Chapter(8)
Sampling Distribution
(Examples)

Review

Example (1)

Compute the mean, Variance, Standard deviation of the following population values:
6, 3, 5, 4, and 2   by using two deferent formulas

Solution:

i	X_i	X_i^2
1	**6**	**36**
2	**3**	**9**
3	**5**	**25**
4	**4**	**16**
5	**2**	**4**
Total	$\sum X_i = 20$	$\sum X_i^{\,2} = 90$

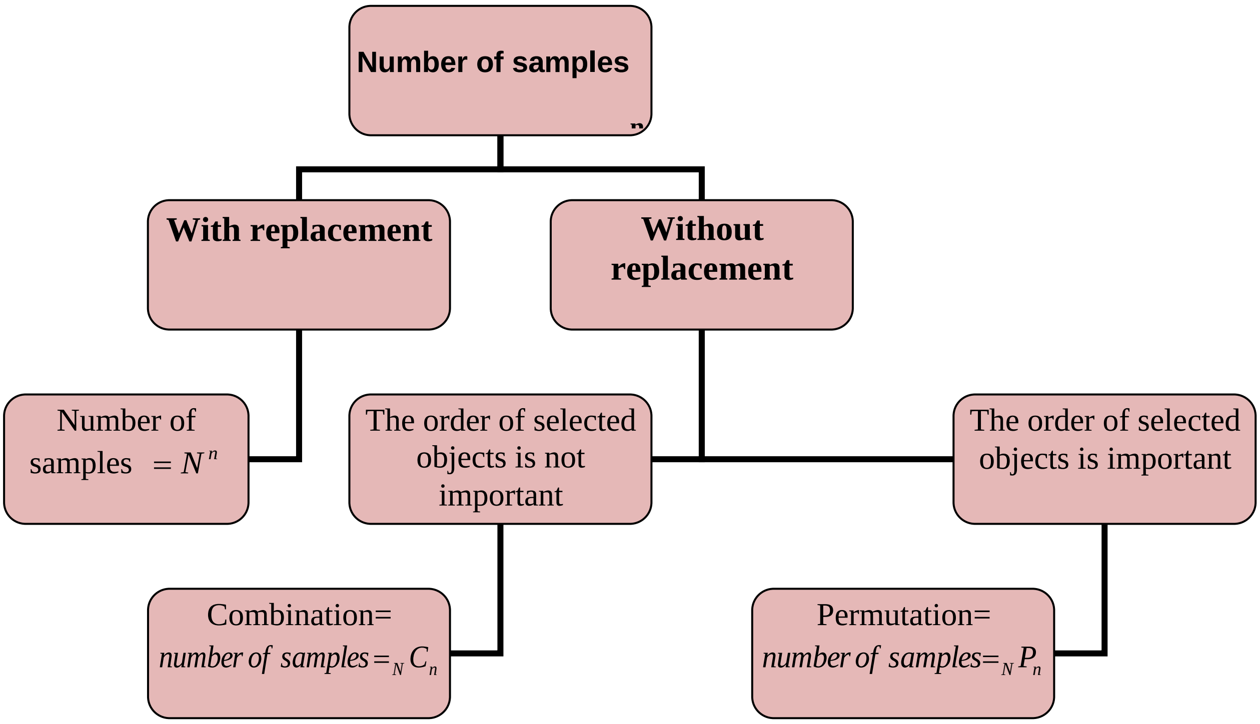
$$\mu_x = \frac{\sum_{i=1}^{N} X_i}{N} = \frac{20}{5} = 4$$

$$\sigma^2 = \frac{1}{N}\left(\sum_{i=1}^{N} X_i^{\,2} - \frac{\left(\sum_{i=1}^{N} X_i\right)^2}{N}\right) = \frac{\sum_{i=1}^{N} X_i^2}{N} - \mu_x^2 = \frac{90}{5} - (4)^2 = 18 - 16 = 2$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} X_i^2}{N} - \mu_x^2} = \sqrt{\frac{90}{5} - (4)^2} = \sqrt{18-16} = \sqrt{2} = 1.414$$

Sampling distribution:

How to draw sample from population



Case (1) the sampling with replacement:

$= K = N^n$ Number of samples

Case (2) the sampling without replacement:

(2-1) the order of selected objects is important

$$Number\ of\ samples = K = {}^N P_n = \frac{N!}{(N-n)!} = N(N-1)(N-2)(N-3)......(N-n+1).$$

(2-2)The order of selected objects is not important

$$Number\ of\ samples = K = {}^N C_n = \frac{N!}{(N-n)!\,n!}.$$

$$= \frac{N(N-1)(N-2)(N-3)..\qquad(N-n+1)........}{n!}$$

Sampling distribution of the sample mean

We study three cases:

Case (1): The sampling distribution with replacement:

Case (2): The sampling distribution without replacement:

(2-1): The order of selected objects is important.

Case (2-1): The order of selected objects is not important.

Case (1): The sampling distribution with replacement:

Example (2)

Refer to example (1) list all possible samples of 2 executives from the population and compute their means **(with replacement)**.**Find:**

1. How many different samples of 2 are possible?

2. The probability of selecting any of the possible simple random samples.

3. The mean of $\bar{X}_i$ (the mean of sampling distribution of the sample means), the variance of $\bar{X}_i$ (the variance of sampling distribution of the sample means), the standard deviation of $\bar{X}_i$ (the standard deviation of sampling distribution of the sample means).

Solution:

With replacement

1. Number of samples=k $= N^n = 5^2 = 25$

2. The probability of selecting any of the possible simple random samples

$$P = \frac{1}{N^n} = \frac{1}{25}$$

The items: 6, 3, 5, 4, 2

The possible samples:

The Mean and The Variance of $\overline{X}_i$

Sample number (i)	**Samples**	$\overline{X}$	$\overline{X}_i^2$
1	**6 ,6**	**6**	**36**
2	**6,3**	**4.5**	**20.25**
3	**6,5**	**5.5**	**30.25**
4	**6,4**	**5**	**25**
5	**6,2**	**4**	**16**
6	**3,6**	**4.5**	**20.25**
7	**3,3**	**3**	**9**
8	**3,5**	**4**	**16**
9	**3,4**	**3.5**	**12.25**
10	**3,2**	**2.5**	**6.25**
11	**5 ,6**	**5.5**	**30.25**
12	**5 ,3**	**4**	**16**
13	**5,5**	**5**	**25**
14	**5, 4**	**4.5**	**20.25**
15	**5,2**	**3.5**	**12.25**
16	**4 ,6**	**5**	**25**
17	**4, 3**	**3.5**	**12.25**
18	**4 ,5**	**4.5**	**20.25**
19	**4,4**	**4**	**16**
20	**4,2**	**3**	**9**
21	**2 ,6**	**4**	**16**
22	**2, 3**	**2.5**	**6.25**
23	**2 ,5**	**3.5**	**12.25**
24	**2,4**	**3**	**9**
25	**2,2**	**2**	**4**
		$\sum \overline{X} = 100$	$\sum \overline{X}^2 = 425$

$$\sum_{i=1}^{25} \overline{X}_i = 100$$

The mean of $\overline{X}_i = \mu_{\bar{x}} = \frac{\sum_{i=1}^{25} \overline{X}_i}{k} = \frac{\bar{x}_1 + \bar{x}_2 + \ldots\ldots\ldots\ldots + \bar{x}_{25}}{K} = \frac{100}{25} = 4$

$\therefore \mu_{\bar{x}} = \mu_x$ **, In this case the statistic is an unbiased estimator of the parameter.**

$$\sigma_{\bar{x}}^{2} = \frac{\sum_{i=1}^{k} \bar{X}_i^{2}}{k} - \mu_{\bar{x}}^{2} = \frac{425}{25} - (4)^2 = 17 - 16 = 1$$

The Standard deviation of $\bar{X}_i$ **=**

$$\sigma_{\bar{x}} = \sqrt{\frac{\sum_{i=1}^{k} \bar{X}_i^{2}}{k} - \mu_{\bar{x}}^{2}} = \sqrt{\frac{425}{25} - (4)^2} = \sqrt{17-16} = \sqrt{1} = 1$$

$$\therefore \sigma_{\bar{x}}^{2} = \frac{\sigma_x^2}{n} = \frac{2}{2} = 1 \qquad \therefore \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{1.414}{\sqrt{2}} = \frac{1.414}{1.414} = 1$$

Conclusion: For case of sampling with replacement, the sample mean is a random variable with mean and variance given by:

$$\therefore \mu_{\bar{x}} = \mu_x$$

$$\sigma_{\bar{x}}^{2} = \frac{\sigma_x^2}{n} \qquad \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}$$

Case (2): The sampling distribution without replacement:

(2-1): The order of selected objects is important.

(2-1): The order of selected objects is not important.

Example (3)

Refer to example (1) list all possible samples of 2 executives from the population and compute their means without replacement if:

- The order of selected objects is important.
- The order of selected objects is not important

Find:

1. How many different samples of 2 are possible?

2. The probability of selecting any of the possible simple random samples.

3. The mean of $\bar{X}_i$ (the mean of sampling distribution of the sample means), the variance of $\bar{X}_i$ (the variance of sampling distribution of the sample means), the standard deviation of $\bar{X}_i$ (the standard deviation of sampling distribution of the sample means).

Solution:

Case (2-1): Without replacement and the order of selected objects is important:

1- **Number of samples**

$$K={}^{N}P_n = \frac{N!}{(N-n)!} = N(N-1)(N-2)(N-3)......(N-n+1).$$

$$={}_5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20$$

1. **The probability of selecting any of the possible simple random samples** $P = \frac{1}{{}_NP_n} = \frac{1}{20}$

2. The possible samples

The mean & variance

Sample number (i)	**Samples**	$\bar{X}$	$\bar{X}_i^2$
1	**6,3**	**4.5**	**20.25**
2	**6,5**	**5.5**	**30.25**
3	**6,4**	**5**	**25**
4	**6,2**	**4**	**16**
5	**3,6**	**4.5**	**20.25**
6	**3,5**	**4**	**16**
7	**3,4**	**3.5**	**12.25**
8	**3,2**	**2.5**	**6.25**
9	**5 ,6**	**5.5**	**30.25**
10	**5 ,3**	**4**	**16**
11	**5, 4**	**4.5**	**20.25**
12	**5,2**	**3.5**	**12.25**
13	**4 ,6**	**5**	**25**
14	**4, 3**	**3.5**	**12.25**
15	**4 ,5**	**4.5**	**20.25**
16	**4,2**	**3**	**9**
17	**2 ,6**	**4**	**16**
18	**2, 3**	**2.5**	**6.25**
19	**2 ,5**	**3.5**	**12.25**
20	**2,4**	**3**	**9**
		$\sum \bar{X} = 80$	$\sum \bar{X}^2 = 335$

$$\sum_{i=1}^{20} \bar{X}_i = 80$$

$$\sum_{i=1}^{20} \bar{X}^2{}_i = 335$$

The mean of $\overline{X}_i = \mu_{\bar{x}} = \frac{\sum_{i=1}^{20} \bar{X}_i}{k} = \frac{\bar{x}_1 + \bar{x}_2 + \ldots\ldots\ldots\ldots + \bar{x}_{20}}{k} = \frac{80}{20} = 4$

$\therefore \mu_{\bar{x}} = \mu_x$ **, In this case the statistic is an unbiased estimator of the parameter**

The Variance of $\overline{X}_i =$

$$\sigma_{\bar{x}}^{2}=\frac{\sum_{i=1}^{k}\overline{X}_{i}^{2}}{k}-\mu_{\bar{x}}^{2}=\frac{335}{20}-(4)^{2}=16.75-16=0.75$$

The Standard deviation of $\overline{X}_i$ **=**

$$\sigma_{\bar{x}}=\sqrt{\frac{\sum_{i=1}^{k}\overline{X}_{i}^{2}}{k}-\mu_{\bar{x}}^{2}}=\sqrt{\frac{335}{20}-(4)^{2}}=\sqrt{16.75-16}=\sqrt{0.75}=0.87$$

$$\therefore\sigma_{\bar{x}}^{2}=\frac{\sigma_{x}^{2}}{n}\left(\frac{N-n}{N-1}\right)=\frac{2}{2}\left(\frac{5-2}{5-1}\right)=(1)\left(\frac{3}{4}\right)=0.75$$

$$\therefore\sigma_{\bar{x}}=\frac{\sigma_{x}}{\sqrt{n}}\sqrt{\left(\frac{N-n}{N-1}\right)}=\frac{1.414}{\sqrt{2}}\sqrt{\left(\frac{5-2}{5-1}\right)}=\frac{1.414}{1.414}\sqrt{\left(\frac{3}{4}\right)}=\sqrt{0.75}=0.87$$

Case (2-2):

Without replacement and the order of selected objects is not important:

1- The Number of samples

$$K={}^{N}C_{n}=\frac{N!}{(N-n)!\,n!}.$$

$$=\frac{N(N-1)(N-2)(N-3)..\qquad(N-n+1)........}{n!}={}_{5}C_{2}=\frac{5!}{(5-2)!\,2!}=10$$

2. The probability of selecting any of the possible simple random samples $=P=\frac{1}{{}_{N}C_{n}}=\frac{1}{10}$

The possible samples

3. The possible samples

The mean & variance

Sample number (i)	**Samples**	$\bar{X}$	$\bar{X}_i^2$
1	**6,3**	**4.5**	**20.25**
2	**6,5**	**5.5**	**30.25**
3	**6,4**	**5**	**25**
4	**6,2**	**4**	**16**
5	**3,5**	**4**	**16**
6	**3,4**	**3.5**	**12.25**
7	**3,2**	**2.5**	**6.25**
8	**5, 4**	**4.5**	**20.25**
9	**5,2**	**3.5**	**12.25**
10	**4,2**	**3**	**9**
		$\sum \bar{X} = 40$	$\sum \bar{X}^2 = 167.5$

$$\sum_{i=1}^{10} \bar{X}_i = 40 \qquad \sum_{i=1}^{10} \bar{X}^2_i = 167.5$$

The mean of $\bar{X}_i = \mu_{\bar{x}} = \frac{\sum_{i=1}^{10} \bar{X}_i}{k} = \frac{\bar{x}_1 + \bar{x}_2 . + \ldots\ldots\ldots\ldots + \bar{x}_{10} ..}{k} = \frac{40}{10} = 4$

$\therefore \mu_{\bar{x}} = \mu_x$ **, In this case the statistic is an unbiased estimator of the parameter**

The Variance of $\bar{X}_i$ **=**

$$\sigma_{\bar{x}}^2 = \frac{\sum_{i=1}^{k} \bar{X}_i^2}{k} - \mu_{\bar{x}}^2 = \frac{167.5}{10} - (4)^2 = 16.75 - 16 = 0.75$$

The Standard deviation of $\bar{X}_i$ **=**

$$\sigma_{\bar{x}} = \sqrt{\frac{\sum_{i=1}^{k} \bar{X}_i^2}{k} - \mu_{\bar{x}}^2} = \sqrt{\frac{167.5}{10} - (4)^2} = \sqrt{16.75 - 16} = \sqrt{0.75} = 0.87$$

$$\therefore \sigma_{\bar{x}}^2 = \frac{\sigma_x^2}{n}\left(\frac{N-n}{N-1}\right) = \frac{2}{2}\left(\frac{5-2}{5-1}\right) = (1)\left(\frac{3}{4}\right) = 0.75$$

$$\therefore \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}\sqrt{\left(\frac{N-n}{N-1}\right)} = \frac{1.414}{\sqrt{2}}\sqrt{\left(\frac{5-2}{5-1}\right)} = \frac{1.414}{1.414}\sqrt{\left(\frac{3}{4}\right)} = \sqrt{0.75} = 0.87$$

Conclusion: For case of sampling without replacement, the sample mean is a random variable with mean and variance given by:

$$\therefore \mu_{\bar{x}} = \mu_x$$

$$\sigma_{\bar{x}}^{2} = \frac{\sigma_x^2}{n}\left(\frac{N-n}{N-1}\right) \qquad \textbf{and} \qquad \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}\sqrt{\left(\frac{N-n}{N-1}\right)}$$

Example (4)

Random sample of size n were selected from populations with the means and variance given here. Find the number of samples, the mean and standard deviation of the sampling distribution of the sample mean in each case:

a. $N=6 \quad n=3 \quad \mu=10 \quad \sigma^2=9 \quad (with\ replacement)$

b.

$N=7 \quad n=2 \quad \mu=5 \quad \sigma^2=4 \quad (without\ replacement\ and\ the\ order\ is\ important)$

c.

$N=8 \quad n=3 \quad \mu=120 \quad \sigma^2=16 \quad (without\ replacement\ and\ the\ order\ is\ not\ important)$

Solution:

a.

The number of samples $=k=N^n=6^3=216$

$$\mu_{\bar{x}}=\mu=10$$

$$\sigma_{\bar{x}}^2=\frac{\sigma^2}{n}=\frac{9}{3}=3$$

$$\sigma_{\bar{x}}=\sqrt{\sigma_{\bar{x}}^2}=\sqrt{3}=1.73$$

or

$$\sigma_{\bar{x}}=\frac{\sigma}{\sqrt{n}}=\frac{3}{\sqrt{3}}=\frac{3}{1.73}=1.73$$

b.

The number of samples $=k={}_NP_n={}_7P_2=7\times6=42$

$$\mu_{\bar{x}}=\mu=5$$

$$\sigma_{\bar{x}}^2=\frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)=\frac{4}{2}\left(\frac{7-2}{7-1}\right)=2\left(\frac{5}{6}\right)=\frac{10}{6}=1.67$$

$$\sigma_{\bar{x}}=\sqrt{\sigma_{\bar{x}}^2}=\sqrt{1.67}=1.29$$

or

$$\sigma_{\bar{x}}=\frac{\sigma}{\sqrt{n}}\left(\sqrt{\frac{N-n}{N-1}}\right)=\frac{2}{\sqrt{2}}\left(\sqrt{\frac{7-2}{7-1}}\right)=\frac{2}{1.4142}\left(\sqrt{\frac{5}{6}}\right)=1.4142\left(\sqrt{0.83}\right)=(1.4142)(0.91)=1.29$$

c.

The number of samples $=k={}_NC_n={}_8C_3=\frac{8\times7\times6}{3\times2\times1}=56$

$$\mu_{\bar{x}}=\mu=120$$

$$\sigma_{\bar{x}}^2=\frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)=\frac{16}{3}\left(\frac{8-3}{8-1}\right)=5.33\left(\frac{5}{7}\right)=5.33(0.71)=3.81$$

$$\sigma_{\bar{x}}=\sqrt{\sigma_{\bar{x}}^2}=\sqrt{3.81}=1.95$$

or

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\left(\sqrt{\frac{N-n}{N-1}}\right) = \frac{4}{\sqrt{3}}\left(\sqrt{\frac{8-3}{8-1}}\right) = \frac{4}{1.73}\left(\sqrt{\frac{5}{7}}\right) = 2.312\left(\sqrt{0.71}\right) = (2.312)(0.84) = 1.95$$

Example (5)

If $N = 6 \quad , \ n = 2$

Find

a. Number of samples and the probability of selecting any of the possible simple random samples in case of drawing with replacement.

b. Number of samples and the probability of selecting any of the possible simple random samples in case of drawing without replacement, order are important.

c. Number of samples and the probability of selecting any of the possible simple random samples in case of drawing without replacement, order are not important.

Solution

a.

$$k = N^n = 6^2 = 36$$

$$P = \frac{1}{N^n} = \frac{1}{36}$$

b.

$$k = {}_N P_n = \frac{N!}{(N-n)!} = N(N-1) = {}_6 P_2 = 6 \times 5 = 30$$

$$P = \frac{1}{{}_N P_n} = \frac{1}{30}$$

c.

$$k = {}_N C_n = \frac{N!}{n!(N-n)!} = \frac{N(N-1)}{n!} = {}_6 C_2 = \frac{6!}{2!(6-2)!} = \frac{6 \times 5}{2 \times 1} = 15$$

Theorem1:

If the population follows a normal probability distribution, then for any sample size the sampling distribution of the sample mean will be also normal.

Suppose that $X \sim N\left(\mu, \sigma_X^2\right)$ **and let** $\overline{X}$ **be the mean of a sample of size** n **then** $\overline{X} \sim N\left(\mu, \sigma_{\overline{X}}^2\right)$ **where;**

$$\sigma_{\overline{X}}^2 = \begin{cases} \dfrac{\sigma_X^2}{n}; & \text{in case of replacment} \\ \\ \dfrac{\sigma_X^2}{n}\left(\dfrac{N-n}{N-1}\right); & \text{in case of without replacment} \end{cases}$$

Theorem2 (The Central limit Theorem):

If all sample of a particular size are selected from any population, the sampling distribution of the sample mean is approximately a normal distribution. This approximation improves with larger sample ($n \geq 30$).

For a population with a mean μ and a variance σ^2 the sampling distribution of the means of all possible samples of size n generated from the population will be approximately normally distributed with mean μ and standard deviation $\sigma / \sqrt{n}$ as the sample size n becomes larger **(**$n \geq 30$**).**

Example (6)

Suppose that the random variable X represent the IQ (Intelligence quotient) score for students at certain university, and that $X \sim N(100, 75^2)$. A random sample of size 25 students is selected. Find the following:

1. Probability that the mean IQ score computed from the sample will be grater than 125.
2. Probability that the mean IQ score computed from the sample will be less than 80.
3. Probability that the mean IQ score computed from the sample will be between 70 and 130.

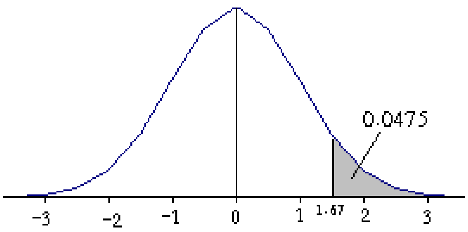
Solution:

1. Note that $\bar{X} \sim N(100, 225)$

   $\mu = 100$, $\sigma^2_{\bar{x}} = \dfrac{75^2}{25} = \dfrac{5625}{25} = 225$

   $\sigma_{\bar{x}} = \sqrt{225} = 15$

   $P(\bar{X} > 125) = P\left(Z > \dfrac{125-100}{15}\right) = 0.0475$



2. $P(\bar{X} < 80) = P\left(Z < \dfrac{80-100}{15}\right) = \Phi(-1.33) = 0.0918$

3. $P(70 < \bar{X} < 130) = P\left(\dfrac{70-100}{15} < Z < \dfrac{130-100}{15}\right) = 0.9546$