0^{.5} = 12[(1/3)(.5)^3 - (1/4)(.5)^4]$$
$$= .3125$$

8-54. For a beta distribution with $\alpha = 2$ and $\beta = 4$:

$$P(X \le 0.3) = \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\int_0^{.3}x(1-x)^3dx$$
$$= 20\int_0^{.3}(x - 3x^2 + 3x^3 - x^4)dx$$
$$= 20(x^2/2 - x^3 + 3x^4/4 - x^5/5)\Big|_0^{.3}$$
$$= 30[(1/2)(.3)^2 - (.3)^3 + (3/4)(.3)^4 - (1/5)(.3)^5]$$
$$= .47178$$

8-55. Let X have a beta distribution with parameters α and β.

$$E(X) = \int_0^1 x f(x)dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^\alpha(1-x)^{\beta-1}dx$$

(and using Equation 8.33)

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}$$

CHAPTER 8

8-2. 50; 833.33

8-3. 1/6

8-4. (a) 3/10 (b) 1/12

8-5. (a) 42.5; 18.75 (b) 44 minutes

8-7. (a) 70; 300 (b) .7167

8-8. (a) .3818 (b) .1455

8-9. (a) .4512 (b) .1653

8-10. $\frac{1}{\lambda} \cdot ln\, 2$

8-11. (a) .5654 (b) .1889

8-12. $\frac{1}{b-t}$

8-13. (a) .4821 (b) .4541

8-15. (a) .0111 (b) .2063

8-16. 1.9179; 9.2420

8-19. (a) .1535 (b) .3679

8-22. 1.20; .48

8-23. 1.50; .1875

8-24. $\alpha = 12$; $\beta = 2/3$

8-25. (a) $1 - e^{-3x}(3x+1)$ (b) .9988 (c) .1818

8-26. 3270

8-27. (a) .8155 (b) .4238 (c) .6826 (d) .0990

8-28. (a) 0.93 (b) -1.90 (c) -1.35 (d) 0.97 (e) 1.645 (f) 1.96

8-29. $1 - \alpha$; $2\alpha - 1$

8-30. .8272 (Table), .82689 (TI-83)

8-31. .9793 (Table), .97939 (TI-83)

8-32. (a) .9270 (Table), .92698 (TI-83)
(b) .9711 (Using Table answer in binomial probability),
.97104 (using TI-83 answer)

8-33. (a) .7881 (Table), .78815(TI-83)
(b) .4895 (Using Table answer in binomial probability),
.48957 (using TI-83 answer)

8-34. .5244 (Table), .524304 (TI-83)

8-35. 3.5

8-36. $E(Y) = 160.77$; $V(Y) = 4{,}484.96$

8-37. .6684 (Table), .6691 (TI-83)

8-38. .1335 (Table), .1330 (TI-83)

8-39. e^{μ}

8-40. (a) .2776(Table), .276668(TI-83)
(b) .1788(Table), .178096(TI-83)

8-41. $\mu = 7.7498$; $\sigma = .3853$

8-43. (a) 5.6 (b) 5.9733 (c) 4.8761 (d) .22054

8-44. (a) 700 (b) 200 (c) 93,333.33

8-45. (a) $(1/2)\pi^{1/2}$ (b) $(3/4)\pi^{1/2}$ (c) $(15/8)\pi^{1/2}$

8-46. (a) .2007 (b) .1666

8-47. $10.5x^2$

8-48. (a) .4737 (b) .0613 (c) .66389

8-50. $315x^3(1-x)^{1/2}/32$

8-51. 105

8-52. .60; .04

8-53. .3125

8-54. .47178

8-56. .42045

8-57. .1915

8-58. 10,256

8-59. .4348

8-60 173.3

8-61. .123

8-62. .8185

8-63. .1587

8-64. .9887

8-65. .7698

8-66. 6,342,547.5

Chapter 9
Applications for Continuous Random Variables

9.7 Exercises

9.1 Expected Value of a Function of a Random Variable

9-1. Suppose the amount of a single loss for an insurance policy has density function $f(x) = .001e^{-.001x}$, for $x > 0$. If this policy has a \$300 per claim deductible, what is the expected amount of a single claim for this policy?

9-2. If the policy in Exercise 9-1 also has a payment cap of \$1500 per claim, what is the expected amount of a single claim?

9-3. Work Example 9.4 using the utility function $u(w) = ln(w)$. What are $E[u(W_1)]$ and $E[u(W_2)]$?

9.2 Moment Generating Functions of Continuous Random Variables

9-4. Let X be the random variable which is uniformly distributed over the interval $[a, b]$. Find $M_X(t)$.

9-5. Find $E(X)$ for the random variable in Exercise 9-4 using its moment generating function.

9-6. Let X be the random variable whose density function is given by $f(x) = 2(1 - x)$, for $0 \le x \le 1$, and $f(x) = 0$ elsewhere. Find $M_X(t)$.

9-7. Find $E(X)$ for the random variable in Exercise 9-6 using its moment generating function. (Note: the derivative of $M(t)$ is not defined at 0, but you can take the limit as t approaches 0 to find $E(X)$. This is a much more difficult way to find $E(X)$ than direct integration for this particular density function.)

9-8. If the moment generating function of X is $\left(\frac{2}{2-t}\right)^5$, identify the random variable X.

9-9. If X is an exponential random variable with $\lambda = 3$, what is the moment generating function of $Y = 2X + 5$?

9-10. Let X be the random variable whose moment generating function is $e^{(t+t^2)}$. Find $E(X)$ and $V(X)$.

9-11. Let X be a normal random variable with parameters μ and σ. Use the moment generating function for X to find $E(X^2)$. Then show that $V(X) = \sigma^2$.

9.3 The Distribution of $Y = g(X)$

9-12. Let X be uniformly distributed over $[0, 1]$ and $Y = e^X$. Find (a) $F_Y(y)$; (b) $f_Y(y)$.

9-13. Let X be a random variable with density function given by $f_X(x) = 3x^{-4}$, for $x \geq 1$ (Pareto with $\alpha = 3$, $\beta = 1$), and let $Y = ln\, X$. Find $F_Y(y)$.

9-14. If X is the random variable defined in Exercise 9-13 and $Y = 1/X$, find (a) $F_Y(y)$; (b) $f_Y(y)$.

9-15. The monthly maintenance cost X of a machine is an exponential random variable with unknown parameter. Studies have determined that $P(X > 100) = .64$. For a second machine the cost Y is a random variable such that $Y = 2X$. Find $P(Y > 100)$.

9.4 Simulation of Continuous Distributions

9-16. For a continuous random variable X, show that $F(X)$ is uniformly distributed over $[0, 1]$. (i.e., show $P[F(X) \leq x] = x$, for $0 \leq x \leq 1$.

For Exercises 9-17 and 9-18, use the following sequence of random numbers in $[0,1)$.

1. .90463	6. .81008	11. .15533	16. .31239
2. .17842	7. .49660	12. .29701	17. .68995
3. .55660	8. .92602	13. .82751	18. .77787
4. .55071	9. .71729	14. .67490	19. .66928
5. .96216	10. .39443	15. .68556	20. .53100

9-17. Let X be uniformly distributed over $[0,4]$, and use the above random numbers to simulate $F(x)$. How many of the transformed values $x = F^{-1}(u)$ are in each subinterval $[0,1)$, $[1,2)$, $[2,3)$ and $[3,4)$?

9-18. Let X have a Pareto distribution with $\alpha = 3$ and $\beta = 3$, and use the above random numbers to simulate $F(x)$. How many of the transformed values $x = F^{-1}(u)$ are in each subinterval $[3,4)$, $[4,5)$, $[5,6)$ and $[6,\infty)$.

9.5 Mixed Distributions

9-19. For a certain type of policy, an insurance company divides its claims into two classes, minor and major. Last year 90 percent of the policyholders filed no claims, 9 percent filed minor claims, and 1 percent filed major claims. The amounts of the minor claims were uniformly distributed over $(0, 1{,}000]$, and the major claims were uniformly distributed over $(1{,}000, 10{,}000]$. Find $F(x)$, for $0 \leq x \leq 10{,}000$.

9-20. Find $E(X)$ for the insurance policy in Exercise 9-19.

9-21. An auto insurance company issues a comprehensive policy with a \$200 deductible. Last year 90 percent of the policyholders filed no claims (either no damage or damage less than the deductible). For the 10 percent who filed claims, the claim amount had a Pareto distribution with $\alpha = 3$ and $\beta = 200$. If X is the random variable of the amount paid by the insurer, what is $F(x)$, for $x \geq 0$?

9.6 Two Useful Identities

9-22. Let X be a random variable with hazard rate $\lambda(x) = \frac{2}{1+x}$, for $x \geq 0$. Find $S(x)$.

9-23. Let X be a random variable with hazard rate $\lambda(x) = \frac{1}{100-x}$, for $0 \leq x < 100$. Find $S(x)$.

9-24. Let X be the random variable defined in Exercise 9-22. Use Equation (9.7) to find $E(X)$.

9-25. Let X be a random variable whose survival function is given by $S(x) = \frac{100-x}{100}$, for $0 \leq x < 100$, and $S(x) = 0$ for $x \geq 100$. Use Equation (9.7) to find $E(X)$.

9.8 Sample Exam Problems

9-26. An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x & \text{for} \quad 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64.

Calculate C.

9-27. A manufacturer's annual losses follow a distribution with density function

$$f(x) = \begin{cases} \dfrac{2.5(0.6)^{2.5}}{x^{3.5}} & \text{for} \quad x > 0.6 \\ 0 & \text{otherwise} \end{cases}$$

To cover its losses, the manufacturer purchases an insurance policy with an annual deductible of 2.

What is the mean of the manufacturer's annual losses not paid by the insurance policy?

9-28. An insurance policy is written to cover a loss, X, where X has a uniform distribution on [0, 1000].

At what level must a deductible be set in order for the expected payment to be 25% of what it would be with no deductible?

9-29. A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made.

At what level must x be set if the expected payment made under this insurance is to be 1000?

9-30. A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$.

Determine $E[X]$.

9-31. An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, Y, follows a distribution with density function:

$$f(y) = \begin{cases} \frac{2}{y^3} & y > 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

9-32. The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X, has density function

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } 0 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

Let Y be the age of the machine at the time of replacement. Determine the variance of Y.

9-33. The owner of an automobile insures it against damage by purchasing an insurance policy with a deductible of 250. In the event that the automobile is damaged, repair costs can be modeled by a uniform random variable on the interval $(0,1500)$.

Determine the standard deviation of the insurance payment in the event that the automobile is damaged.

9-34. An insurance company sells an auto insurance policy that covers losses incurred by a policyholder, subject to a deductible of 100. Losses incurred follow an exponential distribution with mean 300.

What is the 95th percentile of actual losses that exceed the deductible?

9-35. The time, T, that a manufacturing system is out of operation has cumulative distribution function

$$F(t) = \begin{cases} 1-\left(\frac{2}{t}\right)^2 & \text{for } t>2 \\ 0 & \text{otherwise} \end{cases}$$

The resulting cost to the company is $Y = T^2$.

Determine the density function of Y, for $y > 4$.

9-36. An investment account earns an annual interest rate R that follows a uniform distribution on the interval $(0.04, 0.08)$. The value of a 10,000 initial investment in this account after one year is given by $V = 10{,}000e^R$.

Determine the cumulative distribution function, $F(v)$, of V for values of v that satisfy $0 < F(v) < 1$.

9-37. An actuary models the lifetime of a device using the random variable $Y = 10X^{.8}$, where X is an exponential random variable with mean 1 year.

Determine the probability density function $f(y)$, for $y > 0$, of the random variable Y.

9-38. Let T denote the time in minutes for a customer service representative to respond to 10 telephone inquiries. T is uniformly distributed on the interval with endpoints 8 minutes and 12 minutes. Let R denote the average rate, in customers per minute, at which the representative responds to inquiries.

Find the density function of the random variable R on the interval $\left(\frac{10}{12} \le r \le \frac{10}{8}\right)$.

9-39. The monthly profit of Company I can be modeled by a continuous random variable with density function f. Company II has a monthly profit that is twice that of Company I.

Determine the probability density function of the monthly profit of Company II.

9-40. A random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{x^2-2x+2}{2} & \text{for } 1 \le x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$$

Calculate the variance of X.

CHAPTER 9

9-1. 740.82

9-2. 575.52

9-3. $E[u(W_1)] = 2.3009$; $E[u(W_2)] = 2.2574$

9-4. $(e^{bt} - e^{at})/[t(b-a)]$ if $t \neq 0$, 1 if $t = 0$.

9-5. $(b+a)/2$

9-6. $(2e^t - 2t - 2)/t^2$ if $t \neq 0$, 1 if $t = 0$

9-7. 1/3

9-8. Gamma with $\alpha = 5$ and $\beta = 2$

9-9. $e^{5t}[3/(3-2t)]$

9-10. $E(X) = 1$; $V(X) = 2$

9-11. $E(X^2) = \mu^2 + \sigma^2$

9-12. (a) $ln\, y$ (b) $1/y$ (both on $[1, e]$)

9-13. $1 - e^{-3y}$, for $y \geq 0$

9-14. (a) y^3 (b) $3y^2$, for $0 < y \leq 1$

9-15. .80

9-17. 2, 4, 8, 6

9-18. 9, 6, 2, 3

9-19. $F(0) = .90$
$F(x) = .90 + .09x/1000$, for $0 < x \leq 1000$
$F(x) = .99 + .01(x-1000)/9000$, for $1000 < x \leq 10{,}000$

9-20. 100

9-21. $F(0) = .90$
$F(x) = .90 + .10[1 - (200/(x+200))^3]$, for $x > 0$

9-22. $\frac{1}{(1+x)^2}, x \geq 0$

9-23. $\frac{100-x}{100}, 0 \leq x < 100$

9-24. 1

9-25. 50

9-26. .3

9-27. .93427

9-28. 500

9-29. 5644.30

9-30. $2 + 3e^{-2/3}$

9-31. 1.9

9-32. 1.7067

9-33. 403.436

9-34. 998.72

9-35. $\frac{4}{y^2}$

9-36. $25\left[\ln\left(\frac{v}{10{,}000}\right) - .04\right]$

9-37. $.125e^{-(.10_y)^{1.25}}(.1y)^{.25}$

9-38. $\frac{5}{2r^2}$

9-39. $f_X\left(\frac{y}{2}\right)\left[\frac{1}{2}\right]$

9-40. $\frac{5}{36}$

10.6 Exercises

10.1 Joint Distributions for Discrete Random Variables

10-1. Let $p(x,y) = (xy+y)/27$, for $x = 1,2,3$ and $y = 1,2$, be the joint probability for the random variables X and Y. Construct a table of the joint probabilities of X and Y and the marginal probabilities of X and Y.

10-2. A company has 5 CPA's, 3 actuaries, and 2 economists. Two of these 10 professionals are selected at random to prepare a report. Let X be the random variable for the number of CPA's chosen and let Y be the random variable for the number of actuaries chosen. Construct a table of the joint probabilities for X and Y and the marginal probabilities of X and Y.

10-3. For the random variables in Exercise 10-1, find $E(X)$ and $E(Y)$.

10-4. For the random variables in Exercise 10-2, find $E(X)$ and $E(Y)$.

10-5. For the random variables in Exercise 10-2, find $V(X)$ and $V(Y)$.

10.2 Joint Distributions for Continuous Random Variables

10-6. Show that the function $f(x,y) = \frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy$, for $0 \le x \le 1$ and $0 \le y \le 1$, is a joint probability density function. Find $P(0 \le X \le .50, .50 \le Y \le 1)$.

10-7. For the joint density function in Exercise 10-6, find (a) $f_X(x)$; (b) $f_Y(y)$.

10-8. Let $f(x,y) = 2x^2 + 3y$, for $0 \le y \le x \le 1$. Find (a) $f_X(x)$; (b) $f_Y(y)$.

10-9. For the joint density function in Exercise 10-8, use the marginal distributions to find (a) $P(X > .50)$; (b) $P(Y > .50)$.

10-10. For the joint density function in Exercise 10-6, find $E(X)$.

10-11. For the joint density function in Exercise 10-6, find $P(X > Y)$.

10-12. For the joint density function in Exercise 10-8, find $E(X)$ and $E(Y)$.

10-13. An auto insurance company separates its comprehensive claims into two parts: losses due to glass breakage and losses due to other damage. If X is the random variable for losses due to glass breakage and Y the random variable for other damage, $f(x,y) = (30 - x - y)/1875$, for $0 \leq x \leq 5$, $0 \leq y \leq 25$, where x and y are in hundreds of dollars. Find $P(X \geq 4, Y \geq 20)$.

10-14. For the random variables in Exercise 10-13, find (a) $f_X(x)$; (b) $f_Y(y)$.

10-15. For the random variables in Exercise 10-13, find $E(X)$ and $E(Y)$.

10.3 Conditional Distributions

Exercises 10-16, 10-17 and 10-18 refer to Exercise 10-1.

10-16. Find $P(X|Y = 1)$.

10-17. Find $P(Y|X = 1)$.

10-18. Find $E(X|Y = 1)$.

10-19. For the joint density function in Exercise 10-6, find $f(x \mid y)$.

10-20. For the joint density function in Exercise 10-8, find $f(y \mid x)$.

10-21. For the conditional density function in Exercise 10-20, find (a) $f(y \mid .50)$; (b) $E(Y \mid X = .50)$.

10-22. If $f(x,y) = 6x$, for $0 < x < y < 1$ and 0 elsewhere, find (a) $f_Y(y)$; (b) $f(x \mid y)$; (c) $E(X \mid Y = y)$; (d) $E(X \mid Y = .50)$.

10.4 Independence for Random Variables

10-23. Determine if the random variables in Exercise 10-1 are dependent or independent.

10-24. Determine if the random variables in Exercise 10-2 are dependent or independent.

10-25. Determine if the random variables in Exercise 10-6 are dependent or independent.

10-26. Determine if the random variables in Exercise 10-8 are dependent or independent.

10.7 Sample Actuarial Examination Problems

10-27. A doctor is studying the relationship between blood pressure and heartbeat abnormalities in her patients. She tests a random sample of her patients and notes their blood pressures (high, low, or normal) and their heartbeats (regular or irregular). She finds that:

(i) 14% have high blood pressure.
(ii) 22% have low blood pressure.
(iii) 15% have an irregular heartbeat.
(iv) Of those with an irregular heartbeat, one-third have high blood pressure.
(v) Of those with normal blood pressure, one-eighth have an irregular heartbeat.

What portion of the patients selected have a regular heartbeat and low blood pressure?

10-28. A large pool of adults earning their first driver's license includes 50% low-risk drivers, 30% moderate-risk drivers, and 20% high-risk drivers. Because these drivers have no prior driving record, an insurance company considers each driver to be randomly selected from the pool. This month, the insurance company writes 4 new policies for adults earning their first driver's license.

What is the probability that these 4 will contain at least two more high-risk drivers than low-risk drivers?

10-29. A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f(x,y) = \frac{x+y}{8} \quad \text{for} \quad 0<x<2 \quad \text{and} \quad 0<y<2$$

What is the probability that the device fails during its first hour of operation?

10-30. A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f(x,y) = \frac{x+y}{27} \quad \text{for} \quad 0<x<3 \quad \text{and} \quad 0<y<3$$

Calculate the probability that the device fails during its first hour of operation.

10-31. A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s,t)$, where $0<s<1$ and $0<t<1$.

Express the probability that the device fails during the first half hour of operation as a double integral.

10-32. The future lifetimes (in months) of two components of a machine have the following joint density function:

$$f(x,y) = \begin{cases} \frac{6}{125{,}000}(50-x-y) & \text{for} \quad 0<x<50-y<50 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that both components are still functioning 20 months from now? Express your answer as a double integral, but do not evaluate it.

10-33. An insurance company sells two types of auto insurance policies: Basic and Deluxe. The time until the next Basic Policy claim is an exponential random variable with mean two days. The time until the next Deluxe Policy claim is an independent exponential random variable with mean three days.

What is the probability that the next claim will be a Deluxe Policy claim?

10-34. Two insurers provide bids on an insurance policy to a large company. The bids must be between 2000 and 2200. The company decides to accept the lower bid if the two bids differ by 20 or more. Otherwise, the company will consider the two bids further.

Assume that the two bids are independent and are both uniformly distributed on the interval from 2000 to 2200.

Determine the probability that the company considers the two bids further.

10-35. A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car.

Let X denote the number of luxury cars sold in a given day, and let Y denote the number of extended warranties sold.

$$P(X = 0, Y = 0) = 1/6$$

$$P(X = 1, Y = 0) = 1/12$$

$$P(X = 1, Y = 1) = 1/6$$

$$P(X = 2, Y = 0) = 1/12$$

$$P(X = 2, Y = 1) = 1/3$$

$$P(X = 2, Y = 2) = 1/6$$

What is the variance of X?

10-36. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 24xy & \text{for } 0<x<1 \text{ and } 0<y<1-x \\ 0 & \text{otherwise.} \end{cases}$$

Find $P\left(Y < X \mid X = \frac{1}{3}\right)$

10-37. Once a fire is reported to a fire insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant for the fire loss. When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint density function

$$f(x,y) = \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)}, \quad x>1, y>1.$$

Given that the initial claim estimated by the company is 2, determine the probability that the final settlement amount is between 1 and 3.

10-38. A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy.

Let X denote the proportion of employees who purchase the basic policy, and Y the proportion of employees who purchase the supplemental policy. Let X and Y have the joint density function $f(x,y) = 2(x+y)$ on the region where the density is positive.

Given that 10% of the employees buy the basic policy, what is the probability that fewer than 5% buy the supplemental policy?

10-39. Two life insurance policies, each with a death benefit of 10,000 and a one-time premium of 500, are sold to a couple, one for each person. The policies will expire at the end of the tenth year. The probability that only the wife will survive at least ten years is 0.025, the probability that only the husband will survive at least ten years is 0.01, and the probability that both of them will survive at least ten years is 0.96.

What is the expected excess of premiums over claims, given that the husband survives at least ten years?

10-40. A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:

$$P(X=0,\, Y=0) = 0.800$$

$$P(X=1,\, Y=0) = 0.050$$

$$P(X=0,\, Y=1) = 0.025$$

$$P(X=1,\, Y=1) = 0.125$$

Calculate $Var(Y \mid X=1)$.

10-41. The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y) = \begin{cases} 2x & \text{for} \quad 0 < x < 1 \quad \text{and} \quad x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$$

What is the conditional variance of Y given that $X = x$?

10-42. An actuary determines that the annual numbers of tornadoes in counties P and Q are jointly distributed as follows:

Annual number in Q / Annual number in P	0	1	2	3
0	0.12	0.06	0.05	0.02
1	0.13	0.15	0.12	0.03
2	0.05	0.15	0.10	0.02

Calculate the conditional variance of the annual number of tornadoes in *county Q, given that there are no tornadoes in county P.*

10-43. A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f(x,y) = \begin{cases} 6[1-(x+y)] & \text{for } x>0, y>0 \text{ and } x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the probability that the portion of a claim representing damage to the house is less than 0.2.

10-44. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 15y & \text{for } x^2 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

Find g, the marginal density function of Y.

10-45. An auto insurance policy will pay for damage to both the policyholder's car and the other driver's car in the event that the policyholder is responsible for an accident. The size of the payment for damage to the policyholder's car, X, has a marginal density function of 1 for $0 < x < 1$. Given $X = x$, the size of the payment for damage to the other driver's car, Y, has conditional density of 1 for $x < y < x+1$.

If the policyholder is responsible for an accident, what is the probability that the payment for damage to the other driver's car will be greater than 0.500?

10-46. An insurance policy is written to cover a loss X where X has density function

$$f(x) = \begin{cases} \frac{3x^2}{8} & \text{for} \quad 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The time (in hours) to process a claim of size x, where $0 \le x \le 2$, is uniformly distributed on the interval from x to $2x$.

Calculate the probability that a randomly chosen claim on this policy is processed in three hours or more.

10-47. Let X represent the age of an insured automobile involved in an accident. Let Y represent the length of time the owner has insured the automobile at the time of the accident.

X and Y have joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{64}(10 - xy^2) & \text{for} \quad 2 \le x \le 10 \quad \text{and} \quad 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Calculate the expected age of an insured automobile involved in an accident.

10-48. A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails.

Let X and Y be the times at which the first and second circuits fail, respectively. X and Y have joint probability density function.

$$f(x,y) = \begin{cases} 6e^{-x}e^{-2y} & \text{for } \ 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

What is the expected time at which the device fails?

10-49. A study of automobile accidents produced the following data:

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident.

Model	Proportion of All Vehicles	Probability of Involvement in an Accident
1997	0.16	0.05
1998	0.18	0.02
1999	0.20	0.03
Other	0.46	0.04

Determine the probability that the model year of this automobile is 1997.

CHAPTER 10

10-1. The values of $p(x,y)$ can be found by direct substitution, e.g. $p(1,1) = P(X = 1, Y = 1) = [(1)(1) + 1]/27 = 2/27$.
The marginal probabilities can be found by adding the entries in the columns to get $p(x)$ and the row entries to get $p(y)$.

$$P(X = 1) = p(1,1) + p(1,2) = 2/27 + 4/27 = 2/9$$
$$P(Y = 1) = p(1,1) + p(2,1) + p(3,1) = 2/27 + 3/27 + 4/27 = 1/3$$

10-2. Since the team consists of 2 professionals the possible values of X and Y are $0 \le X + Y \le 2$, (2 CPA's and 1 actuary would be impossible). The event $(X = x, Y = y)$, denotes a team of x CPA's, y actuaries, and $2 - (x + y)$ accountants.

$$P(X = 0, Y = 0) = p(0,0) = C(5,0)C(3,0)C(2,2)/C(10,2) = 1/45$$
$$P(X = 0, Y = 1) = p(0,1) = C(5,0)C(3,1)C(2,1)/45 = 6/45$$
$$P(X = 1, Y = 0) = p(1,0) = C(5,1)C(3,0)C(2,1)/45 = 10/45$$
$$P(X = 1, Y = 1) = p(1,1) = C(5,1)C(3,1)C(2,0)/45 = 15/45, \text{ etc.}$$

The marginal probabilities can be found by obtaining row sums and column sums as in 10-1.

$$P(X = 0) = p(0,0) + p(0,1) + p(0,2) = 1/45 + 6/45 + 3/45 = 10/45$$

10-3. Using the table obtained in Exercise 10-1:

$$E(X) = \sum x p_X(x) = 2/9 + 2(3/9) + 3(4/9) = 20/9$$

$$E(Y) = \sum y p_Y(y) = (1/3) + 2(2/3) = 5/3$$

10-4. Using the table obtained in Exercise 10-2:

$$E(X) = \sum x p_X(x) = 0 + 25/45 + 2(10/45) = 1$$

$$E(Y) = \sum y p_Y(y) = 0 + 21/45 + 2(3/45) = 27/45 = 3/5$$

10-5. Using the table from Exercise 10-2:

$$E(X^2) = \sum x^2 p_X(x) = 0 + 1^2(25/45) + 2^2(10/45)$$
$$= 65/45 = 13/9$$

$$V(X) = E(X^2) - E(X)^2 = 13/9 - 1 = 4/9$$

$$E(Y^2) = \sum y^2 p_Y(y) = 0 + 1^2(21/45) + 2^2(3/45)$$
$$= 33/45 = 11/15$$

$$V(Y) = 11/15 - (3/5)^2 = (55 - 27)/75 = 28/75$$

10-6. Clearly the function $f(x,y) = \frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy$, $0 \le x \le 1$ and $0 \le y \le 1$, is greater than or equal to 0 on this range, so we only need to show that the volume under the surface is 1.

$$\int_0^1\int_0^1 \left(\frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy\right)dxdy = \int_0^1 \left(\frac{x}{4} + \frac{x^2}{4} + \frac{xy}{2} + \frac{x^2y}{2}\right)\Bigg|_{x=0}^1 dy$$
$$= \int_0^1 (1/2 + y)dy$$
$$= (y/2 + y^2/2)\Big|_0^1 = 1$$

Therefore $f(x,y)$ is a joint probability density function.

$$P(0 \le X \le 1/2, 1/2 \le Y \le 1) = \int_{1/2}^1\int_0^{1/2} \left(\frac{1}{4} + \frac{x}{2} + \frac{y}{x} + xy\right)dxdy$$
$$= \int_{1/2}^1 \left(\frac{x}{4} + \frac{x^2}{4} + \frac{xy}{2} + \frac{x^2y}{2}\right)\Bigg|_{x=0}^{1/2} dy$$
$$= \int_{1/2}^1 \left(\frac{3}{16} + \frac{3y}{8}\right)dy$$
$$= \left(\frac{3y}{16} + \frac{3y^2}{16}\right)\Bigg|_{1/2}^1 = 15/64$$

10-7. (a) $f_X(x) = \int_0^1 f(x,y)dy = \int_0^1 \left(\frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy\right)dy$

$$= \left(\frac{y}{4} + \frac{xy}{2} + \frac{y^2}{4} + \frac{xy^2}{2}\right)\Bigg|_{y=0}^1 = (1/2 + x),\ 0 \le x \le 1$$

(b) $f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 \left(\frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy\right)dx$

$$= \left(\frac{x}{4} + \frac{x^2}{4} + \frac{xy}{2} + \frac{x^2y}{2}\right)\Bigg|_{x=0}^1 = (1/2 + y),\ 0 \le y \le 1$$

10-8. The density function $f(x,y) = 2x^2 + 3y$ is defined on the region bounded by the x-axis and the lines $y = x$ and $x = 1$.

(a) $f_X(x) = \int_0^x (2x^2 + 3y)dy = \left(2x^2y + \frac{3y^2}{2}\right)\Bigg|_{y=0}^x$

$$= 2x^3 + 3x^2/2,\ 0 \le x \le 1$$

(b) $f_Y(y) = \int_y^1 (2x^2 + 3y)dx = \left(\frac{2x^3}{3} + 3xy\right)\Bigg|_{x=y}^1$

$$= 2/3 + 3y - 3y^2 - (2/3)y^3,\ 0 \le y \le 1$$

10-9. Using the marginal distributions from Exercise 10-8:

(a) $$P(X > 1/2) = \int_{1/2}^{1}(2x^3 + 3x^2/2)dx = \left(\frac{x^4}{2} + \frac{x^3}{2}\right)_{1/2}^{1} = 29/32$$

(b) $$P(Y > 1/2) = \int_{1/2}^{1}(2/3 + 3y - 3y^2 - 2y^3/3)dy = \left(\frac{2y}{3} + \frac{3y^2}{2} - y^3 - \frac{y^4}{6}\right)_{1/2}^{1} = 41/96$$

10-10. From Exercise 10-6, $f_X(x) = ½ + x$ for $0 \le x \le 1$.

$$E(X) = \int_0^1 x(1/2 + x)dx = \int_0^1 (x/2 + x^2)dx = \left(\frac{x^2}{4} + \frac{x^3}{3}\right)_0^1 = 7/12$$

10-11. If $f(x,y) = 1/4 + x/2 + y/2 + xy$, for $0 \le x \le 1$ and $0 \le y \le 1$, then

$$P(X > Y) = \int_0^1\int_0^x (1/4 + x/2 + y/2 + xy)dydx$$
$$= \int_0^1 \left(\frac{y}{4} + \frac{xy}{2} + \frac{y^2}{4} + \frac{xy^2}{2}\right)_{y=0}^{x} dx$$
$$= \int_0^1 \left(\frac{x}{4} + \frac{3x^2}{4} + \frac{x^3}{2}\right)dx$$
$$= \left(\frac{x^2}{8} + \frac{x^3}{4} + \frac{x^4}{8}\right)_0^1 = 1/2$$

10-12. From Exercise 10-8, $f_X(x) = 2x^3 + 3x^2/2$, for $0 \le x \le 1$.

$$E(X) = \int_0^1 x f_X(x)dx = \int_0^1 (2x^4 + 3x^3/2)dx = \left(\frac{2x^5}{5} + \frac{3x^4}{8}\right)_0^1 = 31/40$$

Also $f_Y(y) = 2/3 + 3y - 3y^2 - 2y^3/3$, for $0 \le y \le 1$.

$$E(Y) = \int_0^1 y f_Y(y)dy = \int_0^1 (2y/3 + 3y^2 - 3y^3 + 2y^4/3)dy = \left(\frac{y^2}{3} + y^3 - \frac{3y^4}{4} - \frac{2y^5}{15}\right)_0^1 = 9/20$$

10-13. $f(x,y) = (30 - x - y)/1875$, for $0 \le x \le 5$ and $0 \le y \le 25$.

$$P(X \ge 4, Y \ge 20) = (1/1875)\int_{20}^{25}\int_4^5 (30 - x - y)dxdy$$
$$= (1/1875)\int_{20}^{25}(30x - x^2/2 - xy)\Big|_{x=4}^{5} dy$$
$$= (1/3750)\int_{20}^{25}(51 - 2y)dy$$
$$= (1/3750)(51y - y^2)\Big|_{20}^{25} = 30/3750 = 1/125$$

10-14. (a) $$f_X(x) = \int_0^{25}\left(\frac{30 - x - y}{1875}\right)dy = \left(\frac{30y - xy - y^2/2}{1875}\right)_{y=0}^{25}$$
$$= \left(\frac{60y - 2xy - y^2}{3750}\right)_{y=0}^{25} = \frac{875 - 50x}{3750}$$
$$= (35 - 2x)/150, \text{ for } 0 \le x \le 50$$

10-14. (b) $$f_Y(y) = \int_0^5\left(\frac{30 - x - y}{1875}\right)dx = \left(\frac{30x - x^2/2 - xy}{1875}\right)_{x=0}^{5}$$
$$= \left(\frac{60 - x^2 - 2xy}{3750}\right)_{x=0}^{5} = \frac{275 - 10y}{3750}$$
$$= (55 - 2y)/750, \text{ for } 0 \le y \le 25$$

10-15. For the random variables in Exercise 10-13:

$$E(X) = \int_0^5 x f_X(x)dx = (1/150)\int_0^5 (35x - 2x^2)dx$$
$$= \left(\frac{1}{150}\right)\left(\frac{35x^2}{2} - \frac{2x^3}{3}\right)_0^5 = \frac{2125}{900} = \frac{85}{36}$$
$$E(Y) = \int_0^{25} y f_Y(y)dy = (1/750)\int_0^{25}(55y - 2y^2)dy$$
$$= \left(\frac{1}{750}\right)\left(\frac{55y^2}{2} - \frac{2y^3}{3}\right)_0^{25} = \frac{40{,}625}{6\cdot 750} = \frac{325}{36}$$

10-16. Using the data from the table obtained in Exercise 10-1:

$$P(X = 1|Y = 1) = \frac{p(1,1)}{p_Y(1)} = \frac{2/27}{1/3} = 2/9$$
$$P(X = 2|Y = 1) = \frac{p(2,1)}{p_Y(1)} = \frac{1/9}{1/3} = 1/3$$
$$P(X = 3|Y = 1) = \frac{p(3,1)}{p_Y(1)} = \frac{4/27}{1/3} = 4/9$$

10-17. Using the data from the table obtained in Exercise 10-1:

$$P(Y = 1|X = 1) = \frac{p(1,1)}{p_X(1)} = \frac{2/27}{2/9} = 1/3$$
$$P(Y = 2|X = 1) = \frac{p(1,2)}{p_X(1)} = \frac{4/27}{2/9} = 2/3$$

10-18. Using the data obtained in Exercise 10-16:

$$E(X|Y = 1) = \sum xP(X = x \mid Y = 1)$$
$$= 1(2/9) + 2(1/3) + 3(4/9) = 20/9$$

10-19. For the joint density function in Exercise 10-6:

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1/4 + x/2 + y/2 + xy}{1/2 + y}$$
$$= \frac{(1/2 + x)(1/2 + y)}{(1/2 + y)}$$
$$= 1/2 + x,\ 0 \le x \le 1$$

10-20. For the joint density function in Exercise 10-8:

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2x^2 + 3y}{2x^3 + 3x^2/2}, \text{ for } 0 < y \le x \le 1$$

10-21. (a) $$f(y|1/2) = \frac{2(1/2)^2 + 3y}{2(1/2)^3 + (3/2)(1/2)^2} = \frac{4 + 24y}{5},\ 0 < y \le 1/2$$

(b) $$E(Y|X = 1/2) = \int_0^{1/2} y f(y|1/2)dy$$
$$= (1/5)\int_0^{1/2}(4y + 24y^2)dy$$
$$= (1/5)(2y^2 + 8y^3)\Big|_0^{1/2} = 3/10$$

10-22. Let f(x,y) = 6x, for $0 < x < y < 1$, and 0 elsewhere.

(a) $f_Y(y) = \int_0^y f(x,y)dx = \int_0^y 6x dx = 3y^2$, for $0 < y < 1$

(b) $f(x|y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{6x}{3y^2} = \dfrac{2x}{y^2}$, for $0 < x < y < 1$

(c) $E(X|Y = y) = \int_0^y x f(x|y)dx = (1/y^2)\int_0^y 2x^2 dx = \dfrac{1}{y^2}\cdot\dfrac{2y^3}{3} = 2y/3$

(d) $E(X|Y = 1/2) = 2(1/2)/3 = 1/3$

10-23. Let X and Y be the random variables in Exercise 10-1. From the table obtained in that exercise it can be seen that $p(x,y) = p_X(x)p_Y(y)$ for all possible choices of x and y, e.g.,

$p(1,1) = 2/27 = (2/9)(1/3) = p_X(1)p_Y(1)$
$p(2,1) = 1/9 = (1/3)(1/3) = p_X(2)p_Y(1)$, etc.

Hence X and Y are independent.

10-24. Using the tables obtained for the random variables in Exercise 10-2,

$$p(0,0) = 1/45 \neq (10/45)(21/45) = p_X(0)p_Y(0).$$

Hence X and Y are dependent.

10-25. In Exercise 10-6 the joint density function is

$$f(x,y) = 1/4 + x/2 + y/2 + xy = (1/2 + x)(1/2 + y) = f_X(x)f_Y(y).$$

Hence X and Y are independent.

10-26. For the joint density function f(x,y) in Exercise 10-8,

$$f(x,y) = 2x^2 + 3y \neq (2x^3 + 3x^2/2)(2/3 + 3y - 3y^2 - 2y^3/3)$$
$$= f_X(x)f_Y(y).$$

Hence X and Y are dependent.

CHAPTER 10

10-1.

y \ x	1	2	3	$p(y)$
1	2/27	1/9	4/27	1/3
2	4/27	2/9	8/27	2/3
$p(x)$	2/9	1/3	4/9	

10-2.

y \ x	0	1	2	$p(y)$
0	1/45	10/45	10/45	21/45
1	6/45	15/45	0	21/45
2	3/45	0	0	3/45
$p(x)$	10/45	25/45	10/45	

10-3. $E(X) = 20/9$; $E(Y) = 5/3$

10-4. $E(X) = 1$; $E(Y) = 3/5$

10-5. $V(X) = 4/9$; $V(Y) = 28/75$

10-6. 15/64

10-7. (a) $1/2 + x,\ 0 \leq x \leq 1$ (b) $1/2 + y,\ 0 \leq y \leq 1$

10-8. (a) $2x^3 + (3/2)x^2,\ 0 \leq x \leq 1$
(b) $2/3 + 3y - (2/3)y^3 - 3y^2,\ 0 \leq y \leq 1$

10-9. (a) 29/32 (b) 41/96

10-10. 7/12

10-11. 1/2

10-12. $E(X) = 31/40$; $E(Y) = 9/20$

10-13. 1/125

10-14. (a) $(35 - 2x)/150, 0 \leq x \leq 5$ (b) $(55 - 2y)/750, 0 \leq y \leq 25$

10-15. $E(X) = 85/36$; $E(Y) = 325/36$

10-16.

x	1	2	3
$p(x\|1)$	2/9	1/3	4/9

10-17.

y	1	2
$p(y\|1)$	1/3	2/3

10-18. 20/9

10-19. $1/2 + x,\ 0 \leq x \leq 1$

10-20. $(2x^2 + 3y)/(2x^3 + ((3/2)x^2),\ 0 < y \leq x \leq 1$

10-21. (a) $4/5 + (24/5)y,\ 0 \leq y \leq 1/2$ (b) 3/10

10-22. (a) $3y^2, 0 < y < 1$ (b) $2x/y^2, 0 < x < y < 1$ (c) $2y/3$

(d) 1/3

10-23. Independent

10-24. Dependent

10-25. Independent

10-26. Dependent

10-27. 20%

10-28. .0488

10-29. .625

10-30. .41

10-31. $\int_0^{0.5}\int_{0.5}^{1} f(s,t)\,ds\,dt + \int_0^{1}\int_0^{0.5} f(s,t)\,ds\,dt$

10-32. $\frac{6}{125,000}\int_{20}^{30}\int_{20}^{50-x}(50-x-y)\,dy\,dx$

10.33. 2/5

10-34. .19

10-35. .576

10-36. 1/4

10-37. 8/9

10-38. .4167

10-39. 896.91

10-40. .204

10-41. 1/12

10-42. .9856

10-43. .488

10-44. $15y^{3/2}(1-y^{1/2})$

10-45. 7/8

10-46. .172

10-47. 5.78

10.48. .833

10-49. .45474

Chapter 11
Applying Multivariate Distributions

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11.7 Exercises

11.1 Distributions of Functions of Two Random Variables

11-1. Let $p(x, y)$ be the joint probability function of Exercise 10-1, and let $S = X + Y$. Find the probability function $p_S(s)$.

11-2. Let $f(x, y) = \dfrac{4(1 - xy)}{3}$, for $0 \le x \le 1$, $0 \le y \le 1$. Find $P(X + Y \le 1)$.

11-3. Let X and Y be independent random variables with marginal distribution functions $f_X(x) = 2e^{-2x}$, for $x \ge 0$, and $f_Y(y) = 3e^{-3y}$, for $y \ge 0$, and let $S = X + Y$. Find $f_S(s)$.

11-4. For the joint density function given in Example 11.3, find $P(X + Y \le 1.5)$. Hint: Find $P(X + Y > 1.5)$ first.

11-5. Let $f(x, y)$ be the joint density function given in Example 11.4, and let $S = X + Y$. Use a double integral to find $F_S(s)$, take the derivative of this to get $f_S(s)$, and compare with Example 11.4.

11-6. Let X and Y be the independent random variables in Exercise 10-6. Find $P(min(X, Y) > t)$, for $0 < t < 1$. Note: X and Y are *not* exponential random variables.

11.2 Expected Values of Functions of Random Variables

11-7. For the random variables in Exercise 10-1, find $E(X+Y)$ using the joint probabilities in the table. Then find $E(X+Y)$ using the function $p_S(s)$ found in Exercise 11-1. Show that each of these is equal to $E(X)+E(Y)$, as found in Exercise 10-3.

11-8. Let $f(x,y) = \frac{4(1-xy)}{3}$, for $0 \le x \le 1$ and $0 \le y \le 1$, as in Exercise 11-2. Find $E(X+Y)$ using the joint density function. Show that this is equal to $E(X)+E(Y)$.

11-9. Prove that $E(X+Y) = E(X)+E(Y)$ for continuous random variables.

11-10. For the random variables in Example 11.11, find $E(XY)$ directly.

11-11. For the random variables in Exercise 11-8, find (a) $E(XY)$; (b) $E(X)\cdot E(Y)$; (c) $Cov(X,Y)$.

11-12. For the random variables in Exercise 11-8, find (a) $V(X)$; (b) $V(Y)$; (c) $V(X+Y)$.

11-13. For the random variables in Exercise 10-1, find $V(X+Y)$.

11-14. Let X and Y be random variables whose joint probability distribution and marginal distributions are given below.

$y \backslash x$	1	2	$p_Y(y)$
1	.15	.25	.40
2	.35	.25	.60
$p_X(x)$	.50	.50	

Find (a) $E(X)$; (b) $E(Y)$; (c) $V(X)$; (d) $V(Y)$; (e) $Cov(X,Y)$; (f) $V(X+Y)$.

11-15. Let X and Y be the random variables in Exercise 10-22 with joint density function $f(x,y) = 6x$, for $0 < x < y < 1$, and $f(x,y) = 0$ elsewhere. Find (a) $V(X)$; (b) $V(Y)$; (c) $E(XY)$; (d) $V(X+Y)$.

11-16. For the random variables given in Exercise 11-14, find the correlation coefficient.

11-17. For the random variables given in Exercise 11-15, find the correlation coefficient.

11-18. Let X and Y be random variables with joint density function $f(x, y) = x + y$, for $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and $f(x, y) = 0$ elsewhere. Find the correlation coefficient.

11-19. Let X and Y be random variables whose joint density function is $f(x, y) = \dfrac{3(x^2 + y^2)}{8}$, for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, and $f(x, y) = 0$ elsewhere.

(a) Find $f_X(x)$ and $f_Y(y)$, and show that X and Y are not independent.

(b) Find $E(X)$, $E(Y)$, $E(XY)$ and $Cov(X, Y)$.

11.3 Moment Generating Functions for Sums of Independent Random Variables

11-20. Let X and Y be independent random variables with joint probability function $f(x, y) = x(y + 1)/15$, for $x = 1, 2$ and $y = 1, 2$. Find $M_{X+Y}(t)$.

11-21. Let X and Y be independent random variables, each uniformly distributed over $[0, 2]$. Find $M_{X+Y}(t)$.

11.4 The Sum of More Than Two Random Variables

11-22. The random variable S representing the sum of n fair dice is the sum of n independent random variables, X_i, $i = 1, 2, \ldots, n$, where X_i represents the number of dots on the toss of the i^{th} die. Find $E(S)$ and $V(S)$.

11-23. Let X_1, X_2, X_3 and X_4 be random variables such that for each i, $V(X_i) = 13/162$, and for $i \neq j$, $Cov(X_i, X_j) = -1/81$. Find $V(X_1 + X_2 + X_3 + X_4)$.

11-24. Let $S = X_1 + X_2 + \cdots + X_{10}$ be the sum of random variables such that $V(S) = 500/9$, $V(X_i) = 25/3$ for each i, and all covariances, for $i \neq j$, are the same. Find $Cov(X_i, X_j)$.

11-25. Let $S = X_1 + X_2 + \cdots + X_{500}$, where the X_i are independent and identically distributed with mean .50 and variance .25. Use the Central Limit Theorem to find $P(235 \leq S \leq 265)$.

11.5 Double Expectation Theorems

Exercises 11-26 through 11-30 refer to the random variables and distributions in Examples 11.30 and 11.32.

11-26. Find (a) $E(Y|X = 90)$; (b) $E(Y|X = 100)$; (c) $E(Y|X = 110)$.

11-27. Find $E[E(Y|X)]$.

11-28. Find (a) $V(Y|X = 90)$; (b) $V(Y|X = 100)$; (c) $V(Y|X = 110)$.

11-29. Find $E[V(Y|X)]$.

11-30. Find $V[E(Y|X)]$, and verify the identity

$$E[V(Y|X)] + V[E(Y|X)] = V(Y).$$

11-31. The probability that a claim is filed on an insurance policy is .07, and at most one claim is filed in a year. Claim amounts are for either \$500, \$1000 or \$2000. Given that a claim is filed, the distribution of claim amounts is $P(500) = .60$, $P(1000) = .30$ and $P(2000) = .10$. Find the variance of the claim amount paid to a randomly selected policyholder. (Recall that some policyholders do not file a claim and are paid nothing.)

Exercises 11-32 through 11-36 refer to the random variables in Exercise 10-24, whose joint density function is $f(x, y) = 6x$, for $0 < x < y < 1$, and $f(x, y) = 0$ elsewhere.

11-32. Find (a) $f_X(x)$; (b) $E(X)$; (c) $V(X)$.

11-33. Find $E[E(X|Y)]$. (This should be equal to $E(X)$.)

11-34. Find $V(X|Y = y)$.

11-35. Find $E[V(X|Y)]$.

11-36. Find $V[E(X|Y)]$. Verify that $E[V(X|Y)] + V[E(X|Y)] = V(X)$.

11.6 Applying the Double Expectation Theorem; The Compound Poisson Distribution

11-37. The number of claims received by an insurance company in a month is a Poisson random variable with $\lambda = 20$. The claim amounts are independent of each other, and each is uniformly distributed over $[0, 500]$. S is the random variable for the total amount of claims paid. Find (a) $E(S)$; (b) $V(S)$.

11-38. Let the claim amounts in Exercise 11-37 have a lognormal distribution, whose underlying normal distribution has $\mu = 5$ and $\sigma = .40$. Find (a) $E(S)$; (b)$V(S)$.

Use the normal approximation to the compound Poisson distribution in Exercises 11-39 and 11-40.

11-39. The number of claims received in a year by an insurance company is a Poisson random variable with $\lambda = 500$. The claim amounts are independent and uniformly distributed over $[0, 500]$. If the company has \$140,000 available to pay claims, what is the probability that it will have enough to pay all the claims that come in?

11-40. The number of claims received in a year by an insurance company is a Poisson random variable with $\lambda = 500$. The claim amount distribution has mean $E(X) = 600$ and variance $V(X) = 12{,}000$. What is the minimum amount the company would need so that it would have a .95 probability of being able to pay all claims? (Use the fact that $F_Z(1.645) \approx .95$.)

11.8 Sample Actuarial Examination Problems

11-41. An insurance company determines that N, the number of claims received in a week, is a random variable with $P[N=n]=\frac{1}{2^{n+1}}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week.

Determine the probability that exactly seven claims will be received during a given two-week period.

11-42. A company agrees to accept the highest of four sealed bids on a property. The four bids are regarded as four independent random variables with common cumulative distribution function

$$F(x) = \frac{1}{2}(1+\sin \pi x) \quad \text{for} \quad \frac{3}{2} \leq x \leq \frac{5}{2}$$

Which of the following represents the expected value of the accepted bid?

(A) $\pi \int_{3/2}^{5/2} x \cos \pi x dx$

(B) $\frac{1}{16} \int_{3/2}^{5/2} (1+\sin \pi x)^4 dx$

(C) $\frac{1}{16} \int_{3/2}^{5/2} x(1+\sin \pi x)^4 dx$

(D) $\frac{1}{4}\pi \int_{3/2}^{5/2} \cos \pi x(1+\sin \pi x)^3 dx$

(E) $\frac{1}{4}\pi \int_{3/2}^{5/2} x \cos \pi x(1+\sin \pi x)^3 dx$

11.43. Claim amounts for wind damage to insured homes are independent random variables with common density function

$$f(x) = \begin{cases} \frac{3}{x^4} & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

where x is the amount of a claim in thousands.

Suppose 3 such claims will be made. What is the expected value of the largest of the three claims?

11-44. An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have joint density function

$$f(x) = \begin{cases} \frac{2x+2-y}{4} & \text{for } 0<x<1 \text{ and } 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the total loss is at least 1?

11-45. A family buys two policies from the same insurance company. Losses under the two policies are independent and have continuous uniform distributions on the interval from 0 to 10. One policy has a deductible of 1 and the other has a deductible of 2. The family experiences exactly one loss under each policy.

Calculate the probability that the total benefit paid to the family does not exceed 5.

11-46. Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. The joint density function of T_1 and T_2, $f(t_1,t_2)$, is constant over the region $0<t_1<6$, $0<t_2<6$, $t_1+t_2<10$, and zero otherwise.

Determine $E[T_1+T_2]$, the expected time between a car accident and payment of the claim.

11-47. Let T_1 and T_2 represent the lifetimes in hours of two linked components in an electronic device. The joint density function for T_1 and T_2 is uniform over the region defined by $0 \le t_1 \le t_2 \le L$ where L is a positive constant.

Determine the expected value of the sum of the squares of T_1 and T_2.

11-48. In a small metropolitan area, annual losses due to storm, fire, and theft are assumed to be independent, exponentially distributed random variables with respective means 1.0, 1.5, and 2.4.

Determine the probability that the maximum of these losses exceeds 3.

11-49. A company offers earthquake insurance. Annual premiums are modeled by an exponential random variable with mean 2. Annual claims are modeled by an exponential random variable with mean 1. Premiums and claims are independent.

Let X denote the ratio of claims to premiums.

What is the density function of X?

11-50. Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y:

$$E(X) = 50 \quad Var(X) = 50 \quad E(Y) = 20$$
$$Var(Y) = 30 \quad Cov(X,Y) = 10$$

One hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these one hundred people watch movies or sporting events during this three-month period.

Approximate the value of $P(T < 7100)$.

11-51. The profit for a new product is given by $Z = 3X - Y - 5$. X and Y are independent random variables with $Var(X) = 1$ and $Var(Y) = 2$. What is the variance of Z?

11-52. A company has two electric generators. The time until failure for each generator follows an exponential distribution with mean 10. The company will begin using the second generator immediately after the first one fails.

What is the variance of the total time that the generators produce electricity?

11-53. A joint density function is given by

$$f(x,y) = \begin{cases} kx & \text{for } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. What is $Cov(X,Y)$?

11-54. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{8}{3}xy & \text{for } 0 \le x \le 1,\ x \le y \le 2x \\ 0 & \text{otherwise} \end{cases}$$

Calculate the covariance of X and Y.

11-55. Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval $(0,12)$. Given $X = x$, Y is uniformly distributed on the interval $(0, x)$.

Determine $Cov(X,Y)$ according to this model.

11-56. An actuary determines that the claim size for a certain class of accidents is a random variable, X, with moment generating function

$$M_X(t) = \frac{1}{(1-2500t)^4}.$$

Determine the standard deviation of the claim size for this class of accidents.

11-57. A company insures homes in three cities, J, K, and L. Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are independent.

The moment generating functions for the loss distributions of the cities are:

$$M_J(t) = (1-2t)^{-3} \qquad M_K(t) = (1-2t)^{-2.5} \qquad M_L(t) = (1-2t)^{-4.5}$$

Let X represent the combined losses from the three cities.
Calculate $E(X^3)$

11-58. An insurance policy pays a total medical benefit consisting of two parts for each claim.

Let X represent the part of the benefit that is paid to the surgeon, and let Y represent the part that is paid to the hospital. The variance of X is 5000, the variance of Y is 10,000, and the variance of the total benefit, $X + Y$, is 17,000.

Due to increasing medical costs, the company that issues the policy decides to increase X by a flat amount of 100 per claim and to increase Y by 10% per claim.

Calculate the variance of the total benefit after these revisions have been made.

11-59. Let X denote the size of a surgical claim and let Y denote the size of the associated hospital claim. An actuary is using a model in which $E(X)=5$, $E(X^2)=27.4$, $E(Y)=7$, $E(Y^2)=51.4$, and $Var(X+Y)=8$.

Let $C_1 = X + Y$ denote the size of the combined claims before the application of a 20% surcharge on the hospital portion of the claim, and let C_2 denote the size of the combined claims after the application of that surcharge.

Calculate $Cov(C_1, C_2)$.

11-60. Claims filed under auto insurance policies follow a normal distribution with mean 19,400 and standard deviation 5,000.

What is the probability that the average of 25 randomly selected claims exceeds 20,000?

11-61. A company manufactures a brand of light bulb with a lifetime in months that is normally distributed with mean 3 and variance 1. A consumer buys a number of these bulbs with the intention of replacing them successively as they burn out. The light bulbs have independent lifetimes.

What is the smallest number of bulbs to be purchased so that the succession of light bulbs produces light for at least 40 months with probability at least 0.9772?

11-62. An insurance company sells a one-year automobile policy with a deductible of 2.

The probability that the insured will incur a loss is .05. If there is a loss, the probability of a loss of amount N is K/N, for $N = 1,,...,5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur.

Determine the net premium for this policy.

11-63. An auto insurance company insures an automobile worth 15,000 for one year under a policy with a 1,000 deductible. During the policy year there is a .04 chance of partial damage to the car and a .02 chance of a total loss of the car. If there is partial damage to the car, the amount X of damage (in thousands) follows a distribution with density function

$$f(x) = \begin{cases} .5003e^{-x/2} & \text{for } 0 < x < 15 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected claim payment?

CHAPTER 11

11-1. If X and Y are the random variables in Exercise 10-1, the possible values of X are 1, 2 and 3, and of Y are 1 and 2. The possible values of S = X + Y are 2, 3, 4 and 5. Since X and Y are independent,

$$p_S(s) = \sum_x p_X(x) p_Y(s - x).$$

$p_S(2) = p_X(1)p_Y(1) = (2/9)(1/3) = 2/27$

$p_S(3) = p_X(1)p_Y(2) + p_X(2)p_Y(1) = (2/9)(2/3) + (1/3)(1/3) = 7/27$

$p_S(4) = p_X(2)p_Y(2) + p_X(3)p_Y(1) = (1/3)(2/3) + (4/9)(1/3) = 10/27$

$p_S(5) = p_X(3)p_Y(2) = (4/9)(2/3) = 8/27$

11-2. If $f(x,y) = (4/3)(1 - xy)$, $0 \le x \le 1$ and $0 \le y \le 1$:

$$\begin{aligned} P(X + Y \le 1) &= \int_0^1\int_0^{1-y} f(x, y)dxdy \\ &= \int_0^1\int_0^{1-y} (4/3)(1 - xy)dxdy \\ &= (4/3) \int_0^1 (x - x^2y/2)\Big|_{x=0}^{1-y} dy \\ &= (2/3) \int_0^1 (2 - 3y + 2y^2 - y^3)dy \\ &= (2/3)(2y - 3y^2/2 + 2y^3/3 - y^4/4)\Big|_0^1 \\ &= 11/18 \end{aligned}$$

11-3. If $f_X(x) = 2e^{-2x}$ for $x > 0$ and $f_Y(y) = 3e^{-3y}$ for $y > 0$, and $S = X + Y$:

$$
\begin{aligned}
f_S(s) = \int_{-\infty}^{\infty} f_X(x) f_Y(s-x)dx &= 6 \int_0^s e^{-2x} e^{-3(s-x)}dx \\
&= 6e^{-3s} \int_0^s e^{x}dx = 6e^{-3s}(e^{s} - 1) \\
&= 6(e^{-2s} - e^{-3s}),\ s > 0
\end{aligned}
$$

11-4. From Example 11.3, $f(x,y) = 2 - 1.2x - .8y$, $0 \le x \le 1$, $0 \le y \le 1$.

$$
\begin{aligned}
P(X + Y \le 1.5) &= 1 - P(X + Y > 1.5) \\
P(X + Y > 1.5) &= \int_{.5}^{1}\int_{1.5-y}^{1}(2 - 1.2x - .8y)dxdy \\
&= \int_{.5}^{1}(2x - .6x^2 - .8xy)\Big|_{x=1.5-y}^{1} dy \\
&= \int_{.5}^{1}(-.25 + .6y - .2y^2)dy \\
&= (-.25y + .3y^2 - .2y^3/3)\Big|_{.5}^{1} = .0416\bar{6} \\
P(X + Y \le 1.5) &= 1 - .0416\bar{6} = .9583\bar{3}
\end{aligned}
$$

11-5. From Example 11.4, $f(x,y) = e^{-(x+y)}$, $x \ge 0, y \ge 0$. If $S = X + Y$:

$$
\begin{aligned}
F_S(s) = P(S \le s) &= P(X + Y \le s) \\
&= \int_0^s\int_0^{s-y} e^{-(x+y)}dxdy \\
&= \int_0^s e^{-y}(-e^{-x})\Big|_{x=0}^{s-y} dy \\
&= \int_0^s (e^{-y} - e^{-s})dy \\
&= (-e^{-y} - ye^{-s})\Big|_0^s = 1 - e^{-s}(1 + s)
\end{aligned}
$$

To compare $f_S(s)$ with Example 11.4,

$$f_S(s) = F_S'(s) = e^{-s} - e^{-s} + se^{-s} = se^{-s}.$$

11-6. For the independent random variables in Exercise 10-6, we found that $f_X(x) = 1/2 + x$, for $0 \le x \le 1$, and $f_Y(y) = 1/2 + y$, for $0 \le y \le 1$.

$$
\begin{aligned}
P(\min(X,Y) > t) &= P(X > t \text{ and } Y > t) \\
&= P(X > t)P(Y > t) \\
&= \left(\int_t^1 (1/2 + x)dx\right)\left(\int_t^1 (1/2 + y)dy\right)
\end{aligned}
$$

(Note that both integrals give the same value)

$$
\begin{aligned}
&= \left((x/2 + x^2/2)\Big|_t^1\right)^2 \\
&= (1 - t/2 - t^2/2)^2
\end{aligned}
$$

11-7. Using the table obtained in Exercise 10-1:

$$
\begin{aligned}
E(X + Y) &= \sum_y \sum_x (x + y)p(x,y) \\
&= (1 + 1)(2/27) + (2 + 1)(3/27) + (3 + 1)(4/27) \\
&\quad + (1 + 2)(4/27) + (2 + 2)(6/27) + (2+3)(8/27) \\
&= 105/27 = 35/9
\end{aligned}
$$

Using the probability distribution $f_S(s)$ from Exercise 11-1:

$$
\begin{aligned}
E(S) = E(X + Y) &= \sum_s s f_S(s) \\
&= 2(2/27) + 3(7/27) + 4(10/27) + 5(8/27) = 35/9
\end{aligned}
$$

From Exercise 10-3, $E(X) = 20/9$ and $E(Y) = 5/3 = 15/9$, so

$$E(X) + E(Y) = 20/9 + 15/9 = 35/9 = E(X + Y).$$

11-8. For the joint density function $f(x,y) = (4/3)(1 - xy)$, $0 \le x \le 1$, and $0 \le y \le 1$:

$$
\begin{aligned}
E(X + Y) &= \int_0^1\int_0^1 (x + y)f(x,y)dxdy \\
&= (4/3)\int_0^1\int_0^1 (x + y - x^2y - xy^2)dxdy \\
&= (4/3)\int_0^1 (x^2/2 + xy - x^3y/3 - x^2y^2/2)\Big|_{x=0}^{1} dy \\
&= (4/3)\int_0^1 (1/2 + 2y/3 - y^2/2)dy \\
&= (4/3)(y/2 + y^2/3 - y^3/6)\Big|_0^1 = 16/18 = 8/9
\end{aligned}
$$

$$
\begin{aligned}
f_X(x) = \int_0^1 f(x,y)dy &= (4/3)\int_0^1 (1 - xy)dy \\
&= (4/3)(y - xy^2/2)\Big|_{y=0}^{1} = (4/3)(1 - x/2)
\end{aligned}
$$

$$
\begin{aligned}
E(X) &= \int_0^1 x f_X(x)dx = (4/3)\int_0^1 (x - x^2/2)dx \\
&= (4/3)(x^2/2 - x^3/6)\Big|_0^1 = 4/9
\end{aligned}
$$

A similar calculation will show that $E(Y) = 4/9$, so $E(X + Y) = E(X) + E(Y)$.

11-9. Let X and Y be random variables with joint density function $f(x,y)$.

$$
\begin{aligned}
E(X + Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x + y)f(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f(x,y)dydx + \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y)dxdy \\
&= \int_{-\infty}^{\infty} x f_X(x)dx + \int_{-\infty}^{\infty} y f_Y(y)dy \\
&= E(X) + E(Y)
\end{aligned}
$$

11-10. For the random variables (independent) in Example 11.9:

$$E(XY) = \sum_y \sum_x xy p(x,y) = \sum_y \sum_x xy p_X(x) p_Y(y)$$

The only non-zero terms occur when both $x \ne 0$ and $y \ne 0$, i.e. when $(x,y) = (1,1), (1,2), (2,1)$ or $(2,2)$.

$$
\begin{aligned}
E(XY) &= 1(1/4)(1/4) + 2(1/4)(1/4) + 2(1/4)(1/4) + 4(1/4)(1/4) \\
&= 9/16
\end{aligned}
$$

11-11. For the random variables in Exercise 11-8, $f(x,y) = (4/3)(1 - xy)$, for $0 \le x \le 1$ and $0 \le y \le 1$.

(a)
$$
\begin{aligned}
E(XY) = \int_0^1\int_0^1 xyf(x,y)dxdy &= (4/3)\int_0^1\int_0^1 (xy - x^2y^2)dxdy \\
&= (4/3)\int_0^1 (x^2y/2 - x^3y^2/3)\Big|_{x=0}^{1} dy \\
&= (4/3)\int_0^1 (y/2 - y^2/3)dy = (4/3)(y^2/4 - y^3/9)\Big|_0^1 \\
&= (4/3)(5/36) = 5/27
\end{aligned}
$$

(b) $E(X) = E(Y) = 4/9$ from Exercise 11-8, so $E(X)E(Y) = 16/81$

(c) $Cov(X,Y) = E(XY) - E(X)E(Y) = 5/27 - 16/81 = -1/81$

11-12. For the random variables in Exercise 11-8:

(a) $f_X(x) = (4/3)(1 - x/2)$, for $0 \le x \le 1$

$$
\begin{aligned}
E(X^2) &= \int_0^1 x^2 f_X(x)dx = (4/3)\int_0^1 (x^2 - x^3/2)dx \\
&= (4/3)(x^3/3 - x^4/8)\Big|_0^1 = (4/3)(5/24) = 5/18 \\
V(X) &= E(X^2) - E(X)^2 = 5/18 - 16/81 = 13/162
\end{aligned}
$$

(b) A similar calculation will show $V(Y) = 13/162$.

(c)
$$
\begin{aligned}
V(X + Y) &= V(X) + V(Y) + 2Cov(X,Y) \\
&= 13/162 + 13/162 - 2/81 = 11/81
\end{aligned}
$$

11-13. For the random variables in Exercise 10-1 we have already shown (see Exercise 10-3) that $E(X) = 20/9$ and $E(Y) = 5/3$.

$$E(X^2) = \sum_x x^2 p_X(x) = 1(2/9) + 4(1/3) + 9(4/9) = 50/9$$
$$V(X) = E(X^2) - E(X)^2 = 50/9 - (20/9)^2 = 50/81$$
$$E(Y^2) = \sum_y y^2 p_Y(y) = 1(1/3) + 4(2/3) = 3$$
$$V(Y) = E(Y^2) - E(Y)^2 = 3 - (5/3)^2 = 2/9$$

Since X and Y are independent (see Exercise 10-23),

$$V(X + Y) = V(X) + V(Y) = 50/81 + 2/9 = 68/81.$$

11-14. (a) $E(X) = 1(.5) + 2(.5) = 1.5$

(b) $E(Y) = 1(.4) + 2(.6) = 1.6$

(c) $E(X^2) = 1(.5) + 4(.5) = 2.5$
$V(X) = 2.5 - (1.5)^2 = .25$

(d) $E(Y^2) = 1(.4) + 4(.6) = 2.8$
$V(Y) = 2.8 - (1.6)^2 = .24$

(e) $E(XY) = 1(.15) + 2(.25) + 2(.35) + 4(.25) = 2.35$
$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 2.35 - (1.5)(1.6) = -.05$

(f) $V(X + Y) = V(X) + V(Y) + 2\mathrm{Cov}(X,Y) = .25 + .24 - .10 = .39$

11-16. For the random variables in Exercise 11-14, $V(X) = .25$, $V(Y) = .24$ and $\mathrm{Cov}(X,Y) = -.05$. Hence

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{-.05}{\sqrt{(.25)(.24)}} = -.2041$$

11-17. For the random variables in Exercise 11-15, $V(X) = 1/20$, $V(Y) = 3/80$ and $\mathrm{Cov}(X,Y) = 1/40$. Hence

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{1/40}{\sqrt{(1/20)(3/80)}} = .5774 .$$

11-18. Let X and Y be random variables whose joint density function is $f(x,y) = x + y$, for $0 \le x \le 1$ and $0 \le y \le 1$.

$$f_X(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = (xy + y^2/2)\Big|_{y=0}^1 = x + 1/2, \text{ for } 0 \le x \le 1$$
$$E(X) = \int_0^1 x f_X(x)dx = \int_0^1 (x^2 + x/2)dx = (x^3/3 + x^2/4)\Big|_0^1 = 7/12$$
$$E(X^2) = \int_0^1 x^2 f_X(x)dx = \int_0^1 (x^3 + x^2/2)dx = (x^4/4 + x^3/6)\Big|_0^1 = 5/12$$
$$V(X) = 5/12 - (7/12)^2 = 11/144$$

A similar calculation will show that $E(Y) = 7/12$ and $V(Y) = 11/144$.

$$E(XY) = \int_0^1\int_0^1 xy(x+y)dxdy = \int_0^1\int_0^1 (x^2y + xy^2)dxdy$$
$$= \int_0^1 (x^3y/3 + x^2y^2/2)\Big|_{x=0}^1 dy = \int_0^1 (y/3 + y^2/2)dy$$
$$= (1/6)(y^2 + y^3)\Big|_0^1 = 1/3$$
$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 1/3 - (7/12)(7/12) = -1/144$$
$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{-1/144}{\sqrt{(11/144)^2}} = -1/11$$

11-15. For the random variables in Exercise 10-22, we have already shown $f_Y(y) = 3y^2$, for $0 \le y \le 1$.

$$f_X(x) = \int_x^1 f(x)dy = 6\int_x^1 xdy = 6x(1-x), \text{ for } 0 \le x \le 1$$
$$E(X) = \int_0^1 x(6x)(1-x)dx = 6\int_0^1 (x^2 - x^3)dx = 6(x^3/3 - x^4/4)\Big|_0^1 = 1/2$$
$$E(X^2) = \int_0^1 x^2(6x)(1-x)dx = 6\int_0^1 (x^3 - x^4)dx = 6(x^4/4 - x^5/5)\Big|_0^1 = 3/10$$
$$E(Y) = \int_0^1 3y^3dy = (3y^4/4)\Big|_0^1 = 3/4$$
$$E(Y^2) = \int_0^1 3y^4dy = (3y^5/5)\Big|_0^1 = 3/5$$

(a) $V(X) = 3/10 - (1/2)^2 = 1/20$

(b) $V(Y) = 3/5 - (3/4)^2 = 3/80$

(c) $E(XY) = \int_0^1\int_0^y xyf(x,y)dxdy = \int_0^1\int_0^y 6x^2ydxdy$
$$= \int_0^1 (2x^3y)\Big|_{x=0}^y dy = \int_0^1 2y^4dy = (2y^5/5)\Big|_0^1 = 2/5$$

(d) $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 2/5 - (1/2)(3/4) = 1/40$
$V(X + Y) = V(X) + V(Y) + 2\mathrm{Cov}(X,Y) = 1/20 + 3/80 + 2/40 = 11/80$

11-19. Let $f(x,y) = (3/8)(x^2 + y^2)$, for $-1 \le x \le 1$ and $-1 \le y \le 1$.

(a) $f_x(x) = (3/8)\int_{-1}^1 (x^2 + y^2)dy = (3/8)(x^2y + y^3/3)\Big|_{y=-1}^1 = (3/8)(2x^2 + 2/3) = 3x^2/4 + 1/4$, for $-1 \le x \le 1$

$$f_Y(y) = (3/8)\int_{-1}^1 (x^2 + y^2)dx = 1/4 + 3y^2/4, \text{ for } -1 \le y \le 1$$

$$f_X(x)f_Y(y) = (1/4 + 3x^2/4)(1/4 + 3y^2/4) \ne (x^2 + y^2) = f(x,y)$$

Hence X and Y are not independent.

(b) $E(X) = \int_{-1}^1 x f_X(x)dx = \int_{-1}^1 (x/4 + 3x^3/4)dx = (x^2/8 + 3x^4/16)\Big|_{-1}^1 = 0$

By a similar calculation, $E(Y) = 0$.

$$E(XY) = \int_{-1}^1\int_{-1}^1 xyf(x,y)dxdy = (3/8)\int_{-1}^1\int_{-1}^1 xy(x^2 + y^2)dxdy$$
$$= (3/8)\int_{-1}^1 (x^4y/4 + x^2y^3/2)\Big|_{x=-1}^1 dy = 0$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0$$

11-20. The joint probability function yields the following table.

y \ x	1	2	$p_Y(y)$
1	2/15	4/15	2/5
2	3/15	6/15	3/5
$p_X(x)$	1/3	2/3	

X and Y are clearly independent.

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} p(x) = (1/3)e^t + (2/3)e^{2t}$$

$$M_Y(t) = E(e^{tY}) = \sum_y e^{ty} p(y) = (2/5)e^t + (3/5)e^{2t}$$

By independence: $M_{X+Y}(t) = M_X(t)M_Y(t)$
$= (1/15)(2e^{2t} + 7e^{3t} + 6e^{4t})$

11-21. If X and y are each uniformly distributed over [0, 2], then

$M_X(t) = M_Y(t) = (e^{2t} - 1)/2t$. (Exercise 9-4)

Since X and Y are independent,

$M_{X+Y}(t) = M_X(t)M_Y(t) = (e^{2t} - 1)^2/4t^2$

11-22. By Example 5.25 and Exercise 5-33, for each i, $E(X_i) = 7/2$ and $V(X_i) = 35/12$.

$$E(S) = E(X_1) + E(X_2) + \ldots + E(X_n) = n(7/2)$$

Since the X_i's are independent,

$$V(S) = V(X_1) + V(X_2) + \ldots + V(X_n) = n(35/12).$$

11-23. $V(X_1 + X_2 + X_3 + X_4) = \sum_{i=1}^{4} V(X_i) + 2\sum_{i<j} Cov(X_i X_j)$
$= 4(13/162) + 2(6)(-1/81) = 14/81$

11-24. If $S = X_1 + X_2 + \ldots + X_{10}$, $V(S) = 500/9$, $V(X_i) = 25/3$ for each i, and all covariances $Cov(X_i, X_j)$ for $i \neq j$ are the same, then

$$V(S) = \sum_{i=1}^{10} V(X_i) + 2\sum_{i<j} Cov(X_i, X_j).$$

(The number of terms in the covariance sum is $1 + 2 + \ldots + 9 = 45$.)

$500/9 = 10(25/3) + 2(45)Cov(X_i, X_j)$
$Cov(X_i, X_j) = -25/81$

11-25. S is the sum of 500 independent and identically distributed random variables each with mean .50 and variance .25. Then S is approximately normal with $\mu = 500(.5)$ and $\sigma^2 = 500(.25)$.

$$P(235 \leq S \leq 265) = P\left(\frac{235-250}{\sqrt{125}} \leq Z \leq \frac{265-250}{\sqrt{125}}\right)$$
$$= P(-1.34 \leq Z \leq 1.34) = .8198$$

For Exercises 11-26 to 11-30, we make the following table of conditional probabilities from the table in Example 11.28, e.g.,

$$p(0|90) = P(Y = 0|X = 90) = p(90,0)/p_X(90) = .05/.20 = .25.$$

y	0	10
p(y|90)	.25	.75
p(y|100)	.45	.55
p(y|110)	.90	.10

11-26. (a) $E(Y|X = 90) = 0p(0|90) + 10p(10|90) = 0 + 7.5 = 7.5$

(b) $E(Y|X = 100) = 0p(0|100) + 10p(10|100) = 5.5$

(c) $E(Y|X = 110) = 0p(0|110) + 10p(10|110) = 1$

11-27. $E[E(Y|X)] = E(Y|X = 90)p_X(90) + E(Y|X = 100)p_X(100) + E(Y|X = 110)p_X(100)$

$$= 7.5(.20) + 5.5(.60) + 1(.20) = 5 = E(Y)$$

11-28. (a) $E(Y^2|X = 90) = 0(.25) + 100(.75) = 75$
$V(Y|X = 90) = 75 - (7.5)^2 = 18.75$

(b) $E(Y^2|X = 100) = 0(.45) + 10(.55) = 55$
$V(Y|X = 100) = 55 - (5.5)^2 = 24.75$

(c) $E(Y^2|X = 110) = 0(.90) + 100(.10) = 10$
$V(Y|X = 110) = 10 - 1 = 9$

11-29. $E[V(Y|X)] = V(Y|X = 90)p_X(90) + V(Y|X = 100)p_Y(100) + V(Y|X = 110)p_X(110)$
$= 18.75(.20) + 24.75(.60) + 9(.20) = 20.4$

11-30. $V[E(Y|X)] = \sum_x (E(Y|X = x) - E(Y))^2 p_X(x)$
$= (7.5 - 5)^2(.2) + (5.5 - 5)^2(.6) + (1 - 5)^2(.2)$
$= 4.6$

$V[E(Y|X)] + E[V(Y|X)] = 4.6 + 20.4 = 25 = V(Y)$

11-31. Let S be the random variable indicating whether a claim has been filed (S = 1) or not (S = 0). Then $P(S = 0) = .93$ and $P(S = 1) = .07$. Let X be the random variable of the claim amount paid.

$P(X = 500|S = 1) = .60$
$P(X = 1000|S = 1) = .30$
$P(X = 2000|S = 1) = .10$

$E(X|S = 0) = 0$ and $V(X|S = 0) = 0$

$E(X|S = 1) = 500(.60) + 1000(.30) + 2000(.10) = 800$
$E[E(X|S)] = E(X) = 0(.93) + 800(.07) = 56$

$E(X^2|S = 1) = 500^2(.60) + 1000^2(.30) + 2000^2(.10) = 850{,}000$
$V(X|S = 1) = 850{,}000 - 800^2 = 210{,}000$
$E[V(X|S)] = 0(.93) + 210{,}000(.07) = 14{,}700$

$V[E(X|S)] = (E(X|S = 0) - E(X))^2P(S = 0) + (E(X|S = 1) - E(X))^2P(S = 1)$
$= (0 - 56)^2(.93) + (800 - 56)^2(.07) = 41{,}664$

$V(X) = V[E(X|S)] + E[V(X|S)] = 41{,}664 + 14{,}700 = 56{,}364$

11-32. For $f(x,y) = 6x$, $0 < x < y < 1$ and 0 elsewhere:

(a) $f_X(x) = \int_x^1 6x\,dy = 6x(1 - x)dx$, for $0 < x < 1$

(b) $E(X) = \int_0^1 (6x^2 - 6x^3)dx = 1/2$

(c) $E(X^2) = \int_0^1 (6x^3 - 6x^4)dx = 3/10$
$V(X) = 3/10 - (1/2)^2 = 1/20$

(Note: All these were calculated in Exercise 11-15.)

11-33. In Exercise 10-22 we showed that $f_Y(y) = 3y^2$, $0 < y < 1$.

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{6x}{3y^2} = \frac{2x}{y^2}, \quad 0 < x < y$$

$$E(X|Y=y) = \int_0^y x f(x \mid y)dx = \int_0^y \frac{2x^2}{y^2}dx = \frac{2x^3}{3y^2}\Big|_{x=0}^{y} = 2y/3$$

$$E[E(X|Y)] = \int_0^1 E(X \mid Y=y) f_Y(y)dy = \int_0^1 \frac{2y}{3}(3y^2)dy$$

$$= (2y^4/4)\Big|_0^1 = 1/2 \quad \text{(This equals E(X).)}$$

11-34. $E(X^2 \mid Y=y) = \int_0^y x^2\left(\frac{2x}{y^2}\right)dx = \frac{2x^4}{4y^2}\Big|_0^y = y^2/2$

$V(X|Y=y) = y^2/2 - (2y/3)^2 = y^2/18$

11-35. $E[V(X|Y)] = \int_0^1 V(X \mid Y=y) f_Y(y)dy = \int_0^1 \frac{y^4}{6}dy$

$$= \frac{y^5}{30}\Big|_0^1 = 1/30$$

11-36. $E[E(X|Y)^2] = \int_0^1 \left(\frac{2y}{3}\right)^2 (3y^2)dy = \frac{4y^5}{15}\Big|_0^1 = 4/15$

$V[E(X|Y)] = 4/15 - (1/2)^2 = 1/60$

$E[V(X|Y)] + V[E(X|Y)] = 1/30 + 1/60 = 1/20 = V(X)$

11-37. The number of claims N is a Poisson random variable, so $E(N) = V(N) = \lambda = 20$. The amounts of claim random variable X is uniform over [0, 500], so $E(X) = 250$ and $V(X) = 500^2/12$.

(a) $E(S) = E(N)E(X) = 20(250) = 5000$

(b) $V(S) = E(N)V(X) + V(N)E(X)^2$
$= 20(500^2/12 + 250^2) = 1{,}666{,}666.67$

11-38. If the amount of claim random variable X is lognormal whose underlying normal distribution has $\mu = 5$ and $\sigma = .40$:

$$E(X) = e^{\mu + \frac{\sigma^2}{2}} = e^{5.08}$$
$$V(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) = e^{10.16}(e^{.16} - 1)$$

(a) $E(S) = 20e^{5.08} = 3215.48$

(b) $V(S) = 20[e^{10.16}(e^{.16} - 1) + e^{10.16}) = 606{,}665.15$

11-39. The claim amount S is approximately normal, with

$\mu = E(N)E(X) = 500(250) = 125{,}000$ and
$\sigma^2 = 500(500^2/12 + 250^2)$, so $\sigma = 6454.97$.

$$P(S \le 140{,}000) = P\left(Z \le \frac{140{,}000 - 125{,}000}{6454..97}\right)$$
$$= P(Z \le 2.32) = .9898$$

11-40. If $E(X) = 600$ and $V(X) = 12{,}000$, then S is approximately normal with $\mu = 500(600) = 300{,}000$ and $\sigma^2 = 500(12{,}000 + 600^2)$, so $\sigma = 13{,}638.18$.

We need to find S_o so that $P(S \le S_o) = .95$. From the z-tables we know that $F_Z(1.645) \approx .95$. Hence

$$P(Z \le 1.645) = P\left(\frac{S - 300{,}000}{13{,}638.18} \le 1.645\right) \approx .95.$$

Therefore $S_o = 300{,}000 + 1.645(13{,}638.18) = 322{,}434.81$.

CHAPTER 11

11-1.

s	2	3	4	5
$p_S(s)$	2/27	7/27	10/27	8/27

11-2. 11/18

11-3. $6(e^{-2s} - e^{-3s})$

11-4. .95833

11-5. $F_S(s) = 1 - e^{-s}(1+s)$

11-6. $(1 - t/2 - t^2/2)^2$

11-7. $E(X + Y) = 35/9 = 20/9 + 15/9 = E(X) + E(Y)$

11-8. $E(X + Y) = 8/9$; $E(X) = E(Y) = 4/9$

11-11. (a) 5/27 (b) 16/81 (c) $-1/81$

11-12. (a) 13/162 (b) 13/162 (c) 11/81

11-13. 68/81

11-14. (a) 1.5 (b) 1.6 (c) .25 (d) .24 (e) $-.05$ (f) .39

11-15. (a) 1/20 (b) 3/80 (c) 2/5 (d) 11/80

11-16. $-.2041$

11-17. .5774

11-18. $-\frac{1}{11}$

11-19. (a) $f_X(x) = \frac{1}{4} + \frac{3}{4}x^2$, $f_Y(y) = \frac{1}{4} + \frac{3}{4}y^2$,
$f_X(x) \cdot f_Y(y) \neq f(x, y)$

(b) $E(X) = E(Y) = E(XY) = Cov(X, Y) = 0$

11-20. $(2e^{2t} + 7e^{3t} + 6e^{4t})/15$

11-21. $[(e^{2t} - 1)^2/(4t^2)]$

11-22. $E(S) = n(7/2)$; $V(S) = n(35/12)$

11-23. 14/81

11-24. $-25/81$

11-25. .8198 (Table), .82029 (TI-83)

11-26. (a) 7.5 (b) 5.5 (c) 1

11-27. 5

11-28. (a) 18.75 (b) 24.75 (c) 9

11-29. 20.4

11-30. 4.6

11-31. 56,364

11-32. (a) $6x(1 - x)$, for $0 < x < 1$ (b) 1/2 (c) 1/20

11-33. 1/2

11-34. $y^2/18$

11-35. 1/30

11-36. 1/60

11-37. (a) 5000 (b) 1,666,666.67

11-38. (a) 3215.48 (b) 606,665.15

11-39. .9898(Table), .98993(TI-83)

11-40. 322,434.81

11-41. 1/64

11-42. $\frac{1}{4}\pi \int_{3/2}^{5/2} x \cos \pi x (1 + \sin \pi x)^3 \, dx$

11-43. 2025

11-44. .71

11-45. .295

11-46. 5.72

11-47. $\frac{2L^2}{3}$

11-48. .414

11-49. $\frac{2}{(2x+1)^2}$ for $x > 0$

11-50. .8413

11-51. 11

11-52. 200

11-53. 0

11-54. .041

11-55. 6

11-56. 5,000

11-57. 10,560

11-58. 19,300

11-59. 8.80

11-60. .2743

11-61. 16

11-62 .03139

11-63. 328

Chapter 12
Stochastic Processes

12.6 Exercises

12.1 Simulation Examples

For Exercises 12-1 through 12-3, use the following sequence of random numbers.

1. .57230	6. .82496	11. .02480	16. .78322
2. .85472	7. .52184	12. .99954	17. .00067
3. .37282	8. .49837	13. .81708	18. .24844
4. .77133	9. .76729	14. .90535	19. .14118
5. .20525	10. .50986	15. .76227	20. .47417

12-1. For the two gamblers in Example 12.1, suppose A has 3 coins and B has 5 coins, and the game is played as described in the example. Use the random numbers given above to simulate the game. Which player would win the game, and how many coin tosses were needed to decide the winner?

12-2. For an employee in the pension plan in Example 12.2, the probabilities for staying in a fund or switching funds are given in the following table.

Start in \ End in	0	1
0	.65	.35
1	.25	.75

Use the decision-making process for switching funds described in the example and the random numbers given above to simulate the progress of an employee who is initially in Fund 0. How many times in the next 20 months would he switch to, or stay in, Fund 1?

12-3. Suppose the waiting time in months between accidents at an intersection is exponential with $\lambda = 3$. Use the method in Example 12.3 and the random numbers given above to simulate the time between accidents. How many accidents occur in each of the first three months at this intersection?

12.2 Finite Markov Chains

12-4. For members in a pension plan, the transition matrix of probabilities of switching funds is

$$\mathbf{P} = \begin{bmatrix} .65 & .35 \\ .25 & .75 \end{bmatrix}.$$

If the initial probability distribution is $\mathbf{p}^{(0)} = [.50, \ .50]$, find (a) $\mathbf{p}^{(1)}$; (b) $\mathbf{p}^{(2)}$.

12-5. The transition matrix for a Markov process with 2 states is

$$\mathbf{P} = \begin{bmatrix} .72 & .28 \\ .36 & .64 \end{bmatrix},$$

and the initial probability distribution is $\mathbf{p}^{(0)} = [.40, \ .60]$. Find (a) $\mathbf{p}^{(1)}$; (b) $\mathbf{p}^{(2)}$.

12-6. The transition matrix for a Markov process with 3 states is

$$\mathbf{P} = \begin{bmatrix} .4 & .2 & .4 \\ .2 & .5 & .3 \\ .1 & .3 & .6 \end{bmatrix},$$

and the initial probability distribution is $\mathbf{p}^{(0)} = [.30, .30, .40]$. Find $\mathbf{p}^{(1)}$.

12-7. A mutual fund investor has the choice of a stock fund (Fund 0), a bond fund (Fund 1), and a money market fund (Fund 2). At the end of each quarter she can move her money from fund to fund. The probability that she stays in Fund 0 is .60, in Fund 1, .50, and in Fund 2, .40. If she switches funds, she will move to each of the other funds with equal probability. If she starts with all of her money in the stock fund, what is the probability distribution after two quarters?

12.3 Regular Markov Processes

12-8. For the transition matrix in Exercise 12-4, find the limiting distribution.

12-9. What is the limiting distribution for the Markov process in Exercise 12-5?

12-10. What is the limiting distribution for the Markov process in Exercise 12-6?

12-11. What is the limiting distribution for the investor in Exercise 12-7?

12-12. Prove that if $\mathbf{P}$ is the transition matrix of a regular finite Markov process and $\boldsymbol{\ell}$ is its limiting distribution, then $\boldsymbol{\ell}\mathbf{P} = \boldsymbol{\ell}$. Hint: Write $\boldsymbol{\ell}\mathbf{P}^n = (\boldsymbol{\ell}\mathbf{P}^{n-1})\mathbf{P}$ and take the limit of both sides.

12.4 Absorbing Markov Chains

12-13. In the gambler's ruin example, suppose the game is rigged so that the probability that A wins is 1/3 and the probability that B wins is 2/3. Let the states represent the number of coins that A has at any time, and let the total number of coins between both players be 3.

(a) Find the matrix $\mathbf{N}$.

(b) Find the matrix $\mathbf{A}$.

(c) If A starts with 2 coins, what is the probability that he will lose (end in State 0)?

12-14. Let the gamblers in Exercise 12-13 start with 4 coins between them.

(a) Find the matrix $\mathbf{N}$.

(b) Find the matrix $\mathbf{A}$.

(c) If A starts with 2 coins, what is the probability that he will lose?

CHAPTER 12

The data in the table below is for the simulations in Exercises 12-1 and 12-2. The third column shows the number of coins player A has in 12-1 and the fourth shows which fund the employee in 12-2 is in at the end of each month.

Trial	Random Number	A has	Fund
Start		3	0
1	.57230	4	0
2	.85472	5	1
3	.37282	4	1
4	.77133	5	1
5	.20525	4	0
6	.82496	5	1
7	.52184	6	1
8	.49837	5	1
9	.76729	6	1
10	.50986	7	1
11	.02480	6	0
12	.99954	7	1
13	.81708	8	1
14	.90535		1
15	.76227		1
16	.78322		1
17	.00067		0
18	.24844		0
19	.14118		0
20	.47417		0

12-1. Player A wins a toss if the random number is in [.5,1) and loses otherwise. The game ends when A has all 8 coins (he wins the game) or when he has 0 coins (he loses the game). From the table, A wins in 13 tosses.

12-2. If the employee is in Fund 0, he remains there if the random number is in [0,.65) and switches otherwise. If he is in Fund 1, he switches if the random number is in [0,.25) and stays there otherwise. From the table he is in Fund 1 thirteen times.

12-3. The waiting time X between accidents is exponentially distributed with $\lambda = 3$, so $F(x) = 1 - e^{-3x}$. To simulate waiting times, $x = F^{-1}(u) = -\ln(1 - u)/3$. The following table simulates the waiting time between accidents and the cumulative time to the n^{th} accident. From the table you count the number of accidents in the first few months.

Trial	Random Number	$F^{-1}(u)$ Waiting Time	Total Time
1	0.57230	0.28311	0.28311
2	0.85472	0.64303	0.92614
3	0.37282	0.15551	1.08165
4	0.77133	0.49183	1.57347
5	0.20525	0.07658	1.65005
6	0.82496	0.58091	2.23096
7	0.52184	0.24594	2.47690
8	0.49837	0.22996	2.70686
9	0.76729	0.48599	3.19285
10	0.50986	0.23769	3.43054
11	0.02480	0.00837	3.43891
12	0.99954	2.56143	6.00034
13	0.81708	0.56624	6.56657
14	0.90535	0.78586	7.35243
15	0.76227	0.47887	7.83130
16	0.78322	0.50962	8.34093
17	0.00067	0.00022	8.34115
18	0.24844	0.09520	8.43635
19	0.14118	0.05073	8.48708
20	0.47417	0.21426	8.70134

12-4. (a) $\mathbf{p}^{(1)} = \mathbf{p}^{(0)}\mathbf{P} = [.50, \quad .50]\begin{bmatrix} .65 & .35 \\ .25 & .75 \end{bmatrix} = [.45, \quad .55]$

(b) $\mathbf{p}^{(2)} = \mathbf{p}^{(1)}\mathbf{P} = [.45, \quad .55]\begin{bmatrix} .65 & .35 \\ .25 & .75 \end{bmatrix} = [.43, \quad .57]$

12-5. (a) $\mathbf{p}^{(1)} = \mathbf{p}^{(0)}\mathbf{P} = [.40, \quad .60]\begin{bmatrix} .72 & .28 \\ .36 & .64 \end{bmatrix} = [.504, \quad .496]$

(b) $\mathbf{p}^{(2)} = \mathbf{p}^{(1)}\mathbf{P} = [.504, \quad .496]\begin{bmatrix} .72 & .28 \\ .36 & .64 \end{bmatrix} = [.54144, \quad .45856]$

12-6. $\mathbf{p}^{(1)} = \mathbf{p}^{(0)}\mathbf{P} = [.30, \quad .30, \quad .40]\begin{bmatrix} .4 & .2 & .4 \\ .2 & .5 & .3 \\ .1 & .3 & .6 \end{bmatrix} = [.22, \quad .33, \quad .45]$

12-7. For this Markov process the transition matrix is

$$\mathbf{P} = \begin{bmatrix} .60 & .20 & .20 \\ .25 & .50 & .25 \\ .30 & .30 & .40 \end{bmatrix} \text{ and } \mathbf{p}^{(0)} = [1, \quad 0, \quad 0]$$

$$\mathbf{p}^{(1)} = \mathbf{p}^{(0)}\mathbf{P} = [.60, \quad .20, \quad .20]$$

$$\mathbf{p}^{(2)} = \mathbf{p}^{(1)}\mathbf{P} = [.60, \quad .20, \quad .20]\begin{bmatrix} .60 & .20 & .20 \\ .25 & .50 & .25 \\ .30 & .30 & .40 \end{bmatrix} = [.47, \quad .28, \quad .25]$$

12-8. To find the limiting distribution for the transition matrix in Exercise 12-4, we have to solve the following system of equations.

$$[x, \quad y]\begin{bmatrix} .65 & .35 \\ .25 & .75 \end{bmatrix} = [x, \quad y]$$
$$x + y = 1$$

or

$$\begin{aligned} .65x + .25y &= x \\ .35x + .75y &= y \\ x + \quad y &= 1 \end{aligned}$$

which can be rewritten as

$$\begin{aligned} -.35x + .25y &= 0 \\ .35x - .25y &= 0 \\ x + \quad y &= 1 \end{aligned}$$

The solution is [5/12, 7/12].

12-9. The limiting distribution for Exercise 12-5 is the solution of the following system of equations.

$$[x, \quad y]\begin{bmatrix} .72 & .28 \\ .36 & .64 \end{bmatrix} = [x, \quad y]$$
$$x + y = 1$$

This can be rewritten as

$$\begin{aligned} -.28x + .36y &= 0 \\ .28x - .36y &= 0 \\ x + \quad y &= 1 \end{aligned}$$

The solution is [9/16, 7/16].

12-10. The limiting distribution for Exercise 12-6 is the solution of the following system of equations.

$$[x, \quad y, \quad z]\begin{bmatrix} .4 & .2 & .4 \\ .2 & .5 & .3 \\ .1 & .3 & .6 \end{bmatrix} = [x, \quad y, \quad z]$$

$$x + y + z = 1$$

This can be rewritten as

$$\begin{aligned} -.6x + .2y + .1z &= 0 \\ .2x - .5y + .3z &= 0 \\ .4x + .3y - .4z &= 0 \\ x + y + z &= 1 \end{aligned}$$

The solution is $[11/57, \quad 20/57, \quad 26/57]$.

12-11. The limiting distribution for Exercise 12 –7 is the solution of the following system of equations.

$$[x, \quad y, \quad z]\begin{bmatrix} .60 & .20 & .20 \\ .25 & .50 & .25 \\ .30 & .30 & .40 \end{bmatrix} = [x, \quad y, \quad z]$$

$$x + y + z = 1$$

This can be rewritten as

$$\begin{aligned} -.4x + .25y + .3z &= 0 \\ .2x - .5y + .3z &= 0 \\ .2x + .25y - .6z &= 0 \\ x + y + z &= 1 \end{aligned}$$

The solution is $[15/37, \quad 12/37, \quad 10/37]$.

12-12. Let $\mathbf{P}$ be the transition matrix of a regular Markov process and let ℓ be its limiting distribution.

$$\ell\mathbf{P}^n = \ell\mathbf{P}^{n-1}\mathbf{P}$$

Taking the limit of both sides,

$$\lim_{n\to\infty}\mathbf{P}^n = (\ell\lim_{n\to\infty}\mathbf{P}^{n-1})\mathbf{P}$$
$$\ell\mathbf{L} = (\ell\mathbf{L})\mathbf{P}$$
$$\ell = \ell\mathbf{P}^n$$

(Recall that for any vector $\mathbf{v}$, $\mathbf{v}\mathrm{L} = \ell$.)

12-13. The transition matrix for the states for player A, after rewriting the matrix with the absorbing states first, is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 1/3 & 2/3 & 0 \end{bmatrix}.$$

(a) $$\mathbf{I} - \mathbf{Q} = \begin{bmatrix} 1 & -1/3 \\ -2/3 & 1 \end{bmatrix}$$

$$(\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} 9/7 & 3/7 \\ 6/7 & 9/7 \end{bmatrix} = \mathbf{N}$$

(b) $$\mathbf{A} = \mathbf{NR} = \begin{bmatrix} 9/7 & 3/7 \\ 6/7 & 9/7 \end{bmatrix}\begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 6/7 & 1/7 \\ 4/7 & 3/7 \end{bmatrix}$$

(c) The probability that if player A starts in state 2 (2 coins) he ends in state 0 (he loses) is $a_{20} = 4/7$. Recall that the rows in $\mathbf{A}$ refer to states 1 and 2 and the columns to states 0 and 3.

12-14. The transition matrix for the states for player A, after rewriting the matrix with the absorbing states first, is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 1/3 & 0 & 2/3 & 0 \end{bmatrix}.$$

(a) $$\mathbf{I} - \mathbf{Q} = \begin{bmatrix} 1 & -1/3 & 0 \\ -2/3 & 1 & -1/3 \\ 0 & -2/3 & 1 \end{bmatrix}$$

$$(\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} 7/5 & 3/5 & 1/5 \\ 6/5 & 9/5 & 3/5 \\ 4/5 & 6/5 & 7/5 \end{bmatrix} = \mathbf{N}$$

(b) $$\mathbf{A} = \mathbf{NR} = \begin{bmatrix} 7/5 & 3/5 & 1/5 \\ 6/5 & 9/5 & 3/5 \\ 4/5 & 6/5 & 7/5 \end{bmatrix} \begin{bmatrix} 2/3 & 0 \\ 0 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 14/15 & 1/15 \\ 4/5 & 1/5 \\ 8/15 & 7/15 \end{bmatrix}$$

(c) The probability that if player A starts in state 2 (2 coins) he will end in state 0 (he loses) is $a_{20} = 4/5$.

CHAPTER 12

12-1. A would win in 13 tosses

12-2. 13

12-3. 2, 3, 3

12-4. (a) [.45, .55] (b) [.43, .57]

12-5. (a) [.504, .496] (b) [.54144, .45856]

12-6. [.22, .33, .45]

12-7. [.47, .28, .25]

12-8. [5/12, 7/12]

12-9. [9/16, 7/16]

12-10. [11/57, 20/57, 26/57]

12-11. [15/37, 12/37, 10/37]

12-13. (a) $\begin{bmatrix} 9/7 & 3/7 \\ 6/7 & 9/7 \end{bmatrix}$ (b) $\begin{bmatrix} 6/7 & 1/7 \\ 4/7 & 3/7 \end{bmatrix}$ (c) 4/7

12-14. (a) $\begin{bmatrix} 7/5 & 3/5 & 1/5 \\ 6/5 & 9/5 & 3/5 \\ 4/5 & 6/5 & 7/5 \end{bmatrix}$ (b) $\begin{bmatrix} 14/15 & 1/15 \\ 4/5 & 1/5 \\ 8/15 & 7/15 \end{bmatrix}$ (c) 4/5