الاختبار 332

السؤال الثالث:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad w = f(x,y) = \ln(x^2+y^2+1)$$

طريقة الأولى:

$$f(x,y) = \ln(x^2+y^2+1)$$
$$= \ln\left((r\cos\theta)^2 + (r\sin\theta)^2 + 1\right)$$
$$= \ln\left(r^2(\cos^2\theta + \sin^2\theta) + 1\right)$$
$$= \ln(r^2+1) = g(r,\theta)$$

$$\boxed{\frac{\partial w}{\partial r} = \frac{2r}{r^2+1} \quad \Big| \quad \frac{\partial w}{\partial \theta} = 0}$$

طريقة الثانية (قاعدة السلسلة):

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{2x}{x^2+y^2+1}\cos\theta + \frac{2y}{x^2+y^2+1}\sin\theta$$
$$= \frac{2r\cos^2\theta + 2r\sin^2\theta}{r^2+1}$$
$$\boxed{\frac{\partial w}{\partial r} = \frac{2r}{r^2+1}}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$
$$= \frac{2x}{x^2+y^2+1}(-r\sin\theta) + \frac{2y}{x^2+y^2+1}(r\cos\theta)$$
$$= \frac{-2r^2\cos\theta\sin\theta + 2r^2\sin\theta\cos\theta}{r^2+1}$$
$$\boxed{\frac{\partial w}{\partial \theta} = 0}$$

(ب) $w=f(x,y)$: لها مشتقات جزئية من الرتبة الثانية متصلة

$y=6u-4v \,,\ x=3u+2v$: أثبت $\dfrac{\partial^2 w}{\partial u\partial v}=6\dfrac{\partial^2 w}{\partial x^2}-24\dfrac{\partial^2 w}{\partial y^2}$

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\boxed{\frac{\partial w}{\partial u}=3\frac{\partial w}{\partial x}+6\frac{\partial w}{\partial y}}$$

$$\frac{\partial^2 w}{\partial u\partial v}=\frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right)$$

$$=\frac{\partial}{\partial x}\left(3\frac{\partial w}{\partial x}+6\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial v}+\frac{\partial}{\partial y}\left(3\frac{\partial w}{\partial x}+6\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial v}$$

$$=\left(3\frac{\partial^2 w}{\partial x^2}+6\frac{\partial^2 w}{\partial y\partial x}\right)2+\left(3\frac{\partial^2 w}{\partial x\partial y}+6\frac{\partial^2 w}{\partial y^2}\right)(-4)$$

$$\frac{\partial^2 w}{\partial u\partial v}=6\frac{\partial^2 w}{\partial x^2}+12\frac{\partial^2 w}{\partial y\partial x}-12\frac{\partial^2 w}{\partial x\partial y}-24\frac{\partial^2 w}{\partial y^2}$$

وبما أن f لها مشتقات ثانية متصلة

فإن $\dfrac{\partial^2 w}{\partial x\partial y}=\dfrac{\partial^2 w}{\partial y\partial x}$

وبالتالي:

$$\boxed{\frac{\partial^2 w}{\partial u\partial v}=6\frac{\partial^2 w}{\partial x^2}-24\frac{\partial^2 w}{\partial y^2}}$$

الاختبار 351

الجزء الأول

(2)

$$\lim_{(0,0)} \frac{xy+y^4}{x^2+y^2} = ?$$

لدينا:

$$\lim_{\substack{(0,0)\\ x=0}} \frac{xy+y^4}{x^2+y^2} = \lim_{y\to 0} \frac{0+y^4}{y^2} = \lim_{y\to 0} y^2 = 0$$

ولدينا:

$$\lim_{\substack{(0,0)\\ y=x}} \frac{xy+y^4}{x^2+y^2} = \lim_{x\to 0} \frac{x^2+x^4}{x^2+x^2} = \lim_{x\to 0} \frac{x^2(1+x^2)}{2x^2}$$

$$= \lim_{x\to 0} \frac{1+x^2}{2} = \frac{1}{2}$$

وبالتالي $\lim_{(0,0)} \frac{xy+y^4}{x^2+y^2}$ غير موجودة.

الاختبار 342

الجزء الثاني

(1) $f(x,y) = e^{x^2} + x^2y^2 + \ln(x^2+y^2+1)$

(أ) مجال الدالة f هو $D_f = \mathbb{R}^2$

(ب) $(x,y) \in \mathbb{R}^2$:

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = 2xe^{x^2} + 2xy^2 + \frac{2x}{x^2+y^2+1}$$

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = 2x^2y + \frac{2y}{x^2+y^2+1}$$

(2)

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)$$

$$= 4xy - \frac{2x\,2y}{(x^2+y^2+1)^2}$$

$$\boxed{f_{xy}(x,y) = 4xy\left(1 - \frac{1}{(x^2+y^2+1)^2}\right)}$$

لأن $\boxed{f_{yx}(x,y) = f_{xy}(x,y)}$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)$$

$$= 2x^2 + \frac{2(x^2+y^2+1) - 2y\,2y}{(x^2+y^2+1)^2}$$

$$\boxed{= 2x^2 + \frac{2(x^2-y^2+1)}{(x^2+y^2+1)^2}}$$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y)$$

$$= 2e^{x^2} + 2x \cdot 2xe^{x^2} + 2y^2 + \frac{2(x^2+y^2+1) - 2x(2x)}{(x^2+y^2+1)^2}$$

$$\boxed{f_{xx}(x,y) = 2(1+2x^2)e^{x^2} + 2y^2 + \frac{2(-x^2+y^2+1)}{(x^2+y^2+1)^2}}$$