Proposition: (Useful curvature formula for space curves)
For a regular parametrised space curve $\alpha : I \to \mathbb{R}^3$ the curvature κ can be computed as

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$

Proof of the useful curvature formula for space curves: Let $s = |\alpha'|$ be the speed of α. For the unit tangent $T = \frac{\alpha'}{|\alpha'|} = \frac{\alpha'}{s}$ we have

$$s \cdot T = \alpha'.$$

Differentiating this equation we obtain

$$s' \cdot T + s \cdot T' = \alpha''.$$

Taking the dot-products we obtain

$$(s' \cdot T + s \cdot T') \bullet (s' \cdot T + s \cdot T') = \alpha'' \bullet \alpha''.$$

Expanding the brackets we obtain

$$(s')^2 \cdot (T \bullet T) + 2ss' \cdot (T \bullet T') + s^2 \cdot (T' \bullet T') = \alpha'' \bullet \alpha''.$$

Note that $T \bullet T = |T|^2 = 1$, $T' \bullet T' = |T'|^2$ and $\alpha'' \bullet \alpha'' = |\alpha''|^2$. Also note that $T \bullet T = 1$ implies $T \bullet T' = 0$. Hence we have

$$(s')^2 + s^2 \cdot |T'|^2 = |\alpha''|^2$$

and therefore

$$|T'|^2 = \frac{|\alpha''|^2 - (s')^2}{s^2}.$$

It remains to compute s' in terms of α. We have $s = |\alpha'| = (\alpha' \bullet \alpha')^{1/2}$, hence

$$s' = \frac{(\alpha' \bullet \alpha')'}{2(\alpha' \bullet \alpha')^{1/2}}.$$

Note that $(\alpha' \bullet \alpha')' = \alpha' \bullet \alpha'' + \alpha'' \bullet \alpha' = 2(\alpha' \bullet \alpha'')$, hence

$$s' = \frac{\alpha' \bullet \alpha''}{|\alpha'|}.$$

Substituting s' in

$$|T'|^2 = \frac{|\alpha''|^2 - (s')^2}{s^2}$$

we obtain

$$|T'|^2 = \frac{|\alpha''|^2 - \frac{(\alpha' \bullet \alpha'')^2}{|\alpha'|^2}}{s^2} = \frac{|\alpha'|^2 \cdot |\alpha''|^2 - (\alpha' \bullet \alpha'')^2}{|\alpha'|^2 \cdot s^2}.$$

Note that for any two vectors $a, b \in \mathbb{R}^3$ we have

$$|a|^2 \cdot |b|^2 - (a \bullet b)^2 = |a|^2 \cdot |b|^2 - |a|^2 \cdot |b|^2 \cdot \cos^2 \varphi = |a|^2 \cdot |b|^2 \cdot (1 - \cos^2 \varphi) = |a|^2 \cdot |b|^2 \cdot \sin^2 \varphi = |a \times b|^2,$$

where φ is the angle between the vectors a and b (compare with Exercises 0, Problem 3(ii)). In particular,

$$|\alpha'|^2 \cdot |\alpha''|^2 - (\alpha' \bullet \alpha'')^2 = |\alpha' \times \alpha''|^2$$

and therefore

$$\begin{aligned}
|T'|^2 &= \frac{|\alpha' \times \alpha''|^2}{s^4}, \\
|T'| &= \frac{|\alpha' \times \alpha''|}{s^2}, \\
\kappa &= \frac{|T'|}{s} = \frac{|\alpha' \times \alpha''|}{s^3} = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.
\end{aligned}$$