If $X_1, X_2, \dots, X_n$ random sample from exponential distribution with parameter θ as:

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0, \qquad \theta > 0$$

Then, $Y = \sum_{i=1}^{n} X_i \sim Gamma(n, \theta)$ where,

$$f(y) = \frac{1}{\Gamma(n)\theta^n} y^{n-1} e^{-\frac{y}{\theta}}; \quad y > 0; \quad n, \theta > 0$$

Moreover,

$$\frac{2\sum_{i=1}^{n} X_i}{\theta} \sim Gamma(n, 2) \quad \text{or} \quad \frac{2\sum_{i=1}^{n} X_i}{\theta} \sim \chi^2_{(2n)}$$

Proof:

Since the mgf of the exponential distribution is given as

$$M_X(t) = (1 - \theta t)^{-1}$$

Thus,

$$M_{\sum_{i=1}^{n} X_i}(t) = \left(M_X(t)\right)^n = (1 - \theta t)^{-n}$$

Which is the mgf of gamma distribution with parameters n and θ.

Moreover,

$$M_{\frac{2\sum_{i=1}^{n} X_i}{\theta}}(t) = E\left(e^{\frac{2\sum_{i=1}^{n} X_i}{\theta} t}\right) = \left(M_X\left(\frac{2t}{\theta}\right)\right)^n = (1 - 2t)^{-n}$$

Which is the mgf of chi-square distribution with degrees of freedom 2n.

Relationships Between Distributions:

when X and Y are independent random variables

- If $X \sim Exp(\theta) \Rightarrow X \sim Gamma(1, \theta) \Rightarrow \sum_{i=1}^{n} X_i \sim Gamma(n, \theta) \Rightarrow \bar{X} \sim Gamma\left(n, \frac{\theta}{n}\right)$.
- If $X \sim Gamma(\alpha, \beta) \Rightarrow \sum_{i=1}^{n} X_i \sim Gamma(n\alpha, \beta) \Rightarrow \bar{X} \sim Gamma\left(n\alpha, \frac{\beta}{n}\right)$.
- If $X \sim Gamma(\alpha, \theta)$ and $Y \sim Gamma(\beta, \theta) \Rightarrow \frac{X}{X+Y} \sim Beta(\alpha, \beta)$.
- If $X \sim Normal(\mu, \sigma^2) \Rightarrow \sum_{i=1}^{n} X_i \sim Normal(n\mu, n\sigma^2) \Rightarrow \bar{X} \sim Normal\left(\mu, \frac{\sigma^2}{n}\right)$.
- If $X \sim Normal(\mu, \sigma^2) \Rightarrow \sum_{i=1}^{n} {X_i}^2 \sim \chi^2_{(n)}$
- If $X \sim Poisson\ (\theta) \Rightarrow \sum_{i=1}^{n} X_i \sim Poisson(n\theta)$.
- If $X \sim Bernoulli(p) \Rightarrow \sum_{i=1}^{n} X_i \sim Binomial(n, p)$.
- If $X \sim Binomial(n, p)$ and $Y \sim Binomial(m, p) \Rightarrow X + Y \sim Binomial(n + m, p)$
- If $X \sim \chi^2_{(v)}$ and $Y \sim \chi^2_{(u)} \Rightarrow X + Y \sim \chi^2_{(v+u)}$.
- If $X \sim \chi^2_{(v)}$ and $Y \sim \chi^2_{(u)} \Rightarrow \frac{X}{X+Y} \sim Beta\left(\frac{v}{2}, \frac{u}{2}\right)$.
- If $X \sim \chi^2_{(v)} \Rightarrow cX \sim Gamma\left(\frac{v}{2}, 2c\right), c > 0$.
- If $Z \sim N(0,1)$ and $X \sim \chi^2_{(v)} \Rightarrow T = \frac{Z}{\sqrt{\frac{X}{v}}} \sim t_{(v)}$.
- If $X \sim \chi^2_{(v)}$ and $Y \sim \chi^2_{(u)} \Rightarrow F = \frac{X/v}{Y/u} \sim F_{(v,u)}$.