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Research Article

Quasi-Jordan Banach Algebras

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We initiate a study of quasi-Jordan normed algebras. It is demonstrated that any quasi-Jordan Banach algebra with a norm 1 unit can be given an equivalent norm making the algebra isometrically isomorphic to a closed right ideal of a unital split quasi-Jordan Banach algebra; the set of invertible elements may not be open; the spectrum of any element is nonempty, but it may be neither bounded nor closed and hence not compact. Some characterizations of the unbounded spectrum of an element in a split quasi-Jordan Banach algebra with certain examples are given in the end.

1. Introduction

Looking back at the development of modern mathematics, we see that early formal study of algebra was mostly commutative and associative. With an abstract study of functions and matrices, it became noncommutative but still associative; then introduction of nonassociative structures, such as Lie structures due to Sophus Lie [1] and Jordan structures due to Jordan [2], has led us to the mathematics which at present is noncommutative as well as nonassociative.

There is a strong relationship between Lie algebras and Jordan algebras [3]. Jordan structures have been extensively studied by a large number of mathematicians: P. Jordan, von Neumann, E. Wigner, N. Jacobson, K. McCrimmon, R. Braun, M. Koecher, E. Neher, and O. Loos, to name but a few. A vast literature containing many important results on Jordan algebras has been developed (cf. [3, 4]). After the mid-1960s, people began studying Jordan structures from the point of view of functional analysis. Interesting theories of Jordan Banach algebras, JB-algebras, JB^*-algebras, and JB^*-triples have been developed, which closely resemble that of C^*-algebras and have found surprisingly important applications in a wide range of mathematical disciplines including analysis, physics, and biology (cf. [5–7]).

A Jordan algebra is a nonassociative algebra $\mathcal{J}$ with the product $x \circ y$ satisfying $x \circ y = y \circ x$ and the Jordan identity: $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$, where $x^2 = x \circ x$. Any associative algebra A becomes a Jordan algebra A^+ with the same linear space structure and the Jordan product $x \circ y := (1/2)(xy + yx)$; it becomes a Lie algebra under the skew-symmetric product $[x, y] := xy - yx$, so called the Lie bracket (cf. [4]). For any Jordan algebra $\mathcal{J}$, there is a Lie algebra $\mathcal{L}(\mathcal{J})$ such that $\mathcal{J}$ is a linear subspace of $\mathcal{L}(\mathcal{J})$ and the product of $\mathcal{J}$ can be expressed in terms of the Lie bracket in $\mathcal{L}(\mathcal{J})$. Moreover, the universal enveloping algebra of a Lie algebra has the structure of an associative algebra; see the original work due to Kantor, Koecher, and Tits appearing in [8–10].

Loday introduced a generalization of Lie algebras, called the Leibniz algebras [11, 12], and successfully demonstrated that the relationship between Lie algebras and associative algebras can be translated into an analogous relationship between Leibniz algebras and the so-called dialgebras (cf. [13]): a *dialgebra* over a field K is a K-module D equipped with two bilinear associative maps $\dashv, \vdash$: $D \times D \rightarrow D$ satisfying $x \dashv (y \vdash z) = x \dashv (y \dashv z)$; $(x \vdash y) \dashv z = x \vdash (y \dashv z)$; and $(x \dashv y) \vdash z = (x \vdash y) \vdash z$. The maps $\dashv$ and $\vdash$ are called the left product and the right product, respectively. Any dialgebra $(D, \dashv, \vdash)$ becomes a Leibniz algebra under the Leibniz bracket $[x, y] := x \dashv y - y \vdash x$, and the universal enveloping algebra of a Leibniz algebra has the structure of a dialgebra; for details, see [12, 13].

Recently in [14], Velásquez and Felipe introduced the notion of quasi-Jordan algebras, which have relation with the Leibniz algebras similar to the existing relationship between the Jordan algebras and the Lie algebras [15]. The quasi-Jordan algebras are a generalization of Jordan algebras where

the commutative law is replaced by a quasicommutative identity and a special form of the Jordan identity is retained. These facts indicate the significance of studying the quasi-Jordan algebras; within a few years time, many mathematicians, including M. K. Kinyon, M. R. Bremner, L. A. Peresi, J. Sanchez-Ortega, and V. Voronin, have got their interests in this new area. In [16], Felipe made an attempt to study dialgebras from the functional analytic point of view.

A Jordan Banach algebra is a real or complex Jordan algebra with a complete norm $\|\cdot\|$ satisfying $\|xy\| \leq \|x\|\|y\|$; basics of Jordan Banach algebras may be seen in [7]. In this paper, we initiate a study of the *quasi-Jordan normed algebras*. The class of complete quasi-Jordan normed algebras, called *quasi-Jordan Banach algebras*, properly includes all Jordan Banach algebras and hence all C^*-algebras (cf. [7]). This study may provide a better mathematical foundation for some important areas such as quantum mechanics. We are interested specially in extending, as much as possible, the theory of Jordan Banach algebras to the general setting of quasi-Jordan Banach algebras. We investigate the notions of invertibility and spectrum of elements in the setting of unital quasi-Jordan Banach algebras. Among other results, we demonstrate that any quasi-Jordan Banach algebra $\mathfrak{J}$ with a norm 1 unit can be given an equivalent norm that makes the algebra $\mathfrak{J}$ isometrically isomorphic to a norm closed right ideal of a unital split quasi-Jordan Banach algebra. We show that the set of invertible elements in a unital quasi-Jordan Banach algebra generally is not open and that the spectrum of any element is nonempty but it may be neither bounded nor closed, hence not compact. Moreover, if the spectrum of an element in a complex unital split quasi-Jordan Banach algebra (see below) is unbounded then it coincides with the whole complex plane and vice versa. Some examples are also given in the end.

2. Quasi-Jordan Banach Algebras

We begin by recalling some basics of the quasi-Jordan algebra theory from [14, 17]. A *quasi-Jordan algebra* is a vector space $\mathfrak{J}$ over a field K of characteristic $\neq 2$ equipped with a bilinear map $\lhd$: $\mathfrak{J} \times \mathfrak{J} \rightarrow \mathfrak{J}$, called the *quasi-Jordan product*, satisfying $x \lhd (y \lhd z) = x \lhd (z \lhd y)$ *(right commutativity)* and $(y \lhd x) \lhd x^2 = (y \lhd x^2) \lhd x$ *(right Jordan identity)*, for all $x, y, z \in \mathfrak{J}$. Here, $x^2 := x \lhd x$ and $x^n := x^{n-1} \lhd x$ for $n \geq 2$; in the sequel, we will see that $x^2 \lhd x$ may not coincide with $x \lhd x^2$. Every quasi-Jordan algebra $\mathfrak{J}$ includes two important sets: $\mathfrak{J}^{\text{ann}}$:= *the linear span of the elements* $x \lhd y - y \lhd x$ *with* $x, y \in \mathfrak{J}$ and $Z(\mathfrak{J}) := \{z \in \mathfrak{J} : x \lhd z = 0, \forall x \in \mathfrak{J}\}$, respectively, called the *annihilator* and the *zero part* of $\mathfrak{J}$. It follows from the right commutativity that $\mathfrak{J}^{\text{ann}} \subseteq Z(\mathfrak{J})$ and that the quasi-Jordan algebra $\mathfrak{J}$ is a Jordan algebra if and only if $\mathfrak{J}^{\text{ann}} = \{0\}$.

Example 1. Let $(D, \dashv, \vdash)$ be a dialgebra over a field K of characteristic $\neq 2$. One can define another product $\lhd$: $D \times D \rightarrow D$ by $x \lhd y := (1/2)(x \dashv y + y \vdash x)$ for all $x, y \in D$, which satisfies the identities $x \lhd (y \lhd z) = x \lhd (z \lhd y)$; $(y \lhd x) \lhd x^2 = (y \lhd x^2) \lhd x$; and $x^2 \lhd (x \lhd y) = x \lhd (x^2 \lhd y)$; however, this quasi-Jordan product generally is not commutative. Therefore, $(D, \lhd)$ is a quasi-Jordan algebra, which may not be a Jordan algebra (cf. [14]). This quasi-Jordan algebra is denoted by D^+, called a *plus quasi-Jordan algebra*. Any quasi-Jordan algebra which is a homomorphic image of a plus quasi-Jordan algebra is called *special*.

We define a *quasi-Jordan normed algebra* as a quasi-Jordan algebra $(\mathfrak{J}, \lhd)$ over the field $\mathbb{C}$ of complex numbers endowed with a norm $\|\cdot\|$ satisfying $\|x \lhd y\| \leq \|x\|\|y\|$, for all $x \in \mathfrak{J}$. Thus, the quasi-Jordan product "$\lhd$" in a normed quasi-Jordan algebra $\mathfrak{J}$ is continuous. A quasi-Jordan normed algebra is called a *quasi-Jordan Banach algebra* if it is complete as a normed space.

If there is an element e in a quasi-Jordan algebra $\mathfrak{J}$ satisfying $e \lhd x = x$ for all $x \in \mathfrak{J}$, called the left unit, then $\mathfrak{J}$ becomes commutative and hence a Jordan algebra. Due to this fact, we will consider only the *right unit* elements; henceforth, unit in a quasi-Jordan algebra would mean a right unit. An element e in a quasi-Jordan algebra $\mathfrak{J}$ is called a *unit* if $x \lhd e = x$, for all $x \in \mathfrak{J}$. A quasi-Jordan algebra may have many (right) units (cf. [15, 17]). If the dialgebra D has a *bar-unit* e (i.e., $x \dashv e = x = e \vdash x$) then $x \lhd e = (1/2)(x \dashv e + e \vdash x) = x$, for all $x \in D$, and hence e is a unit of the plus quasi-Jordan algebra D^+.

For any quasi-Jordan algebra $\mathfrak{J}$ with unit e, we know from [17] that $\mathfrak{J}^{\text{ann}}$ and $Z(\mathfrak{J})$ are two-sided ideals of $\mathfrak{J}$, $\mathfrak{J}^{\text{ann}} = \{x \in \mathfrak{J} : e \lhd x = 0\} = Z(\mathfrak{J})$, and $U(\mathfrak{J}) = \{x + e : x \in Z(\mathfrak{J})\}$, where $U(\mathfrak{J})$ denotes the set of all (right) units in $\mathfrak{J}$.

One can always attach a (two-sided) unit to any Jordan algebra by following the standard unitization process. This unitization process no longer works for quasi-Jordan algebras (cf. [17, pages 210-211]). Adding a unit to a quasi-Jordan algebra is yet an open problem. In giving a partial solution to this unitization problem, Velásquez and Felipe [17] introduced the following special class of quasi-Jordan algebras, called split quasi-Jordan algebras: let $\mathfrak{J}$ be a quasi-Jordan algebra and let I be an ideal in $\mathfrak{J}$ such that $\mathfrak{J}^{\text{ann}} \subseteq I \subseteq Z(\mathfrak{J})$. We say that $\mathfrak{J}$ is a *split quasi-Jordan algebra* (more precisely, $\mathfrak{J}$ is *split over I*) if there exists a subalgebra J of $\mathfrak{J}$ such that $\mathfrak{J} = J \oplus I$, the direct sum of J and I. It is easily seen that such a subalgebra J is a Jordan algebra. One can attach a unit to any split quasi-Jordan algebra; details of such a unitization process are given in [17]. If the algebra $\mathfrak{J}$ has a unit then $\mathfrak{J}^{\text{ann}} = Z(\mathfrak{J})$. Thus, a quasi-Jordan algebra $\mathfrak{J}$ with unit is a *split quasi-Jordan algebra* if and only if $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ for some subalgebra J of $\mathfrak{J}$; the algebra J is called the *Jordan part* of $\mathfrak{J}$. In such a case, each element $x \in \mathfrak{J}$ has a unique representation $x = x_J + x_Z$ with $x_J \in J$ and $x_Z \in Z(\mathfrak{J})$, respectively, called the *Jordan part* and the *zero part* of x. Moreover, if $\mathfrak{J}$ is split quasi-Jordan algebra with unit e, then there exists a unique element $e_J \in J$ which acts as a unit of the quasi-Jordan algebra $\mathfrak{J}$ and at the same time a unit of the Jordan algebra J.

Proposition 2. *Let $\mathfrak{J} := J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan Banach algebra with unit $e \in J$. Then the Jordan part J is a norm closed subalgebra of $\mathfrak{J}$ and hence a unital Jordan Banach algebra.*

Proof. From the above discussion, it is clear that the Jordan part J is a unital Jordan normed algebra. Next, let $\{x_n\}$ be any fixed Cauchy sequence in J. Then, the same $\{x_n\}$ is Cauchy sequence also in the quasi-Jordan Banach algebra $\mathfrak{I}$ since J is a normed subspace of $\mathfrak{I}$. Hence, $x_n \to x$ for some $x \in \mathfrak{I}$.

Now, since each $x_n \in J$ and since e is the unit of the Jordan algebra J under the (restricted) product "$\triangleleft$," we get $e \triangleleft x_n = x_n$ for all n. So, by the continuity of the product, $x_n = e \triangleleft x_n \to e \triangleleft x$. Hence, by the uniqueness of the limit of any convergent sequence in a normed space, $e \triangleleft x = x$. However, $x = e \triangleleft (x_J + x_Z) = e \triangleleft x_J + e \triangleleft x_Z = e \triangleleft x_J + 0 = x_J \in J$. Thus, the required result follows. □

We know from [17] that for any quasi-Jordan algebra $\mathfrak{I}$, $\mathfrak{I}_1 := \{(x, R_y) : x, y \in \mathfrak{I}\}$ equipped with the sum $(x, R_y) + (a, R_b) := (x + a, R_{y+b})$, scalar multiplication $\lambda(x, R_y) := (\lambda x, R_{\lambda y})$, and product $(x, R_y) \triangleleft (a, R_b) := (x \triangleleft b, R_{y \triangleleft b})$ is a split quasi-Jordan algebra and the map $x \mapsto \varphi(x) := (x, R_x)$ is an embedding of $\mathfrak{I}$ into $\mathfrak{I}_1$, where R_z stands for the usual right multiplication operator on $\mathfrak{I}$. It is easily seen that $\{0\} \times R(\mathfrak{I})$ is the Jordan part of $\mathfrak{I}_1$ and the zero part $Z(\mathfrak{I}_1) = \mathfrak{I} \times \{0\}$. Moreover, the embedding φ preserves the units: clearly, $(a, R_b) \triangleleft (e, R_e) = (a \triangleleft e, R_{b \triangleleft e}) = (a, R_b)$ for all $(a, R_b) \in \mathfrak{I}_1$ so that (e, R_e) is a unit in $\mathfrak{I}_1$ whenever e is a unit in $\mathfrak{I}$. In fact, (x, R_e) is a unit in $\mathfrak{I}_1$, for all $x \in \mathfrak{I}$; it may be noted here that $(0, R_e)$ is the only unit in the Jordan part of $\mathfrak{I}_1$.

From [17], we also know that $R(\mathfrak{I}) := \{R_x : x \in \mathfrak{I}\}$, with product "$\bullet$" defined by $R_x \bullet R_y = R_{x \triangleleft y}$, for all $x, y \in \mathfrak{I}$, is a quasi-Jordan algebra. Moreover, $\mathfrak{I}_1$ is the direct product of the quasi-Jordan algebras $\mathfrak{I}$ and $R(\mathfrak{I})$.

Indeed, the split quasi-Jordan algebra $\mathfrak{I}_1$ is a quasi-Jordan Banach algebra with unit (e, R_e) whenever $\mathfrak{I}$ is a quasi-Jordan Banach algebra with a norm 1 unit e, and that $\varphi(\mathfrak{I}) = \{(x, R_x) : x \in \mathfrak{I}\}$ is a closed unital quasi-Jordan normed subalgebra of $\mathfrak{I}_1$. To justify this claim, we need the following result.

Proposition 3. *Suppose $\mathfrak{I}$ is a quasi-Jordan Banach algebra with a norm 1 unit e. Then the algebra $R(\mathfrak{I})$ as above is a quasi-Jordan Banach algebra with unit R_e of norm 1.*

Proof. Clearly, each R_x is a bounded linear operator with $\|R_x\| \le \|x\|$. Hence, the usual operator norm is a norm on the quasi-Jordan algebra $R(\mathfrak{I})$ with the quasi-Jordan product $R_x \bullet R_y = R_{x \triangleleft y}$. Further, we observe that

$$\begin{aligned}
\left\|R_x \bullet R_y\right\| &= \left\|R_{x \triangleleft y}\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|R_{x \triangleleft y}(z)\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft (x \triangleleft y)\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft ((e \triangleleft x) \triangleleft y)\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft \left(R_y\left(R_x(e)\right)\right)\right\| \\
&\le \sup_{z \in \mathfrak{I}, \|z\|=1} \|z\| \left\|R_y\left(R_x(e)\right)\right\| \\
&= \left\|R_y\left(R_x e\right)\right\| \le \left\|R_x\right\| \left\|R_y\right\| \|e\| \\
&= \left\|R_x\right\| \left\|R_y\right\|.
\end{aligned} \tag{1}$$

Alternately, by exploiting the right commutativity of the quasi-Jordan product in $\mathfrak{I}$, we get

$$\begin{aligned}
\left\|R_x \bullet R_y\right\| &= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft (x \triangleleft y)\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft ((x \triangleleft e) \triangleleft (y \triangleleft e))\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft ((x \triangleleft e) \triangleleft (e \triangleleft y))\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft ((e \triangleleft y) \triangleleft (x \triangleleft e))\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft ((e \triangleleft y) \triangleleft (e \triangleleft x))\right\| \\
&= \sup_{z \in \mathfrak{I}, \|z\|=1} \left\|z \triangleleft \left(R_y(e) \triangleleft R_x(e)\right)\right\| \\
&\le \sup_{z \in \mathfrak{I}, \|z\|=1} \|z\| \left\|R_y\right\| \left\|R_x\right\| \|e\|^2 = \left\|R_x\right\| \left\|R_y\right\|.
\end{aligned} \tag{2}$$

Thus, $R(\mathfrak{I})$ together with the operator norm is a quasi-Jordan normed algebra. Moreover, for any $x \in R(\mathfrak{I})$, we have $R_x \bullet R_e = R_{x \triangleleft e} = R_x$ and $1 = \|e\| \ge \|R_e\| = \sup_{0 \ne z \in \mathfrak{I}}(\|R_e(z)\|/\|z\|) \ge \|R_e(e)\|/\|e\| = 1$; that is, R_e is a norm 1 (right) unit in $R(\mathfrak{I})$.

Suppose $\{R_{x_n}\}$ is any fixed Cauchy sequence in $R(\mathfrak{I})$. Then, for any fixed $a \in \mathfrak{I}$, $\|a \triangleleft x_m - a \triangleleft x_n\| = \|R_{x_m}(a) - R_{x_n}(a)\| = \|(R_{x_m} - R_{x_n})(a)\| \le \|R_{x_m} - R_{x_n}\|\|a\| \to 0$ as $m, n \to \infty$, and so $\{a \triangleleft x_n\}$ is a Cauchy sequence in $\mathfrak{I}$. But, $\mathfrak{I}$ is complete. Hence, the sequence $\{a \triangleleft x_n\}$ for any $a \in \mathfrak{I}$ is convergent in $\mathfrak{I}$. In particular, the sequence $\{e \triangleleft x_n\}$ converges to some $y \in \mathfrak{I}$. Moreover, the Cauchy sequence $\{R_{x_n}\}$ converges to $R_y \in R(\mathfrak{I})$ in the operator norm because $\|(R_{x_n} - R_y)(z)\| = \|z \triangleleft (x_n - y)\| = \|z \triangleleft x_n - z \triangleleft y\| = \|z \triangleleft (x_n \triangleleft e) - z \triangleleft y\| = \|z \triangleleft (e \triangleleft x_n) - z \triangleleft y\| = \|z \triangleleft (e \triangleleft x_n - y)\| \le \|z\|\|e \triangleleft x_n - y\| \to 0$ as $n \to \infty$, for all $z \in \mathfrak{I}$. Thus, the quasi-Jordan normed algebra $R(\mathfrak{I})$ is complete.

□

Now, we show that the corresponding algebra $\mathfrak{I}_1$ is a unital split quasi-Jordan Banach algebra.

Proposition 4. *Let $\mathfrak{I}$ be a quasi-Jordan Banach algebra with a norm 1 unit e and let $\mathfrak{I}_1$ be as above. Then $\mathfrak{I}_1$ is the direct product of the quasi-Jordan algebras $\mathfrak{I}$ and $R(\mathfrak{I})$. Moreover, $\mathfrak{I}_1$ equipped with the product norm $\|(x, R_y)\| := \|x\| + \|R_y\|$ is a split quasi-Jordan Banach algebra with unit (e, R_e), and $\varphi(\mathfrak{I}) = \{(x, R_x) : x \in \mathfrak{I}\}$ is a closed right ideal in $\mathfrak{I}_1$ in the norm topology.*

Proof. For the first part, see [17]. Clearly, $\|(x, R_y)\| := \|x\| + \|R_y\|$ is a norm, and it satisfies

$$\begin{aligned}\left\|\left(x, R_y\right) \lhd \left(z, R_w\right)\right\| &= \left\|\left(x \lhd w, R_{y \lhd w}\right)\right\| \\ &= \left\|\left(x \lhd w, R_y \bullet R_w\right)\right\| \\ &= \left\|R_w(x)\right\| + \left\|R_y \bullet R_w\right\| \\ &\leq \left(\|x\| + \left\|R_y\right\|\right)\left\|R_w\right\| \\ &\leq \left(\|x\| + \left\|R_y\right\|\right)\left(\|z\| + \left\|R_w\right\|\right) \\ &= \left\|\left(x, R_y\right)\right\| \left\|\left(z, R_w\right)\right\|,\end{aligned} \tag{3}$$

for all $w, x, y, z \in \mathfrak{J}$. Keeping in view Proposition 3, we deduce that $\mathfrak{J}_1$ being the product of complete spaces $\mathfrak{J}$ and $R(\mathfrak{J})$ is complete in the product norm. Thus, $\mathfrak{J}_1$ is a split quasi-Jordan Banach algebra with (right) unit (e, R_e).

Clearly, $\varphi(\mathfrak{J})$ is a subspace of $\mathfrak{J}_1$ with $(x, R_x) \lhd (y, R_z) = (x \lhd z, R_{x \lhd z}) \in \varphi(\mathfrak{J})$ for all $(x, R_x) \in \varphi(\mathfrak{J})$, $(y, R_z) \in \mathfrak{J}_1$, so that $\varphi(\mathfrak{J})$ is a quasi-Jordan normed subalgebra with the unit (e, R_e) included. Further, let $\{(x_n, R_{x_n})\}$ be any fixed Cauchy sequence in $\varphi(\mathfrak{J})$. Then $\|x_m - x_n\| \leq \|x_m - x_n\| + \|R_{x_m} - R_{x_n}\| = \|(x_m - x_n, R_{x_m} - R_{x_n})\| = \|(x_m, R_{x_m}) - (x_n, R_{x_n})\| \to 0$ as $m, n \to \infty$, so that $\{x_n\}$ is a Cauchy sequence in (the complete space) $\mathfrak{J}$ and hence it converges to some $x \in \mathfrak{J}$. Now, by using the fact $\| R_z \| \leq \| z \|$ for all $z \in \mathfrak{J}$ (since e is a norm 1 unit in $\mathfrak{J}$; see above), we get the convergence of arbitrarily fixed Cauchy sequence $\{(x_n, R_{x_n})\}$ to $(x, R_x) \in \varphi(\mathfrak{J})$ since $\|(x_n, R_{x_n}) - (x, R_x)\| = \|(x_n - x, R_{x_n - x})\| = \|x_n - x\| + \|R_{x_n - x}\| \leq 2\|x_n - x\| \to 0$ as $n \to \infty$. Thus, $\varphi(\mathfrak{J})$ being complete is a closed right ideal in $\mathfrak{J}_1$ in the norm topology. □

Next, we observe the isometry between $\mathfrak{J}$ and $\varphi(\mathfrak{J})$.

Proposition 5. *Let $\mathfrak{J}$ be a quasi-Jordan Banach algebra with norm 1 unit. Then there exists an equivalent norm that makes $\mathfrak{J}$ isometrically isomorphic to a quasi-Jordan closed subalgebra $\varphi(\mathfrak{J})$ of the split quasi-Jordan Banach algebra $\mathfrak{J}_1$.*

Proof. Clearly, $\|x\|_\circ := \|x\| + \|R_x\|$ defines a norm on the quasi-Jordan algebra $\mathfrak{J}$. It follows that $(\mathfrak{J}, \|\cdot\|_\circ)$ is a quasi-Jordan Banach algebra. Moreover, $\mathfrak{J}$ is isomorphic to the subalgebra $\{(x, R_x) : x \in \mathfrak{J}\}$ of $\mathfrak{J}_1$ under the isomorphism $\varphi : x \mapsto (x, R_x)$, as seen above. Further, we observe that φ is an isometry since $\|x\|_\circ = \|x\| + \|R_x\| = \|(x, R_x)\| = \|\varphi(x)\|$. Finally, we note that the two norms $\|\cdot\|$ and $\|\cdot\|_\circ$ on $\mathfrak{J}$ are equivalent since $\|x\| \leq \|x\| + \|R_x\| = \|x\|_\circ = \|x\| + \|R_x\| \leq 2\|x\|$. □

3. Invertible Elements

As in [17], an element x in a quasi-Jordan algebra $\mathfrak{J}$ is called *invertible with respect to a unit* $e \in \mathfrak{J}$ if there exists $y \in \mathfrak{J}$ such that $y \lhd x = e + (e \lhd x - x)$ and $y \lhd x^2 = x + (e \lhd x - x) + (e \lhd x^2 - x^2)$; such an element y is called an *inverse of x with respect to* e. Let $e_\lhd(x)$ denote the element $e \lhd x - x$. Then, x has an inverse y with respect to $e \Leftrightarrow y \lhd x = e + e_\lhd(x)$ and $y \lhd x^2 = x + e_\lhd(x) + e_\lhd(x^2)$.

We know from the above discussion that the embedding $x \mapsto \varphi(x) := (x, R_x)$ of a quasi-Jordan algebra $\mathfrak{J}$ into the split quasi-Jordan algebra $\mathfrak{J}_1$ preserves the units. The embedding φ also preserves the corresponding invertible elements: if y is an inverse of x with respect to a unit e in $\mathfrak{J}$, then $y \lhd x = e + (e \lhd x - x)$ and $y \lhd x^2 = x + (e \lhd x - x) + (e \lhd x^2 - x^2)$. Hence,

$$\begin{aligned}\left(y, R_y\right) \lhd \left(x, R_x\right) &= \left(y \lhd x, R_{y \lhd x}\right) \\ &= \left(e + (e \lhd x - x), R_{e + (e \lhd x - x)}\right) \\ &= \left(e + (e \lhd x - x), R_e + R_{e \lhd x} - R_x\right) \\ &= \left(e, R_e\right) + \left(e \lhd x, R_{e \lhd x}\right) - \left(x, R_x\right) \\ &= \left(e, R_e\right) + \left(e, R_e\right) \lhd \left(x, R_x\right) - \left(x, R_x\right), \\ \left(y, R_y\right) \lhd \left(x, R_x\right)^2 &= \left(y \lhd x^2, R_{y \lhd x^2}\right) \\ &= \left(x, R_x\right) + \left(\left(e \lhd x, R_{e \lhd x}\right) - \left(x, R_x\right)\right) \\ &\quad + \left(\left(e \lhd x^2, R_{e \lhd x^2}\right) - \left(x^2, R_{x^2}\right)\right) \\ &= \left(x, R_x\right) + \left(\left(e, R_e\right) \lhd \left(x, R_x\right) - \left(x, R_x\right)\right) \\ &\quad + \left(\left(e, R_e\right) \lhd \left(x, R_x\right)^2 - \left(x, R_x\right)^2\right).\end{aligned} \tag{4}$$

Thus, (y, R_y) is an inverse of (x, R_x), with respect to the unit (e, R_e), in $\mathfrak{J}_1$.

Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan algebra. Then, $R_x(y) = y \lhd x = y \lhd (x_J + x_Z) = y \lhd x_J + y \lhd x_Z = y \lhd x_J + 0 = R_{x_J}(y)$, for all $x, y \in \mathfrak{J}$. Thus, $R_x = R_{x_J}$ for all $x \in \mathfrak{J}$.

In this section, we demonstrate that the set of invertible elements, with respect to a fixed unit, in a quasi-Jordan Banach algebra, may not be open. For this, we proceed as follows.

Proposition 6. *Let $\mathfrak{J} := J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan algebra with unit $e \in J$. If $x \in \mathfrak{J}$ is invertible with respect to e, then so is λx for all $\lambda \neq 0$.*

Proof. Let y be an inverse of x in $\mathfrak{J}$ with respect to e. We show that $y' := (1/\lambda) y_J + y_Z$ is an inverse of λx with respect to e. Observe that

$$\begin{aligned}y_J \lhd x + y_Z \lhd x &= \left(y_J + y_Z\right) \lhd x \\ &= y \lhd x = e + e_\lhd(x) \\ &= e + e_\lhd\left(x_J + x_Z\right) = e + \left(-x_Z\right),\end{aligned}$$

$$
\begin{aligned}
y_J \lhd x^2 + y_Z \lhd x^2 &= y \lhd x^2 \\
&= x + e_{\lhd}(x_J + x_Z) + e_{\lhd}\left((x_J + x_Z)^2\right) \\
&= x_J + (-x_Z \lhd x).
\end{aligned} \tag{5}
$$

Hence, by the uniqueness of the representation as sum of Jordan and zero parts, we get $y_J \lhd x = e$, $y_Z \lhd x = -x_Z$, $y_J \lhd x^2 = x_J$, and $y_Z \lhd x^2 = -x_Z \lhd x$. Therefore, $y' \lhd \lambda x = \lambda(((1/\lambda)y_J + y_Z) \lhd x) = y_J \lhd x + \lambda y_Z \lhd x = e - \lambda x_Z = e + e_{\lhd}(\lambda x)$ and $y' \lhd (\lambda x)^2 = \lambda^2((1/\lambda)y_J \lhd x^2 + y_Z \lhd x^2) = \lambda x_J - \lambda^2 x_Z \lhd x = \lambda x + e_{\lhd}(\lambda x) + e_{\lhd}((\lambda x)^2)$ because $e_{\lhd}(\lambda x) = e \lhd (\lambda x) - \lambda x = -\lambda x_Z$ and $e_{\lhd}((\lambda x)^2) = e \lhd \lambda^2 x^2 - \lambda^2 x^2 = \lambda^2(e_{\lhd}(x^2)) = -\lambda^2 x_Z \lhd x$. □

Proposition 7. *Let $\mathfrak{J}$ be a quasi-Jordan normed algebra with a unit e. Let $G_e(\mathfrak{J}) := \{x \in \mathfrak{J} : x$ is invertible with respect to $e\}$ be an open set and $x \in G_e(\mathfrak{J})$. Then $x + z \in G_e(\mathfrak{J})$ for all $z \in Z(\mathfrak{J})$.*

Proof. Suppose $x \in G_e(\mathfrak{J})$ and $z \in Z(\mathfrak{J})$. If $z = 0$ then $x + z = x \in G_e(\mathfrak{J})$. Next, suppose $z \neq 0$. Since $G_e(\mathfrak{J})$ is an open set, there exists $\epsilon > 0$ such that $a \in G_e(\mathfrak{J})$ whenever $\|x - a\| < \epsilon$. Hence, $x + z_\circ \in G_e(\mathfrak{J})$ with $z_\circ = (\epsilon/2\|z\|)z$.

Let y and $y_\circ$ be inverses of x and $x + z_\circ$ with respect to e, respectively. Then

$$
\begin{aligned}
y_\circ \lhd (x + z_\circ) &= e + e_{\lhd}(x + z_\circ) \\
&= e + e_{\lhd}(x) - z_\circ \\
&= y \lhd x - z_\circ, \\
y_\circ \lhd (x + z_\circ)^2 &= x + z_\circ + e_{\lhd}(x + z_\circ) + e_{\lhd}\left((x + z_\circ)^2\right) \\
&= x + e_{\lhd}(x) + e_{\lhd}\left(x^2\right) - z_\circ \lhd x \\
&= y \lhd x^2 - z_\circ \lhd x.
\end{aligned} \tag{6}
$$

Hence, by setting $y_1 = y + \alpha(y_\circ - y)$ with $\alpha = 2\|z\|/\epsilon$, we see that

$$
\begin{aligned}
y_1 \lhd (x + z) &= (y + \alpha(y_\circ - y)) \lhd x \\
&= y \lhd x + \alpha\left(y_\circ \lhd (x + z_\circ) - y \lhd x\right) \\
&= e + e_{\lhd}(x) + \alpha(-z_\circ) \\
&= e + e_{\lhd}(x) - z \\
&= e + e_{\lhd}(x + z), \\
y_1 \lhd (x + z)^2 &= (y + \alpha(y_\circ - y)) \lhd x^2 \\
&= y \lhd x^2 + \alpha\left(y_\circ \lhd (x + z_\circ)^2 - y \lhd x^2\right) \\
&= x + e_{\lhd}(x) + e_{\lhd}\left(x^2\right) - \alpha(z_\circ \lhd x) \\
&= x + e_{\lhd}(x) + e_{\lhd}\left(x^2\right) - z \lhd x \\
&= (x + z) + e_{\lhd}(x + z) + e_{\lhd}\left((x + z)^2\right),
\end{aligned} \tag{7}
$$

since $z \in Z(\mathfrak{J})$. Thus, y_1 is an inverse of $x + z$ with respect to the unit e. □

Corollary 8. *Under the hypothesis of Proposition 7, $\lambda e - x \in G_e(\mathfrak{J})$ implies $\lambda e - (x + z) \in G_e(\mathfrak{J})$ for all $z \in Z(\mathfrak{J})$.*

Proposition 9. *Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a split quasi-Jordan normed algebra and let $e \in J$ be a unit in $\mathfrak{J}$ such that the set $G_e(\mathfrak{J})$ is open. Then $x \lhd x^2 = x^2 \lhd x$ for all $x \in \mathfrak{J}$.*

Proof. Of course, the element e is the unit of the Jordan algebra J. Then, for any fixed element $a \in J$, there exists $\lambda \in \mathbb{C}$ such that $\lambda e - a$ is invertible in J; otherwise, we would get the negation of the well-known fact that spectrum of an element of a unital Jordan algebra is bounded. That is, there exists $y \in J$ such that $y \lhd (\lambda e - a) = e$ and $y \lhd (\lambda e - a)^2 = (\lambda e - a)$. However, $e_{\lhd}(a) = 0 = e_{\lhd}(a^2)$. Hence, y is an inverse of a in the quasi-Jordan algebra $\mathfrak{J}$ with respect to the unit e. By Corollary 8, $\lambda e - (a + z)$ is also invertible for any $z \in Z(\mathfrak{J})$. This in turn gives the existence of $b \in \mathfrak{J}$ satisfying $b \lhd (\lambda e - (a + z)) = e + z$ and $b \lhd (\lambda e - (a + z))^2 = (\lambda e - a) + z \lhd (\lambda e - a)$. Multiplying the first equation from the right by $(\lambda e - (a + z))^2$, the second equation by $(\lambda e - (a + z))$, and using the right Jordan identity, we get

$$
\begin{aligned}
&(e + z) \lhd (\lambda e - (a + z))^2 \\
&\quad = ((\lambda e - a) + z \lhd (\lambda e - a)) \lhd (\lambda e - (a + z)).
\end{aligned} \tag{8}
$$

Hence,

$$
(e + z) \lhd (\lambda e - a)^2 = ((\lambda e - a) + z \lhd (\lambda e - a)) \lhd (\lambda e - a), \tag{9}
$$

so that

$$
\begin{aligned}
&e \lhd (\lambda e - a)^2 + \lambda^2 z - 2\lambda (z \lhd a) + z \lhd a^2 \\
&\quad = (\lambda e - a)^2 + \lambda^2 z - 2\lambda z \lhd a + (z \lhd a) \lhd a.
\end{aligned} \tag{10}
$$

This last equation reduces to $e \lhd (\lambda e - a)^2 = (\lambda e - a)^2$ since $(\lambda e - a)^2 \in J$ and e is the unit of J. Hence, $z \lhd a^2 = (z \lhd a) \lhd a$ for all $a \in J$ and $z \in Z(\mathfrak{J})$. Now, for any $x \in \mathfrak{J}$, the last equation with $a = x_J$ and $z = x_Z$ gives $x_Z \lhd x_J^2 = (x_Z \lhd x_J) \lhd x_J$. Thus, $x \lhd x^2 = x_J \lhd x_J^2 + x_Z \lhd x_J^2 = x_J^2 \lhd x_J + (x_Z \lhd x_J) \lhd x_J = x^2 \lhd x$ for all $x \in \mathfrak{J}$. □

Corollary 10. *If a unital split quasi-Jordan algebra has an element x with $x \lhd x^2 \neq x^2 \lhd x$ then the set of invertible elements, with respect to the unit of the Jordan part, is not open.*

In the sequel, we will show the existence of a unital split quasi-Jordan Banach algebra with elements x such that $x \lhd x^2 \neq x^2 \lhd x$. Thus, the above result establishes that the set of invertible elements, with respect to a fixed unit, in a quasi-Jordan Banach algebra may not be open.

4. The Spectrum of Elements in a Unital Quasi-Jordan Algebra

As usual, we define the spectrum of an element x in a unital quasi-Jordan algebra $(\mathfrak{J}, e)$, denoted by $\sigma_{(\mathfrak{J},e)}(x)$, to be the collection of all complex numbers λ for which $\lambda e - x$ is not invertible. Thus, $\sigma_{(\mathfrak{J},e)}(x) := \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible}\}$. Here, the subscript e indicates that the invariability depends on the choice of unit e, which generally is not unique.

Proposition 11. *Let $\mathfrak{J}$ be a unital quasi-Jordan algebra. Then $\sigma_{(\mathfrak{J},e)}(e') = \{1\}$ for all $e, e' \in U(\mathfrak{J})$.*

Proof. Let $e, e' \in U(\mathfrak{J})$. Then, for any $\lambda \neq 1$, $y := (1/(\lambda-1))e'$ is an inverse of $\lambda e - e'$ with respect to the unit e because $y \lhd (\lambda e - e') = (1/(\lambda-1))e' \lhd (\lambda e - e') = e + e_{\lhd}(\lambda e - e')$ and $y \lhd (\lambda e - e')^2 = (\lambda e - e') + e_{\lhd}(\lambda e - e') + e_{\lhd}((\lambda e - e')^2)$. □

Proposition 12. *Let $\mathfrak{J}$ be a quasi-Jordan algebra with unit e. Then $\sigma_{(\mathfrak{J},e)}(z) = \{0\}$ for all $z \in Z(\mathfrak{J})$.*

Proof. For any fixed $z \in Z(\mathfrak{J})$ and nonzero scalar λ, the vector $y := (1/\lambda)(e+z)$ satisfies $y \lhd (\lambda e - z) = \lambda y = e + z = e + e_{\lhd}(\lambda e - z)$ and $y \lhd (\lambda e - z)^2 = \lambda^2 y = \lambda e + \lambda z = (\lambda e - z) + e_{\lhd}(\lambda e - z) + e_{\lhd}((\lambda e - z)^2)$. So $(1/\lambda)(e+z)$ is an inverse of $\lambda e - z$ with respect to e. This means $\lambda \notin \sigma_{(\mathfrak{J},e)}(z)$ for all $\lambda \neq 0$. However, the zero vector is not invertible. Thus, $\sigma_{(\mathfrak{J},e)}(z) = \{0\}$. □

Proposition 13. *Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a split quasi-Jordan algebra with unit $e \in J$. Then $\sigma_{(\mathfrak{J},e)}(p) \subseteq \{0, 1\}$ for all idempotents $p \in \mathfrak{J}$ (i.e., $p^2 = p$).*

Proof. Let p be any fixed idempotent in $\mathfrak{J}$. Since $p \in \mathfrak{J}$, p has a unique representation $p = p_J + p_Z$ with $p_J \in J$ and $p_Z \in Z(\mathfrak{J})$. Clearly, $p_J^2 + p_Z \lhd p_J = p^2 = p = p_J + p_Z$. Then, by uniqueness of the representation in the split quasi-Jordan algebra $\mathfrak{J}$, $p_Z \lhd p_J = p_Z$ and $p_J^2 = p_J$; this means p_J is an idempotent in the Jordan algebra J. Hence, $\sigma_{(J,e)}(p_J) \subseteq \{0, 1\}$. Thus, $p_\lambda := \lambda e - p_J$ is invertible in J with the unique inverse p_λ^{-1}, for all $\lambda \notin \{0, 1\}$.

We show that $y := p_\lambda^{-1} + (1/(\lambda-1))p_Z$ is an inverse of $\lambda e - p$ in $\mathfrak{J}$ with respect to the unit e; for this, we note that $\lambda e - p = p_\lambda - p_Z$, $(p_\lambda - p_Z)^2 = p_\lambda^2 - p_Z \lhd p_\lambda$, $e_{\lhd}(\lambda e - p) = e_{\lhd}(p_\lambda - p_Z) = e \lhd (p_\lambda - p_Z) - (p_\lambda - p_Z) = p_Z$, and $e_{\lhd}((\lambda e - p)^2) = e_{\Delta}((p_\lambda - p_Z)^2) = e_{\lhd}(p_\lambda^2 - p_Z \lhd p_\lambda) = p_Z \lhd p_\lambda = p_Z \lhd (\lambda e - p_J) = \lambda p_Z - p_Z \lhd p_J = \lambda p_Z - p_Z = (\lambda - 1)p_Z$. Hence, $y \lhd (\lambda e - p) = y \lhd (p_\lambda - p_Z) = y \lhd p_\lambda = (p_\lambda^{-1} + (1/(\lambda-1))p_Z) \lhd p_\lambda = p_\lambda^{-1} \lhd p_\lambda + (1/(\lambda-1))p_Z \lhd p_\lambda = e + (1/(\lambda-1))p_Z \lhd (\lambda e - p_J) = e + (1/(\lambda-1))(\lambda p_Z - p_Z \lhd p_J) = e + (1/(\lambda-1))(\lambda p_Z - p_Z) = e + p_Z = e + e_{\lhd}(\lambda e - p)$ and $y \lhd (\lambda e - p)^2 = y \lhd (p_\lambda^2 - p_Z \lhd p_\lambda) = y \lhd p_\lambda^2 = (p_\lambda^{-1} + (1/(\lambda-1))p_Z) \lhd p_\lambda^2 = p_\lambda + (1/(\lambda-1))p_Z \lhd (\lambda e - p_J)^2 = p_\lambda + (1/(\lambda-1))p_Z \lhd (\lambda^2 e - 2\lambda p_J + p_J^2) = p_\lambda + (1/(\lambda-1))p_Z \lhd (\lambda^2 e - 2\lambda p_J + p_J) = p_\lambda + (1/(\lambda-1))(\lambda^2 p_Z \lhd e - 2\lambda p_Z \lhd p_J + p_Z \lhd p_J) = p_\lambda + (1/(\lambda-1))(\lambda^2 p_Z - 2\lambda p_Z + p_Z) = p_\lambda + (1/(\lambda-1))(\lambda-1)^2 p_Z = \lambda e - p_J + (\lambda-1)p_Z = (\lambda e - p) + e_{\lhd}(\lambda e - p) + e_{\lhd}((\lambda e - p)^2)$. □

As mentioned in Section 2, if $\mathfrak{J}$ is a quasi-Jordan algebra with a unit e then the set $\{e + z : z \in Z(\mathfrak{J})\}$ coincides with the set $U(\mathfrak{J})$ of all units in $\mathfrak{J}$.

Proposition 14. *Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan algebra with unit $e \in J$ and $x \in \mathfrak{J}$ invertible with respect to some $e' \in U(\mathfrak{J})$. Then x_J is invertible, with respect to the unit e, in the Jordan algebra J.*

Proof. Clearly, $e \lhd x - x = -x_Z$ and $e \lhd x^2 - x^2 = -x_Z \lhd x_J$. Since $e' \in U(\mathfrak{J})$, we have $e' := e + z$ for some $z \in Z(\mathfrak{J})$. Hence, the invertibility of x in $\mathfrak{J}$, with respect to the unit e', gives the existence of $y \in \mathfrak{J}$ such that $y_J \lhd x_J + y_Z \lhd x_J = y \lhd x_J = y \lhd x = e + z + (e+z) \lhd x - x = e + z + z \lhd x - x_Z$ and $y_J \lhd x_J^2 + y_Z \lhd x_J^2 = y \lhd x_J^2 = y \lhd x^2 = (e+z) \lhd x + (e+z) \lhd x^2 - x^2 = x_J + z \lhd x + z \lhd x^2 - x_Z \lhd x_J$. So, by the uniqueness of the representations in the split algebra $\mathfrak{J}$, we get $y_J \lhd x_J = e$ and $y_J \lhd x_J^2 = x_J$. Thus, y_J is the inverse of x_J in the Jordan algebra J. □

Next, we observe that the spectrum of x in a unital split quasi-Jordan algebra with respect to any unit includes the spectrum of x_J in the Jordan part J.

Corollary 15. *Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan algebra with unit $e \in J$, and let $x \in \mathfrak{J}$. Then $\sigma_{(J,e)}(x_J) \subseteq \sigma_{(\mathfrak{J},e')}(x)$ for all $e' \in U(\mathfrak{J})$.*

Proof. Let $e' := e + z$ with $z \in Z(\mathfrak{J})$, and let $\lambda \notin \sigma_{(\mathfrak{J},e')}(x)$. Then, $\lambda e' - x$ is invertible in $\mathfrak{J}$ with respect to the unit e'. Hence, its Jordan part $\lambda e - x_J$ is invertible in the Jordan algebra J by Proposition 14. Thus, $\lambda \notin \sigma_{(J,e)}(x_J)$. □

It is well known that the spectrum of any element in a unital Jordan Banach algebra is nonempty (cf. [7]). This together with Proposition 2 and Corollary 15 gives the following result.

Corollary 16. *The spectrum of any element in a unital split quasi-Jordan Banach algebra is nonempty.*

The next result extends Corollary 16 to any quasi-Jordan Banach algebra with a norm 1 unit.

Proposition 17. *The spectrum of any element in a quasi-Jordan Banach algebra with a norm 1 unit is nonempty.*

Proof. Let $\mathfrak{J}$ be a unital quasi-Jordan Banach algebra with a norm 1 unit e. From Section 2, we know that the map $\varphi : x \mapsto \varphi(x) = (x, R_x)$ embeds $\mathfrak{J}$ into the unital split quasi-Jordan Banach algebra $\mathfrak{J}_1 := \{(x, R_y) : x, y \in \mathfrak{J}\}$, equipped with the sum $(x, R_y) + (a, R_b) := (x + a, R_{y+b})$, scalar multiplication $\lambda(x, R_y) := (\lambda x, R_{\lambda y})$, and product $(x, R_y) \lhd (a, R_b) := (x \lhd b, R_{y \lhd b})$, and the image $\varphi(\mathfrak{J})$ is a norm closed right ideal isomorphic to $\mathfrak{J}$ with norm 1 unit (e, R_e). Moreover, it is seen in Section 3 that the embedding

φ also preserves the corresponding invertible elements; that is, (y, R_y) is an inverse of (x, R_x), with respect to the unit (e, R_e), in $\mathfrak{J}_1$ whenever y is an inverse of x, with respect to a unit e, in $\mathfrak{J}$. Hence, by Corollary 16, it follows that $\phi \neq \sigma_{(\mathfrak{J}_1,(e,R_e))}(x, R_x) \subseteq \sigma_{(\varphi(\mathfrak{J}),(e,R_e))}(x, R_x) = \sigma_{(\mathfrak{J},e)}(x)$, for all $x \in \mathfrak{J}$. □

Proposition 18. *Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan algebra with unit $e \in J$ and let $x \in J$. Then $\sigma_{(J,e)}(x) = \sigma_{(\mathfrak{J},e)}(x)$.*

Proof. By Corollary 15, $\sigma_{(J,e)}(x) \subseteq \sigma_{(\mathfrak{J},e)}(x)$. For the reverse inclusion, let $\lambda \notin \sigma_{(J,e)}(x)$, then $\lambda e - x$ is invertible in J; that is, there exists $y \in J$ such that $y \lhd (\lambda e - x) = e$ and $y \lhd (\lambda e - x)^2 = \lambda e - x$. However, $e_{\lhd}(\lambda e - x) = 0 = e_{\lhd}((\lambda e - x)^2)$. Hence, y is an inverse of $\lambda e - x$ in $\mathfrak{J}$ with respect to the unit e; that is, $\lambda \notin \sigma_{(\mathfrak{J},e)}(x)$. Thus, $\sigma_{(\mathfrak{J},e)}(x) \subseteq \sigma_{(J,e)}(x)$. □

Proposition 19. *Let $\mathfrak{J}$ be a unital quasi-Jordan normed algebra, and let e be a unit in $\mathfrak{J}$ for which $G_e(\mathfrak{J})$ is open. Then $\sigma_{(\mathfrak{J},e)}(x) = \sigma_{(\mathfrak{J},e)}(x + z)$, for all $z \in Z(\mathfrak{J})$.*

Proof. By Corollary 8, $\lambda e - x$ is invertible if and only if $\lambda e - (x + z)$ is invertible, for all $z \in Z(\mathfrak{J})$. Thus, $\lambda \notin \sigma_{(\mathfrak{J},e)}(x)$ if and only if $\lambda \notin \sigma_{(\mathfrak{J},e)}(x + z)$ for all $z \in Z(\mathfrak{J})$. □

Corollary 20. *Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan normed algebra, and let e be a unit in $\mathfrak{J}$ such that $G_e(\mathfrak{J})$ is open. Then $\sigma_{(\mathfrak{J},e)}(x) = \sigma_{(\mathfrak{J},e)}(x_J)$ for all $x = x_J + x_Z \in \mathfrak{J}$. Further, if the unit $e \in J$ then $\sigma_{(\mathfrak{J},e)}(x) = \sigma_{(\mathfrak{J},e)}(x_J) = \sigma_{(J,e)}(x_J)$.*

Lemma 21. *Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan algebra with a unit $e \in J$, and let $x \in \mathfrak{J}$ be invertible with respect to e. Then $x_Z \lhd x^2 = (x_Z \lhd x) \lhd x$.*

Proof. As x is invertible, there exists $y \in \mathfrak{J}$ such that $y \lhd x = e + (e \lhd x - x) = e - x_Z$ and $y \lhd x^2 = x + (e \lhd x - x) + (e \lhd x^2 - x^2) = x_J - x_Z \lhd x$. Hence, by the uniqueness of the representation in a split quasi-Jordan algebra, we obtain

$$y_Z \lhd x = -x_Z, \tag{11}$$

$$y_Z \lhd x^2 = -x_Z \lhd x. \tag{12}$$

Thus,

$$\begin{aligned} x_Z \lhd x^2 &= -\left(y_Z \lhd x\right) \lhd x^2 \quad (\text{by } (11)) \\ &= -\left(y_Z \lhd x^2\right) \lhd x \quad (\text{by the right Jordan identity}) \\ &= \left(x_Z \lhd x\right) \lhd x \quad (\text{by } (12)). \end{aligned} \tag{13}$$

□

Proposition 22. *Let $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ be a unital split quasi-Jordan algebra with a unit $e \in J$; $x = x_J + x_Z \in \mathfrak{J}$ satisfies $\sigma_{(\mathfrak{J},e)}(x) \neq \mathbb{C}$. Then,*

(1) $x_Z \lhd x^2 = (x_Z \lhd x) \lhd x$,

(2) $x^2 \lhd x = x \lhd x^2$.

Proof. (1) Let $\lambda \notin \sigma_{(\mathfrak{J},e)}(x)$. Then $\lambda e - x = (\lambda e - x_J) + (-x_Z)$ is invertible with respect to e. By Lemma 21, we have

$$x_Z \lhd (\lambda e - x)^2 = \left(x_Z \lhd (\lambda e - x)\right) \lhd (\lambda e - x), \tag{14}$$

since the zero part of $\lambda e - x$ is $-x_Z$. However,

$$\begin{aligned} &x_Z \lhd (\lambda e - x)^2 \\ &\quad = x_Z \lhd \left(\lambda^2 e - \lambda e \lhd x - \lambda x \lhd e + x^2\right) \\ &\quad = \lambda^2 \left(x_Z \lhd e\right) - \lambda \left(x_Z \lhd (e \lhd x)\right) \\ &\qquad - \lambda \left(x_Z \lhd (x \lhd e)\right) + x_Z \lhd x^2 \\ &\quad = \lambda^2 x_Z - 2\lambda x_Z \lhd x \\ &\qquad + x_Z \lhd x^2 \quad (\text{by the right commutativity of } \lhd), \\ &\left(x_Z \lhd (\lambda e - x)\right) \lhd (\lambda e - x) \\ &\quad = \left(\lambda \left(x_Z \lhd e\right) - x_Z \lhd x\right) \lhd (\lambda e - x) \\ &\quad = \left(\lambda x_Z - x_Z \lhd x\right) \lhd (\lambda e - x) \\ &\quad = \lambda^2 \left(x_Z \lhd e\right) - \lambda \left(x_Z \lhd x\right) \lhd e \\ &\qquad - \lambda x_Z \lhd x + \left(x_Z \lhd x\right) \lhd x \\ &\quad = \lambda^2 x_Z - 2\lambda x_Z \lhd x + \left(x_Z \lhd x\right) \lhd x. \end{aligned} \tag{15}$$

Therefore, (14) becomes

$$\begin{aligned} &\lambda^2 x_Z - 2\lambda x_Z \lhd x + x_Z \lhd x^2 \\ &\quad = \lambda^2 x_Z - 2\lambda x_Z \lhd x + \left(x_Z \lhd x\right) \lhd x, \end{aligned} \tag{16}$$

which after simplification reduces to the required equation $x_Z \lhd x^2 = (x_Z \lhd x) \lhd x$.

(2) Since $x^2 = (x_J + x_Z)^2 = x_J^2 + x_Z \lhd x_J$, we have

$$\begin{aligned} x^2 \lhd x &= \left(x_J^2 + x_Z \lhd x\right) \lhd x \\ &= x_J^2 \lhd x + \left(x_Z \lhd x\right) \lhd x \\ &= x_J^2 \lhd x_J + \left(x_Z \lhd x\right) \lhd x \\ &= x_J^2 \lhd x_J + x_Z \lhd x^2 \end{aligned} \tag{17}$$

by the part (1). But, $x_J^2 \lhd x_J = x_J \lhd x_J^2$ since x_J is in the Jordan algebra J. Therefore, $x^2 \lhd x = x_J^2 \lhd x_J + x_Z \lhd x^2 = x_J \lhd x_J^2 + x_Z \lhd x^2 = x \lhd x^2$. □

Remark 23. In any quasi-Jordan algebra, if an element x satisfies $x^3 = x^2 \lhd x = x \lhd x^2$, then $x^n \lhd x^2 = x^{n+2}$ for all positive integers n. For this, suppose x satisfies $x^2 \lhd x = x \lhd x^2$ and $x^m \lhd x^2 = x^{m+2}$ for any fixed $m \geq 1$. Then $x^{m+1} \lhd x^2 = (x^m \lhd x) \lhd x^2 = (x^m \lhd x^2) \lhd x$ (by the right Jordan identity) $= x^{m+2} \lhd x = x^{m+3}$.

Proposition 24. *Let $\mathfrak{J}$ be a unital quasi-Jordan Banach algebra with unit e, and let $x \in \mathfrak{J}$ satisfy $(e - x) \lhd (e - x)^2 = (e - x)^2 \lhd (e - x)$. Then $x \in G_e(\mathfrak{J})$ whenever $\|e - x\| < 1$.*

Proof. First note that $\|e-x\| < 1$ gives $\|(e-x)^n\| \le \|e-x\|^n < 1$ for all $n = 1, 2, 3, \ldots$. Hence, the infinite geometric series $e+\sum_{n=1}^{\infty}(e-x)^n$ converges absolutely to some element $y \in \mathfrak{J}$. We show that the geometric series sum y is an inverse of x, with respect to the unit e. For any fixed positive integer n, let $y_n := e+\sum_{k=1}^{n}(e-x)^k$. Then, the sequence $\{y_n\}$ of partial sums converges to y. By setting $w = e-x$, we get

$$\begin{aligned} y_n \lhd x &= \left(e + \sum_{k=1}^{n} w^k\right) \lhd (e-w) \\ &= e + \sum_{k=1}^{n} w^k - \left(e \lhd w + \sum_{k=2}^{n+1} w^k\right) \\ &= e + w - e \lhd w - w^{n+1} \\ &= e + (e-x) - e \lhd (e-x) - (e-x)^{n+1} \\ &= e + (e - x - e + e \lhd x) - (e-x)^{n+1} \\ &= e + (e \lhd x - x) - (e-x)^{n+1}. \end{aligned} \tag{18}$$

Thus, by allowing $n \to \infty$, we obtain $x \lhd y = e + (e \lhd x - x) = e + e_{\lhd}(x)$ since $\|e-x\| < 1$.

Next, by Remark 23, we have

$$\begin{aligned} y_n \lhd x^2 &= y_n \lhd (e-w)^2 \\ &= y_n \lhd \left(e - e \lhd w - w \lhd e + w^2\right) \\ &= y_n \lhd \left(e - 2w + w^2\right) \\ &= y_n \lhd e - 2y_n \lhd w + y_n \lhd w^2 \\ &= y_n - 2y_n \lhd w + y_n \lhd w^2 \\ &= \left(e + \sum_{k=1}^{n} w^k\right) - 2\left(e + \sum_{k=1}^{n} w^k\right) \lhd w \\ &\quad + \left(e + \sum_{k=1}^{n} w^k\right) \lhd w^2 \\ &= \left(e + \sum_{k=1}^{n} w^k\right) - 2\left(e \lhd w + \sum_{k=2}^{n+1} w^k\right) \\ &\quad + \left(e \lhd w^2 + \sum_{k=3}^{n+2} w^k\right) \\ &= e + w - 2e \lhd w - w^2 + e \lhd w^2 - w^{n+1} + w^{n+2}. \end{aligned} \tag{19}$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} y \lhd x^2 &= e + w - 2e \lhd w - w^2 + e \lhd w^2 \\ &= e + (e-x) - 2e \lhd (e-x) - (e-x)^2 + e \lhd (e-x)^2 \\ &= 2e - x - 2e + 2e \lhd x - e + e \lhd x + x - x^2 \\ &\quad + e - 2e \lhd x + e \lhd x^2 \\ &= x + (e \lhd x - x) + \left(e \lhd x^2 - x^2\right). \end{aligned} \tag{20}$$

□

Proposition 25. *Let $\mathfrak{J}$ be a unital split quasi-Jordan Banach algebra with unit e. If $x \in \mathfrak{J}$ with $(e-x) \lhd (e-x)^2 = (e-x)^2 \lhd (e-x)$, then $|\lambda| \le \|x\|$ for all $\lambda \in \sigma_{(\mathfrak{J},e)}(x)$.*

Proof. If $\lambda \in \sigma_{(\mathfrak{J},e)}(x)$ with $\lambda \ne 0$, then, the noninvertibility of $\lambda e - x$ means the noninvertibility of $e-(1/\lambda)x$, with respect to the unit e. However, by Proposition 24, $e-(1/\lambda)x$ must be invertible with respect to the unit e, whenever $(1/|\lambda|)\|x\| < 1$. It follows that $|\lambda| \le \|x\|$ for all $\lambda \in \sigma_{(\mathfrak{J},e)}(x)$. □

5. Unbounded and Nonclosed Spectrum

In this section, we show that the spectrum of an element in a split quasi-Jordan Banach algebra may be neither bounded nor closed, and hence not compact. The following result gives a couple of characterizations of the unbounded spectrum of an element in a split quasi-Jordan Banach algebra.

Proposition 26. *Let x be an element of a unital split quasi-Jordan Banach algebra $\mathfrak{J} = J \oplus Z(\mathfrak{J})$ with a unit $e \in J$. Then, the following statements are equivalent.*

(1) $\sigma_{(\mathfrak{J},e)}(x) \ne \mathbb{C}$.

(2) $x^2 \lhd x = x \lhd x^2$.

(3) $|\lambda| \le \|x\|$, *for all* $\lambda \in \sigma_{(\mathfrak{J},e)}(x)$.

Proof. (1⇒2): See Proposition 22.

(2⇒3): Suppose $x^2 \lhd x = x \lhd x^2$. Then,

$$\begin{aligned} (e-x)^2 \lhd (e-x) &= \left(e - e \lhd x - x + x^2\right) \lhd (e-x) \\ &= e - 2e \lhd x - x + 2x^2 \\ &\quad + (e \lhd x) \lhd x - x^2 \lhd x \\ &= e - 2x_J - x + 2x^2 + e \lhd x^2 - x^3, \\ (e-x) \lhd (e-x)^2 &= (e-x) \lhd \left(e - 2x + x^2\right) \\ &= e - 2(e \lhd x) + e \lhd x^2 - x \\ &\quad + 2x^2 - x \lhd x^2 \\ &= e - 2x_J + e \lhd x^2 - x + 2x^2 - x^3. \end{aligned} \tag{21}$$

From (21), we get

$$(e-x)^2 \lhd (e-x) = (e-x) \lhd (e-x)^2. \tag{22}$$

Hence, $|\lambda| \le \|x\|$, for all $\lambda \in \sigma_{(\mathfrak{J},e)}(x)$ by Proposition 25.

(3⟹1): Immediate. □

Remark 27. There do exist unital split quasi-Jordan algebras containing elements that have the spectrum, with respect to the unit of the Jordan part, equal to the whole of $\mathbb{C}$, and hence unbounded. To justify this claim, we proceed as follows.

Let A be a unital associative algebra and let M be an A-bimodule. Let $f : M \to A$ be an A-bimodule map (i.e., an additive map satisfying $f(ax) = af(x)$ and $f(xa) = f(x)a$, for all $a \in A$, $x \in M$). Then, one can put a dialgebra structure on M as follows: $x \dashv y := xf(y)$ and $x \vdash y := f(x)y$ (cf. [13, Example 2.2(d)]). Hence, M^+ is a quasi-Jordan algebra under the quasi-Jordan product "$\lhd$" given by $x \lhd y := (1/2)(xf(y) + f(y)x)$. Further, for any $x \in M$, we observe that

$$\begin{aligned} x^2 \lhd x &= \frac{1}{2}\left(xf(x) + f(x)x\right) \lhd x \\ &= \frac{1}{4}\big(\left(xf(x) + f(x)x\right)f(x) \\ &\qquad + f(x)\left(xf(x) + f(x)x\right)\big) \\ &= \frac{1}{4}\left(xf(x)f(x) + 2f(x)xf(x) + f(x)f(x)x\right), \\ x \lhd x^2 &= \frac{1}{2} x \lhd \left(xf(x) + f(x)x\right) \\ &= \frac{1}{4}\left(xf\left(xf(x) + f(x)x\right) + f\left(xf(x) + f(x)x\right)x\right) \\ &= \frac{1}{4}\left(2xf(x)f(x) + 2f(x)f(x)x\right). \end{aligned} \tag{23}$$

However, the right hand sides of the above equations (23) may not be equal; see the following example (Example 28). For such elements x, we have $x^2 \lhd x \ne x \lhd x^2$. Hence, by Proposition 26, the spectrum of x is unbounded whenever M^+ is a unital split quasi-Jordan Banach algebra.

Example 28. Let M be the collection of 2×2 matrices with entries from the field $\mathbb{C}$, and let A be the algebra of all matrices of the form $\left[\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right]$ with $\alpha, \beta \in \mathbb{C}$. Then, it is easily seen that M is an A-bimodule. Next, we define $f : M \to A$ by $f\left[\begin{smallmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{smallmatrix}\right] := \left[\begin{smallmatrix}a_{11} & 0\\ 0 & a_{22}\end{smallmatrix}\right]$. Of course, f is an additive map satisfying

$$\begin{aligned} f\left(\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right) &= \begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} f\left(\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right), \\ f\left(\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}\right) &= f\left(\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right)\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}. \end{aligned} \tag{24}$$

Hence, f is an A-bimodule map. Thus, by Remark 27, M^+ is a quasi-Jordan algebra with the quasi-Jordan product as below:

$$\begin{aligned} \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \lhd \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} &:= \frac{1}{2}\left(\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} f\left(\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\right)\right. \\ &\qquad \left. + f\left(\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\right)\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right) \\ &= \begin{bmatrix} a_{11}b_{11} & a_{12}\dfrac{b_{11}+b_{22}}{2} \\ a_{21}\dfrac{b_{11}+b_{22}}{2} & a_{22}b_{22}\end{bmatrix}. \end{aligned} \tag{25}$$

Indeed, $M^+ = J \oplus Z(M^+)$, where $J = \{\left[\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right] : a, b \in \mathbb{C}\}$ is a subalgebra of M^+ and $Z(M^+) = \{\left[\begin{smallmatrix}0 & a\\ b & 0\end{smallmatrix}\right] : a, b \in \mathbb{C}\}$. Any matrix of the form $\left[\begin{smallmatrix}1 & a\\ b & 1\end{smallmatrix}\right]$ with $a, b \in \mathbb{C}$ is a (right) unit in M^+. Thus, M^+ is a unital split quasi-Jordan algebra with the matrix $I = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ as the unit of its Jordan part J.

Further, a natural norm is defined on M^+ as follows:

$$\left\|\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right\| := |a_{11}| + |a_{12}| + |a_{21}| + |a_{22}|. \tag{26}$$

This norm also satisfies

$$\begin{aligned} \left\|\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \lhd \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\right\| &= \frac{1}{2}\left(\left\|\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}b_{11} & 0\\ 0 & b_{22}\end{bmatrix}\right.\right. \\ &\qquad \left.\left. + \begin{bmatrix}b_{11} & 0\\ 0 & b_{22}\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right\|\right) \\ &\le \left\|\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right\| \left\|\begin{bmatrix}b_{11} & 0\\ 0 & b_{22}\end{bmatrix}\right\| \\ &\le \left\|\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\right\| \left\|\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\right\|. \end{aligned} \tag{27}$$

Next, for any $x = \left[\begin{smallmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{smallmatrix}\right] \in M^+$, we observe that

$$\begin{aligned} x^2 &= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \lhd \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \\ &= \begin{bmatrix} a_{11}^2 & a_{12}\dfrac{a_{11}+a_{22}}{2} \\ a_{21}\dfrac{a_{11}+a_{22}}{2} & a_{22}^2\end{bmatrix} \end{aligned} \tag{28}$$

so that

$$\begin{aligned} x^2 \lhd x &= \begin{bmatrix} a_{11}^3 & \dfrac{a_{12}}{4}(a_{11}+a_{22})^2 \\ \dfrac{a_{21}}{4}(a_{11}+a_{22})^2 & a_{22}^3\end{bmatrix}, \\ x \lhd x^2 &= \begin{bmatrix} a_{11}^3 & \dfrac{a_{12}}{2}\left(a_{11}^2+a_{22}^2\right) \\ \dfrac{a_{21}}{2}\left(a_{11}^2+a_{22}^2\right) & a_{22}^3\end{bmatrix}. \end{aligned} \tag{29}$$

So, for any $x = \left[\begin{smallmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{smallmatrix}\right] \in M^+$, $x^2 \lhd x = x \lhd x^2 \Leftrightarrow a_{12} = a_{21} = 0$ or $a_{11} = a_{22}$. Thus, by Proposition 26,

$\sigma_{(M^+,I)}(\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]) = \mathbb{C}$ whenever $a_{11} \neq a_{22}$ and $\{a_{12}, a_{21}\} \neq \{0\}$. In particular, for $x := \left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right] \in M^+$, we have

$$x^2 \lhd x = \begin{bmatrix} 1^3 & \frac{1}{4}(1+2)^2 \\ \frac{0}{4}(1+2)^2 & 2^3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{9}{4} \\ 0 & 8 \end{bmatrix},$$
$$x \lhd x^2 = \begin{bmatrix} 1^3 & \frac{1}{2}\left(1^2+2^2\right) \\ \frac{0}{2}\left(1^2+2^2\right) & 2^3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & 8 \end{bmatrix}. \tag{30}$$

Concerning the inequality between the right hand sides of (23) in Remark 27, we observe for x as above that

$$\begin{aligned}&\frac{1}{4}\left(xf(x)f(x) + 2f(x)xf(x) + f(x)f(x)x\right) \\ &\quad = \frac{1}{4}\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right. \\ &\qquad \left. + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{9}{4} \\ 0 & 8 \end{bmatrix},\end{aligned} \tag{31}$$

but

$$\begin{aligned}&\frac{1}{4}\left(2xf(x)f(x) + 2f(x)f(x)x\right) \\ &\quad = \frac{1}{4}\left(2\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right) \\ &\quad = \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & 8 \end{bmatrix}.\end{aligned} \tag{32}$$

Hence, $x^2 \lhd x \neq x \lhd x^2$. Thus, $\sigma_{(M^+,I)}(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right]) = \mathbb{C}$ by Proposition 26.

Further, suppose the matrix $\left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right] \in M^+$ is invertible with respect to the unit I; that is, $\left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right] \in G_I(M^+)$. Then there exists $\left[\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right] \in M^+$ such that

$$\begin{aligned}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \lhd \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} &= I + I_\lhd\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & -a_2 \\ -a_3 & 1 \end{bmatrix}, \\ \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \lhd \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^2 &= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + I_\lhd\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) \\ &\quad + I_\lhd\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^2\right) \\ &= \begin{bmatrix} a_1 & -a_2\dfrac{a_1+a_4}{2} \\ -a_3\dfrac{a_1+a_4}{2} & a_4 \end{bmatrix}.\end{aligned} \tag{33}$$

However,

$$\begin{aligned}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \lhd \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} &= \begin{bmatrix} b_1a_1 & b_2\dfrac{a_1+a_4}{2} \\ b_3\dfrac{a_1+a_4}{2} & b_4a_4 \end{bmatrix}, \\ \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \lhd \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^2 &= \begin{bmatrix} b_1a_1 & b_2\dfrac{a_1^2+a_4^2}{2} \\ b_3\dfrac{a_1^2+a_4^2}{2} & b_4a_4^2 \end{bmatrix}.\end{aligned} \tag{34}$$

It follows that $a_1 \neq 0$, $a_4 \neq 0$, $b_1 = 1/a_1$, $b_4 = 1/a_4$, $b_2((a_1+a_4)/2) = -a_2$, $b_3((a_1+a_4)/2) = -a_3$, $b_2((a_1^2+a_4^2)/2) = -a_2((a_1+a_4)/2)$, and $b_3((a_1^2+a_4^2)/2) = -a_3((a_1+a_4)/2)$. From these equations, we get $b_2(a_1-a_4)^2 = 0$ and $b_3(a_1-a_4)^2 = 0$, so that $(b_2-b_3)(a_1-a_4)^2 = 0$. Then, for $a_1 \neq a_4$, we obtain $b_2 = b_3 = 0$ and hence $a_2 = a_3 = 0$. Therefore,

$$\begin{aligned}G_I(M^+) = \Bigg\{ &\begin{bmatrix} a & b \\ c & a \end{bmatrix}, x = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in M^+ : a \neq 0, \\ &\alpha \neq 0, \beta \neq 0, \alpha \neq \beta \Bigg\}.\end{aligned} \tag{35}$$

The set $G_I(M^+)$ is not open: clearly $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in G_I(M^+)$; for any $\epsilon > 0$,

$$\left\|\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1-\dfrac{\epsilon}{4} & 1 \\ 0 & 1+\dfrac{\epsilon}{4} \end{bmatrix}\right\| = \left\|\begin{bmatrix} \dfrac{\epsilon}{4} & 0 \\ 0 & -\dfrac{\epsilon}{4} \end{bmatrix}\right\| = \frac{\epsilon}{2} < \epsilon \tag{36}$$

but $\left[\begin{smallmatrix} 1-\epsilon/4 & 1 \\ 0 & 1+\epsilon/4 \end{smallmatrix}\right] \notin G_I(M^+)$.

Now, if $A = \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, then $\lambda I - A = \left[\begin{smallmatrix} \lambda+1 & -1 \\ 0 & \lambda-1 \end{smallmatrix}\right] \notin G_I(M^+)$, and so $\lambda \in \sigma_{(M^+,I)}(A)$ for all $\lambda \in \mathbb{C}$. Thus, $\sigma_{(M^+,I)}(A) = \mathbb{C}$, an unbounded spectrum.

Next, we observe that the spectrum of an element with respect to a unit is closed whenever the corresponding set of invertibles is open.

Proposition 29. *Let $\mathfrak{J}$ be a quasi-Jordan normed algebra with a unit e such that $G_e(\mathfrak{J})$ is open. Then $\sigma_{(\mathfrak{J},e)}(x)$ is closed, for all $x \in \mathfrak{J}$.*

Proof. Define $f : \mathbb{C} \to \mathfrak{J}$ by $f(\lambda) = \lambda e - x$. Since f is continuous, the inverse image of the open set $G_e(\mathfrak{J})$ is open in $\mathbb{C}$ and so its complement $\sigma_{(\mathfrak{J},e)}(x)$ is closed. □

We conclude this paper with the following example of a nonclosed spectrum.

Example 30. Let M and M^+ be as in Example 28. Let $E = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$ with $a \neq b$ both different from 1. Then E is a unit in M^+. We show that $\sigma_{(M^+,E)}(A) = \mathbb{C} \setminus \{1\}$. For this, let us first investigate when can an element of the form $B = \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \gamma \end{smallmatrix}\right]$ be invertible? Assuming that B is invertible, we get the existence of an element $C \in D^+$ such that

$$\begin{aligned}C \lhd B &= E + E_\lhd(B), \\ C \lhd B^2 &= B + E_\lhd(B) + E_\lhd\left(B^2\right).\end{aligned} \tag{37}$$

From these equations, we get $\alpha \neq 0$, $\gamma \neq 0$, and

$$C = \begin{bmatrix} \frac{1}{\alpha} & \beta_\circ \\ 0 & \frac{1}{\gamma} \end{bmatrix}, \tag{38}$$

where $\beta_\circ$ satisfies the following two equations:

$$\begin{aligned} \beta_\circ\left(\frac{\alpha+\gamma}{2}\right) &= 1-\beta+\frac{\alpha+\gamma}{2}, \\ \beta_\circ\left(\frac{\alpha^2+\gamma^2}{2}\right) &= \frac{\alpha+\gamma}{2}+\frac{\alpha^2+\gamma^2}{2}-\beta\frac{\alpha+\gamma}{2}. \end{aligned} \tag{39}$$

Multiplying the last equation by $2(\alpha+\gamma)/(\alpha^2+\gamma^2)$ and then using the other equation, we get

$$2(1-\beta)+(\alpha+\gamma) = (1-\beta)\frac{(\alpha+\gamma)^2}{\alpha^2+\gamma^2}+(\alpha+\gamma), \tag{40}$$

or equivalently

$$2(1-\beta) = (1-\beta)\frac{(\alpha+\gamma)^2}{\alpha^2+\gamma^2}; \tag{41}$$

this equation is satisfied for $\beta = 1$ or $\alpha = \gamma$. Hence, the matrix B is invertible if and only if $B = \left[\begin{smallmatrix} \alpha & 1 \\ 0 & \gamma \end{smallmatrix}\right]$ or $B = \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right]$ for $\alpha, \gamma \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$.

We conclude that $A_\lambda := \lambda E - A = \left[\begin{smallmatrix} \lambda-a & \lambda \\ 0 & \lambda-b \end{smallmatrix}\right]$ is invertible with respect to E if and only if $\lambda - b \neq 0$, $\lambda - a \neq 0$, and $\lambda = 1$; that is, A_λ is invertible if $a \neq \lambda$ and $b \neq \lambda$ and $\lambda = 1$. Hence, A_λ is invertible only if $\lambda = 1$ as we assumed that $a \neq b$ and both are not 1. So, for all $\lambda \neq 1$, $A_\lambda \notin G_E(M^+)$. Thus, $\sigma_{(M^+,E)}(A) = \mathbb{C} \setminus \{1\}$, which is neither bounded nor closed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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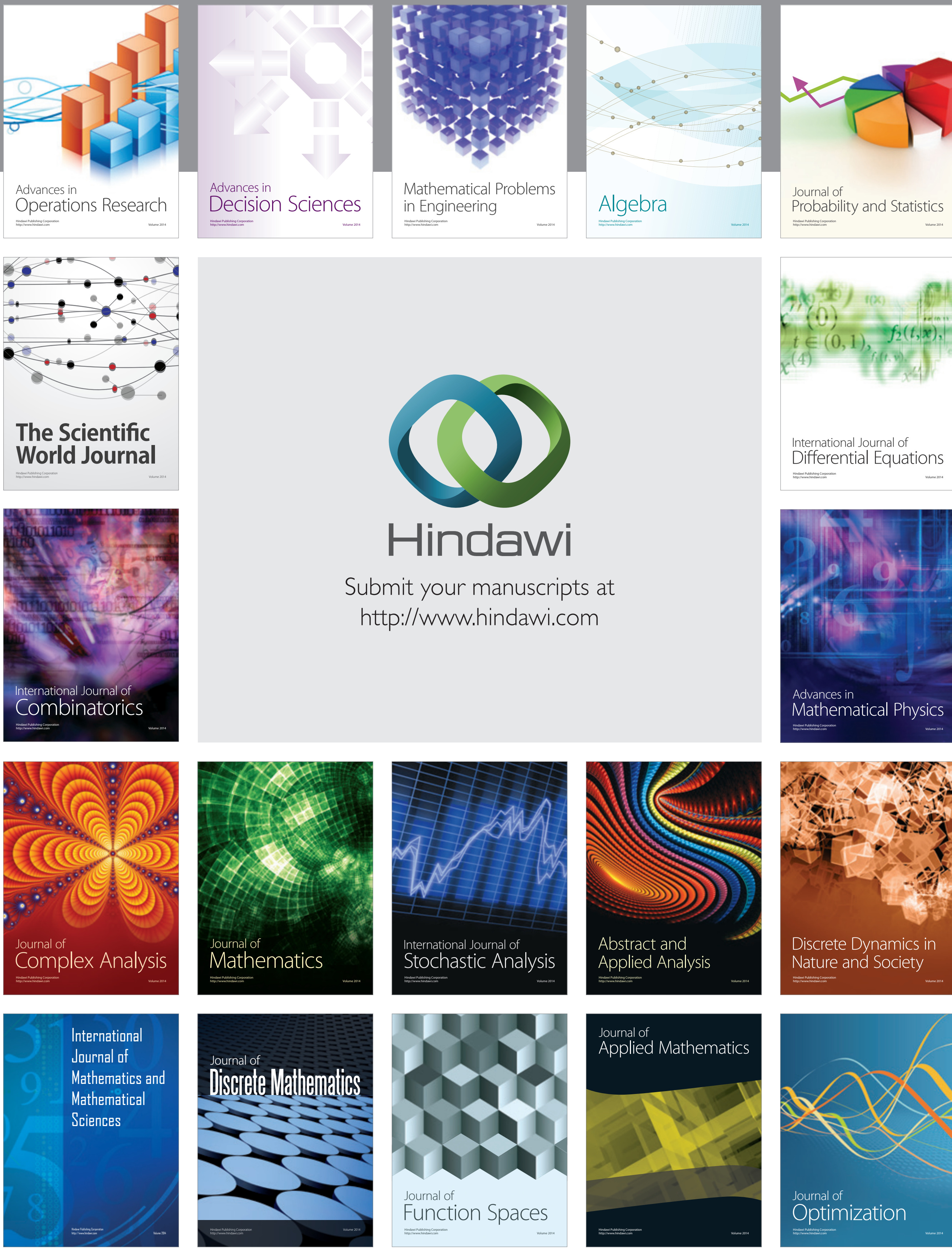
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