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ON THE ASYMPTOTIC INDEPENDENT REPRESENTATIONS FOR SUMS OF SOME WEAKLY DEPENDENT RANDOM VARIABLES

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Abstract

Let, for each $n \in \mathbb{N}$, $(X_{i,n})_{0 \leqq i \leqq n}$ be a triangular array of stationary, centered, square integrable and associated real valued random variables satisfying the weakly dependence condition

$$\lim_{N \to N_0} \limsup_{n \to +\infty} n \sum_{r=N}^{n} \operatorname{Cov}\left(X_{0,n}, X_{r,n}\right) = 0;$$

where N_0 is either infinite or the first positive integer N for which the limit of the sum $n \sum_{r=N}^{n} \operatorname{Cov}\left(X_{0,n}, X_{r,n}\right)$ vanishes as n goes to infinity. The purpose of this paper is to build, from $(X_{i,n})_{0 \leqq i \leqq n}$, a sequence of independent random variables $(\tilde{X}_{i,n})_{0 \leqq i \leqq n}$ such that the two sums $\sum_{i=1}^{n} X_{i,n}$ and $\sum_{i=1}^{n} \tilde{X}_{i,n}$ have the same asymptotic limiting behavior (in distribution).

1. Introduction

Let, for $n \in \mathbb{N}$, $(X_{i,n})_{0 \leqq i \leqq n}$ be a triangular array of row-wise stationary, centered and square integrable real valued random variables. Put $S_n = \sum_{i=1}^{n} X_{i,n}$. Our main task in this paper is to find a sequence $(\tilde{X}_{i,n})$ of

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independent random variables such that, for any real number t,

$$\lim_{n\to+\infty} \left|\mathbb{E}(e^{itS_n}) - \mathbb{E}(e^{it\tilde{S}_n})\right| = 0,$$

with $\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_{i,n}$. Such a sequence $(\tilde{X}_{i,n})$ will be called in Jakubowski's terminology (cf. Jakubowski [6]) *an asymptotic independent representation* (a.i.r.) for sums of $(X_{i,n})$.

We suppose that the sequence is of associated random variables. Recall that the vector $X = (X_1, X_2, \ldots, X_n)$ is *associated* if for all non-decreasing functions f, g defined on $\mathbb{R}^n$

$$\operatorname{Cov}\big(f(X), g(X)\big) \geqq 0, \tag{1}$$

we refer the reader to Esary *et al.* for this definition and for its main properties. For such sequences, the well known Newman's inequality [cf. Newman [8]] states

$$\left|\mathbb{E}\big(\exp(izS_n)\big) - \prod_{j=1}^{n} \big(\mathbb{E}\exp(izX_{j,n})\big)\right| \leqq \frac{z^2}{2}(\operatorname{Var} S_n - n \operatorname{Var} X_{0,n}). \tag{2}$$

Hence, under the Condition

$$\lim_{n\to+\infty} n \sum_{r=1}^{n} \operatorname{Cov}(X_{0,n}, X_{r,n}) = 0, \tag{3}$$

the sum S_n behaves as $\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_{i,n}$, where the sequence $(\tilde{X}_{i,n})$ is the independent version of $(X_{i,n})$, that means that $(\tilde{X}_{i,n})$ are i.i.d. and distributed like $(X_{0,n})$.

Let $R(N)$ denotes the like Cox–Grimmett coefficient defined by:

$$R(N) = \limsup_{n\to+\infty}\ n \sum_{r=N}^{n} \operatorname{Cov}(X_{0,n}, X_{r,n}),$$

and let $N_0 = \inf\big\{N,\ R(N) = 0\big\}$ (N_0 may be infinite). Finally, we denote by the limit as N goes to N_0 of a function $f(N)$, its value at N_0, which is $f(N_0)$ if N_0 is finite or simply the limit as N goes to infinity if N_0 is infinite.

We suppose now, instead of (3), that

$$\lim_{N\to N_0} \limsup_{n\to+\infty} n \sum_{r=N}^{n} \operatorname{Cov}(X_{0,n}, X_{r,n}) = 0, \qquad \lim_{n\to+\infty} \operatorname{Var} S_n =: \sigma^2, \tag{4}$$

for a finite positive real number σ^2. Our task is to describe the asymptotic independent representations of S_n under this Condition (4). For this, we need some notations. Let $\eta > 0$ be fixed and consider the two continuous and 1-Lipschitz decoupling functions f_η and $f^{(\eta)}$ defined respectively by $f_\eta(x) = (\eta \wedge x) \vee (-\eta)$ and $f^{(\eta)}(x) = x - f_\eta(x)$. Set

$$X_{i,n}(\eta) = f_\eta(X_{i,n}) - \mathbb{E}\big(f_\eta(X_{i,n})\big), \quad X_{i,n}^{(\eta)} = X_{i,n} - X_{i,n}(\eta),$$

and let, $S_n^{(\eta)}$ and $S_n(\eta)$ be respectively the sums

$$S_n^{(\eta)} = \sum_{i=1}^{n} X_{i,n}^{(\eta)}, \quad S_n(\eta) = \sum_{i=1}^{n} X_{i,n}(\eta).$$

Clearly $S_n = S_n^{(\eta)} + S_n(\eta)$. Finally, for any $t \in [0,1]$, we denote by $S_n(t) = \sum_{i=1}^{[nt]} X_{i,n}$, where $[.]$ denotes the integer part as usual.

The paper is organized as follows. In the following section, we give our main result [cf. Theorem 1 below]. Theorem 1 is followed by three remarks. In remark 1, we comment the case when $N_0 = 1$. In remark 2, we explain how we can deduce, using Theorem 1, a central limit theorem for triangular arrays of sums of associated random variables from the analogous result already well known in the independent setting. In remark 3, we discuss a case of associated and m-dependent random variables. Section three is dedicated to the proofs of our results.

2. Results

Let $\mathcal{F}_C$ be the class of three-times continuously differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) = f'(0) = 0$ and $\|f''\|_\infty \leqq C$, $\|f'''\|_\infty \leqq C$ and let $\mathcal{F}_{0,C}$ be a set of functions belonging to $\mathcal{F}_C$ and fulfilling moreover $f''(0) = 0$. Finally, we denote by $\tilde{\mathcal{F}}_{0,C}$ the subset of $\mathcal{F}_{0,C}$ defined by

$$\tilde{\mathcal{F}}_{0,C} = \left\{ x \to \cos(zx) - 1 + \frac{z^2x^2}{2},\ x \to \sin(zx) - zx,\ \text{for } z \in \mathbb{R} \right\} \tag{5}$$

Our main result is the following.

THEOREM 1. *Let $(X_{i,n})$ be a triangular array of row-wise stationary, centered and associated square integrable real valued random variables, such that* $\sup_{n\in\mathbb{N}} n\mathbb{E}(X_{0,n}^2) < \infty$. *Suppose that* (4) *holds. Let $(\tilde{X}_{i,n})$ be a sequence*

of centered and i.i.d. random variable for which, letting for positive integer N, $S_{N,n}^{(\eta)} = \sum_{i=1}^{N} X_{i,n}^{(\eta)}$, *one has*

(6)
$$\lim_{N\to N_0} \limsup_{\eta\to 0} \limsup_{n\to+\infty} \sup_{h\in\tilde{\mathcal{F}}_{0,C}} n\Big|\mathbb{E}h\big(\tilde{X}_{0,n}^{(\eta)}\big) - \mathbb{E}\big[h\big(S_{N,n}^{(\eta)}\big) - h\big(S_{N-1,n}^{(\eta)}\big)\big]\Big| = 0,$$

and

$$\lim_{n\to+\infty} n \operatorname{Var} \tilde{X}_{0,n} = \sigma^2. \tag{7}$$

Then $(\tilde{X}_{i,n})$ *is an a.i.r. for sums of* $(X_{i,n})$

Remark 1. Suppose that $N_0 = 1$, that means that Condition (3) holds. Suppose moreover that there exists a sequence of centered and i.i.d. random variables $(\tilde{X}_{i,n})$ for which

$$\lim_{\eta\to 0} \limsup_{n\to+\infty} \sup_{h\in\mathcal{F}_{0,C}} n\Big|\mathbb{E}h\big(\tilde{X}_{0,n}^{(\eta)}\big) - \mathbb{E}h\big(X_{0,n}^{(\eta)}\big)\Big| = 0, \tag{8}$$

and

$$\lim_{n\to+\infty} \big|\, n \operatorname{Var} \tilde{X}_{0,n} - n \operatorname{Var} X_{0,n}\big| = 0. \tag{9}$$

Then, we deduce from the conclusion of Theorem 1 that $(\tilde{X}_{i,n})$ is an a.i.r. for sums of $(X_{i,n})$. Now, a sequence $(\tilde{X}_{i,n})$ of i.i.d. random variables fulfilling (8) and (9) exists: it remains to take a sequence of i.i.d. random variables distributed as $X_{0,n}$. This result is also deduced using Newman's inequality stated in (2).

Remark 2. Let us now explain how we can deduce a central limit theorem for S_n. Let $(\tilde{X}_{i,n})_{i\in\mathbb{N},\, n\in\mathbb{N}}$ be a row-wise stationary array of i.i.d. square-integrable and centered random variables. Suppose that $n\mathbb{E}(\tilde{X}_{0,n}^2)$ converges to some positive real number σ^2 as n tends to infinity. If moreover

$$\lim_{n\to+\infty} n\mathbb{E}\big(\tilde{X}_{0,n}^2 \mathbb{1}_{|\tilde{X}_{0,n}|\geqq\varepsilon}\big) = 0 \quad \text{for all} \quad \varepsilon > 0, \tag{10}$$

then we know that $\tilde{S}_n$ converges in distribution to a centered normal law with variance σ^2 (cf. for instance Billingsley [1]). If this sequence $(\tilde{X}_{i,n})_{i\in\mathbb{N},\, n\in\mathbb{N}}$ fulfills moreover Conditions (6) and (7) of Theorem 1, then we deduce from

the conclusion of Theorem 1 that S_n converges also in distribution to a centered normal law with variance σ^2.

So our task is to give sufficient conditions on the sequence $(X_{i,n})_{i\in\mathbb{N},\ n\in\mathbb{N}}$ under which Conditions (6) and (7) of Theorem 1 are satisfied. Clearly Condition (7) is satisfied as soon as $\operatorname{Var} S_n$ converges to σ^2 as n goes to infinity. Now, we prove that Condition (6) is satisfied as soon as

$$\lim_{n\to+\infty} n\mathbb{E}\big(X_{0,n}^2 \mathbb{I}_{|X_{0,n}|\geqq\varepsilon}\big) = 0 \quad \text{for all} \quad \varepsilon > 0. \tag{11}$$

In fact, we deduce since $h \in \mathcal{F}_{0,C}$ that for any random variable X with finite variance,

$$\big|\mathbb{E}\, h(X)\big| \leqq C\,\mathbb{E}(X^2).$$

The last constatation together with the following inequality (due to the property of association),

$$\mathbb{E}\big(S_{N-1,n}^{(\eta)}\big)^2 \leqq \mathbb{E}\big(S_{N,n}^{(\eta)}\big)^2,$$

gives

$$\sup_{h\in\mathcal{F}_{0,C}} n\Big|\mathbb{E}h\big(\tilde{X}_{0,n}^{(\eta)}\big) - \mathbb{E}\big[h\big(S_{N,n}^{(\eta)}\big) - h\big(S_{N-1,n}^{(\eta)}\big)\big]\Big| \leqq C\Big(n\mathbb{E}\big(\tilde{X}_{0,n}^{(\eta)}\big)^2 + n\mathbb{E}\big(S_{N,n}^{(\eta)}\big)^2\Big). \tag{12}$$

We have

$$n\mathbb{E}\big(\tilde{X}_{0,n}^{(\eta)}\big)^2 \leqq n\mathbb{E}\big(\tilde{X}_{0,n}^2 \mathbb{I}_{|\tilde{X}_{0,n}|\geqq\eta}\big)$$

$$n\mathbb{E}\big(S_{N,n}^{(\eta)}\big)^2 \leqq N^2 n\mathbb{E}\big(X_{0,n}^{(\eta)}\big)^2 \leqq N^2 n\mathbb{E}\big(X_{0,n}^2 \mathbb{I}_{|X_{0,n}|\geqq\eta}\big).$$

The two last inequalities together with (12) prove that Condition (6) is true as soon as the limits in (10) and (11) are satisfied. We summarize this remark in the following corollary.

Corollary 1. *Let $(X_{i,n})_{i\in\mathbb{N},\ n\in\mathbb{N}}$ be a row-wise stationary array of associated square-integrable and centered random variables. Suppose that $\mathbb{E}(X_{0,n}^2)$ tends to zero as n tends to infinity and that Conditions* (4) *and* (11) *are satisfied with $\sigma^2 = 1$. Then S_n converges in distribution to the standard normal law.*

For stationary and centered associated sequence with $\sum_{r=1}^{\infty} \mathrm{Cov}\,(X_0, X_r) < +\infty$, Corollary 1 applied to the arrays $X_{i,n} = X_i/\sqrt{\mathrm{Var}\, S_n}$ yields the CLT theorem already proved by Newman and Wright [9].

We prove in Dedecker and Louhichi [3] that the conditions stated in Corollary 1 ensure the convergence of the process $\{S_n(t),\, t \in [0,1]\}$ in distribution and in $\big(D\big([0,1]\big), d\big)$ to the standard Wiener process.

Remark 3. We suppose now that $X_{i,n} = \frac{X_i}{B_n}$ where (X_i) is a centered sequence of stationary and associated random variables having finite second moment and $B_n = \sqrt{\mathrm{Var}\,(X_1 + \ldots + X_n)}$. We suppose moreover that this sequence is of m-dependent random variables, fulfilling (4) and

$$\lim_{\eta \to 0} \limsup_{n \to +\infty} n\mathbb{E}\big(X_{0,n}(\eta)\big)^2 = 0. \tag{13}$$

The following corollary gives sufficient conditions for the existence of an i.i.d. sequence $\tilde{X}_{i,n}$ as described in Theorem 1.

Corollary 2. *Let* (X_i) *be a sequence of* m*-dependent associated random variables as defined in Remark 3, fulfilling* (13) *and* (4). *Let* $(\tilde{X}_{i,n} = \frac{\tilde{X}_i}{B_n})$, *where* $\tilde{X}_i$ *is a sequence of centered and i.i.d. random variables, for which*

$$\lim_{\eta \to 0} \limsup_{n \to +\infty} n\mathbb{E}\big(\tilde{X}_{0,n}(\eta)\big)^2 = 0. \tag{14}$$

If the characteristic function $\mathbb{E}(e^{iz\tilde{X}_1})$ *of* $\tilde{X}_1$ *fulfills*

$$\mathbb{E}\big(e^{iz\tilde{X}_1}\big) = \frac{\mathbb{E}\Big(e^{iz(X_1+\ldots+X_{m+1})}\Big)}{\mathbb{E}\Big(e^{iz(X_1+\ldots+X_m)}\Big)}, \tag{15}$$

then Conditions (6) *and* (7) *are satisfied and the conclusion of Theorem 1 applies.*

We prove Corollary 2 at the end of the paper. Let us note that the fact that the ratio

$$\frac{\mathbb{E}\Big(e^{iz(X_1+\ldots+X_{m+1})}\Big)}{\mathbb{E}\Big(e^{iz(X_1+\ldots+X_m)}\Big)} \tag{16}$$

is a characteristic function of some random variable is not necessarily true ; we refer the reader to Remark 3.1 in Harrelson and Houdré [5].

Let us comment Condition (13). This Condition (respectively (14)) is true as soon as $X_{0,n}$ (respectively $\tilde{X}_{0,n}$) is in the domain of attraction of an

infinitely divisible distribution with non-Gaussian part. In fact, let μ_F be an infinitely divisible distribution with characteristic function given by:

$$\hat{\mu}_F(z) = \exp\left(\int (e^{izx} - 1 - izx)\frac{1}{x^2}dF(x)\right), \quad \text{for any} \quad z \in \mathbb{R}, \tag{17}$$

where F is the distribution function of some finite measure (i.e. F is bounded, nondecreasing, $F(-\infty) = 0$ and $F(\infty)$ is finite). We know from Theorem 2 of Chapter 4 in Gnedenko and Kolmogorov [4] that if $(X_{j,n})_{1\leqq j\leqq n}$ is a triangular array of i.i.d. centered random variables such that $\mathbb{E}(X_{0,n}^2)$ converges to zero as n tends to infinity, then $S_n = X_{1,n} + \cdots + X_{n,n}$ converges in distribution to μ_F if and only if, for any continuity point x of F,

$$\lim_{n\to\infty} n\mathbb{E}(X_{0,n}^2 \mathbb{1}_{X_{0,n}\leqq x}) = F(x).$$

The last limit together with the continuity of F at 0 ensures Condition (13) (respectively (14)) as soon as $X_{0,n}$ (respectively $\tilde{X}_{0,n}$) is in the domain of attraction of μ_F. Now the continuity of F at 0 is equivalent to the fact that there is non-Gaussian part in (17).

Corollary 2 is analogous to Theorem 3.1 in Harrelson and Houdré [5] stated in other context of dependence. In fact, Theorem 3.1 in Harrelson and Houdré [5] gives example of m-dependent and stationary infinitely divisible, purely non-Gaussian sequence $(X_j)_{j\in\mathbb{Z}}$ for which the ratio (16) is the characteristic function of some random variable $\tilde{X}$; they proved moreover that the i.i.d. sequence $(\tilde{X}_i)$, having the common law $\tilde{X}$, is an a.i.r. for the sums $\sum_{i=1}^n X_i$ suitably centered and normalized.

3. Proofs

3.1. Some auxiliary results

We first recall the following lemma, already discussed in Dedecker and Louhichi [3], which is an analogous of Lemma 3.2 in Jakubowski [7].

Lemma 1. *Let $(X_{i,n})$ be a triangular array of row-wise stationary and associated real valued random variables. Let h be a fixed function of the set $\mathcal{F}_C$. Then, for any positive integers p, N and positive real number η, there*

holds

$$\begin{aligned}(18)\quad &\Big|p\mathbb{E}h\big(S_n^{(\eta)}(1/p)\big)-p[n/p]\mathbb{E}\big(h(S_{N,n}^{(\eta)})-h(S_{N-1,n}^{(\eta)})\big)\Big|\\ &\qquad\leqq C\Big\{Np\mathbb{E}\big(S_{N,n}^{(\eta)}\big)^2+Cn\sum_{k=N}^{n}\operatorname{Cov}\big(X_{0,n}^{(\eta)},X_{k,n}^{(\eta)}\big)\\ &\qquad\qquad+\big(\operatorname{Var}S_{[n/p]}\big)^{1/2}\frac{2n}{\eta}\mathbb{E}X_{0,n}^2\Big\}.\end{aligned}$$

From Lemma 1, we deduce the following corollary.

Corollary 3. *Suppose that all the requirements of Lemma 1 are satisfied. Suppose moreover that Condition* (4) *is satisfied. If* $\mathbb{E}(X_{0,n})^2$ *tends to* 0 *as* n *goes to infinity, then for any* $\eta>0$ *there holds*

$$\lim_{N\to N_0}\limsup_{p\to+\infty}\limsup_{n\to+\infty}\Big|p\mathbb{E}h\big(S_n^{(\eta)}(1/p)\big)-n\mathbb{E}\big(h\big(S_{N,n}^{(\eta)}\big)-h\big(S_{N-1,n}^{(\eta)}\big)\big)\Big|=0.$$

Proof of Corollary 3. We deduce, since the random variables are associated, that:

$$\begin{aligned}(19)\qquad &\operatorname{Var}S_{[n/p]}\leqq\frac{\operatorname{Var}(S_n)}{p},\\ &0\leqq\operatorname{Cov}\big(X_{0,n}^{(\eta)},X_{k,n}^{(\eta)}\big)\leqq\operatorname{Cov}(X_{0,n},X_{k,n}),\\ &\mathbb{E}\big(S_{N,n}^{(\eta)}\big)^2\leqq\operatorname{Var}(S_{N,n})\leqq N^2\mathbb{E}X_{0,n}^2.\end{aligned}$$

Those bounds together with Lemma 1, Condition (4) and the asymptotic negligibility of $\mathbb{E}(X_{0,n})^2$ ensure, for any fixed $\eta>0$,

$$\begin{aligned}(20)\quad \lim_{N\to N_0}\limsup_{p\to+\infty}\limsup_{n\to+\infty}\Big|&p\mathbb{E}h\big(S_n^{(\eta)}(1/p)\big)-p\left[\frac{n}{p}\right]\mathbb{E}\Big(h\big(S_{N,n}^{(\eta)}\big)\\ &-h\big(S_{N-1,n}^{(\eta)}\big)\Big)\Big|=0.\end{aligned}$$

We have, since the function h belongs to the set $\mathcal{F}_C$,:

$$\big|h(x)\big|\leqq\frac{C}{2}x^2,$$

The last bound, together with (19), gives

$$\left|\left(n - p\left[\frac{n}{p}\right]\right)\mathbb{E}\Big(h\big(S_{N,n}^{(\eta)}\big) - h\big(S_{N-1,n}^{(\eta)}\big)\Big)\right| \leqq pCN^2\mathbb{E}(X_{0,n})^2.$$

Corollary 3 is proved using the last inequality, the asymptotic negligibility of $\mathbb{E}(X_{0,n}^2)$ and the limit in (20). □

We also need the following lemmas, which are proved in Dedecker and Louhichi [3].

Lemma 2. *Let $(X_{i,n})$ be a triangular array of row-wise stationary and centered real valued random variables. Suppose that $\sup_{n\in\mathbb{N}}\mathbb{E}(X_{0,n}^2) < +\infty$. Let h be a fixed function of the set $\mathcal{F}_C$. Then, for any positive integer p and positive real numbers ε and η, there holds*

$$\begin{aligned}&\left|p\mathbb{E}h\big(S_n(1/p)\big) - p\mathbb{E}h\big(S_n^{(\eta)}(1/p)\big) - \frac{h''(0)}{2}p\Big(\operatorname{Var}S_n(1/p) - \operatorname{Var}S_n^{(\eta)}(1/p)\Big)\right|\\&\leqq 2Cp\mathbb{E}\big(S_{n,\eta}^2(1/p)\mathbb{1}_{|S_n(\eta)(1/p)|\geqq\varepsilon}\big) + 3C\varepsilon\big(V_{n,\eta}(p) + V_n^{(\eta)}(p)\big)\\&+ 2C\Big(p\mathbb{E}\big(S_{n,\eta}^2(1/p)\mathbb{1}_{|S_n(\eta)(1/p)|\geqq\varepsilon}\big)\Big)^{1/2}\big(V_n^{(\eta)}(p)\big)^{1/2},\end{aligned}$$

where $V_{n,\eta}(p) = p\operatorname{Var}S_n(\eta)(1/p)$ and $V_n^{(\eta)}(p) = p\operatorname{Var}S_n^{(\eta)}(1/p)$.

Lemma 3. *Let $(X_{i,n})_{i\in\mathbb{N},\ n\in\mathbb{N}}$ be an array of row-wise stationary centered and associated real valued random variables, with finite second moment. If Conditions (4) is satisfied, then for any $\varepsilon > 0$,*

$$\lim_{\eta\to0}\limsup_{p\to+\infty}\limsup_{n\to+\infty} p\mathbb{E}\big(S_{n,\eta}^2(1/p)\mathbb{1}_{|S_n(\eta)(1/p)|\geqq2\varepsilon}\big) = 0. \tag{21}$$

The limit in (21) remains true if instead of $\lim_{\eta\to0}\limsup_{p\to+\infty}$, we write $\lim_{p\to+\infty}\limsup_{\eta\to0}$.

We obtain combining Lemma 2 and Lemma 3 the following corollary.

Corollary 4. *Let $(X_{i,n})$ be a triangular array of row-wise stationary, centered and associated real valued random variables fulfilling (4). Suppose that $\sup_{n\in\mathbb{N}}\mathbb{E}(X_{0,n}^2) < +\infty$. Let h be a fixed function of the set $\mathcal{F}_C$ such that $h''(0) = 0$. Then*

$$\lim_{\eta\to0}\limsup_{p\to+\infty}\limsup_{n\to+\infty}\left|p\mathbb{E}h\big(S_n(1/p)\big) - p\mathbb{E}h\big(S_n^{(\eta)}(1/p)\big)\right| = 0.$$

We are now in position to prove Theorem 1.

3.2. Proof of Theorem 1

The use of Newman's inequality (cf. Inequality (2)), together with some elementary estimations, ensures for any $z \in \mathbb{R}$:

$$\begin{aligned}(22)\quad &\Big|\mathbb{E}(e^{izS_n}) - \big\{\mathbb{E}\big(e^{izS_n(1/p)}\big)\big\}^p\Big| \\ &\qquad\leqq |z|p\mathbb{E}|X_{0,n}| + \frac{z^2}{2}\big(\operatorname{Var} S_n - p\operatorname{Var} S_n(1/p)\big).\end{aligned}$$

Hence the asymptotic negligibility of $\mathbb{E}|X_{0,n}|$ together with the second limit in (4) leads to

$$(23)\qquad \lim_{p\to+\infty}\limsup_{n\to+\infty}\Big|\mathbb{E}(e^{izS_n}) - \big\{\mathbb{E}\big(e^{izS_n(1/p)}\big)\big\}^p\Big| = 0,$$

we refer the reader to the proof of Lemma 11 in Dedecker and Louhichi [3] for further details. Now the fact

$$(24)\quad |x_1\dots x_m - y_1\dots y_m| \leqq \sum_{i=1}^{m}|x_i - y_i|, \quad \text{for} \quad x_i, y_i \in \mathbb{C}, \quad |x_i|,\ |y_i| \leqq 1,$$

and some elementary estimations, yields

$$\begin{aligned}(25)\qquad &\Big|\big\{\mathbb{E}\big(e^{izS_n(1/p)}\big)\big\}^p - \big\{\mathbb{E}\big(e^{iz\tilde{S}_n(1/p)}\big)\big\}^p\Big| \\ &\leqq p\Big|\mathbb{E}\big(e^{izS_n(1/p)}\big) - \mathbb{E}\big(e^{iz\tilde{S}_n(1/p)}\big)\Big| \\ &\leqq 2\sup_{h\in\tilde{\mathcal{F}}_{0,C}}\Big|p\mathbb{E}h\big(S_n(1/p)\big) - p\mathbb{E}h\big(\tilde{S}_n(1/p)\big)\Big| \\ &+ \frac{z^2}{2}\left|p\operatorname{Var} S_n(1/p) - p\left[\frac{n}{p}\right]\operatorname{Var}\tilde{X}_{0,n}\right|,\end{aligned}$$

here C is a positive constant depending on z and $\tilde{\mathcal{F}}_{0,C}$ is the set of functions defined by (5).

Let us evaluate the first term on the right hand side of Inequality (25). For this let h be a fixed function of the set $\tilde{\mathcal{F}}_{0,C}$. Clearly

$$\begin{aligned}
(26)\qquad & \Big|p\mathbb{E}h\big(S_n(1/p)\big) - p\mathbb{E}h\big(\tilde{S}_n(1/p)\big)\Big| \\
&\leqq \Big|p\mathbb{E}h\big(S_n(1/p)\big) - n\mathbb{E}\Big(h(S_{N,n}^{(\eta)}) - h\big(S_{N-1,n}^{(\eta)}\big)\Big)\Big| \\
&+ \Big|p\mathbb{E}h\big(\tilde{S}_n(1/p)\big) - n\mathbb{E}h\big(\tilde{X}_{0,n}^{(\eta)}\big)\Big| \\
&+ \Big|n\mathbb{E}\Big(h\big(S_{N,n}^{(\eta)}\big) - h\big(S_{N-1,n}^{(\eta)}\big)\Big) - n\mathbb{E}h\big(\tilde{X}_{0,n}^{(\eta)}\big)\Big|.
\end{aligned}$$

Corollaries 3 and 4 ensure:

$$\begin{aligned}
(27)\qquad \lim_{N\to N_0}\limsup_{\eta\to 0}\limsup_{p\to+\infty}\limsup_{n\to+\infty}\sup_{h\in\tilde{\mathcal{F}}_{0,C},}\Big|&p\mathbb{E}h\big(S_n(1/p)\big) \\
&- n\mathbb{E}\Big(h\big(S_{N,n}^{(\eta)}\big) - h\big(S_{N-1,n}^{(\eta)}\big)\Big)\Big| = 0.
\end{aligned}$$

The last limit can be applied to independent random variables, with $N_0 = 1$; in fact independent sequences are also associated (cf. Esary *et al.* [2]) and clearly they fulfill the first limit in (4) with $N_0 = 1$. Hence the limit in (27) gives:

$$(28)\qquad \lim_{\eta\to 0}\limsup_{p\to+\infty}\limsup_{n\to+\infty}\sup_{h\in\tilde{\mathcal{F}}_{0,C}}\Big|p\mathbb{E}h\big(\tilde{S}_n(1/p)\big) - n\mathbb{E}h\big(\tilde{X}_{0,n}^{(\eta)}\big)\Big| = 0.$$

Inequality (26), the limits in (27), (28) and Condition (6) of Theorem 1 ensure

$$(29)\qquad \lim_{p\to+\infty}\limsup_{n\to+\infty}\sup_{h\in\tilde{\mathcal{F}}_{0,C}}\Big|p\mathbb{E}h\big(S_n(1/p)\big) - p\mathbb{E}h\big(\tilde{S}_n(1/p)\big)\Big| = 0.$$

The limit in (29), Condition (7), the second limit in (4) and Inequality (25) ensure

$$(30)\qquad \lim_{p\to+\infty}\limsup_{n\to+\infty}\Big|\Big\{\mathbb{E}\Big(e^{izS_n(1/p)}\Big)\Big\}^p - \Big\{\mathbb{E}\Big(e^{iz\tilde{S}_n(1/p)}\Big)\Big\}^p\Big| = 0.$$

We obtain, combining the two limits in (30) and (23)

$$(31)\qquad \lim_{n\to+\infty}\Big|\mathbb{E}(e^{izS_n}) - \mathbb{E}\big(e^{iz\tilde{S}_n}\big)\Big| = 0,$$

from this the proof of Theorem 1 is complete. □

3.3. Proof of Corollary 2

Let $\tilde{X}_1$ be a random variable with characteristic function given by (15). We first deduce from (15) that

$$\operatorname{Var}(\tilde{X}_1) = \operatorname{Var}(X_1 + \ldots + X_{m+1}) - \operatorname{Var}(X_1 + \ldots + X_m).$$

From this relation and from $\lim_{n\to+\infty} \mathbb{E}(X_{0,n}^2) = 0$, we deduce (7) (recall that the sequence $(X_{i,n})_i$ is m-dependent, that $\tilde{X}_{1,n} = \frac{\tilde{X}_1}{B_n}$, $X_{1,n} = \frac{X_1}{B_n}$ and that $B_n = \sqrt{\operatorname{Var}(X_1 + \ldots + X_n)}$).

Our task now is to prove the limit (6). We deduce from Conditions (13) and (14), that it is enough to prove that

$$\lim_{n\to+\infty} \sup_{h\in\tilde{\mathcal{F}}_{0,C}} n\Big|\mathbb{E}h\big(\tilde{X}_{0,n}\big) - \mathbb{E}\big[h(S_{m+1,n}) - h(S_{m,n})\big]\Big| = 0, \tag{32}$$

where $\tilde{\mathcal{F}}_{0,C}$ is the set defined by (5). We deduce from (15) that

$$\mathbb{E}(\cos t\tilde{X}_{0,n}) = \frac{\mathbb{E}(\cos tS_{m+1,n})\mathbb{E}(\cos tS_{m,n}) + \mathbb{E}(\sin tS_{m+1,n})\mathbb{E}(\sin tS_{m,n})}{\big(\mathbb{E}(\cos tS_{m,n})\big)^2 + \big(\mathbb{E}(\sin tS_{m,n})\big)^2}. \tag{33}$$

Let $h_1(x) = \cos(tx) - 1 + \frac{t^2x^2}{2}$. Clearly

$$\begin{aligned}\mathbb{E}\big(h_1(\tilde{X}_{0,n})\big) &- \mathbb{E}\big(h_1(S_{m+1,n}) - h_1(S_{m,n})\big)\\ &= \Big(\mathbb{E}\big(\cos(t\tilde{X}_{0,n})\big) - \mathbb{E}\big(\cos(tS_{m+1,n}) - \cos(tS_{m,n})\big) - 1\Big)\\ &+ \frac{t^2}{2}\Big(\operatorname{Var}\big(\tilde{X}_{0,n}^2\big) + \operatorname{Var}(S_{m,n}) - \operatorname{Var}(S_{m+1,n})\Big).\end{aligned} \tag{34}$$

We have two quantities to control. We first deduce, from (7) and (4) that

$$\lim_{n\to+\infty} n\Big(\operatorname{Var}\big(\tilde{X}_{0,n}^2\big) + \operatorname{Var}(S_{m,n}) - \operatorname{Var}(S_{m+1,n})\Big) = 0. \tag{35}$$

Let us now prove that

$$\lim_{n\to+\infty} n\Big(\mathbb{E}\big(\cos(t\tilde{X}_{0,n})\big) - \mathbb{E}\big(\cos(tS_{m+1,n}) - \cos(tS_{m,n})\big) - 1\Big) = 0. \tag{36}$$

For this, let

$$D_{m,n} = \big(\mathbb{E}(\cos tS_{m,n})\big)^2 + \big(\mathbb{E}(\sin tS_{m,n})\big)^2,$$

we have

$$\lim_{n\to+\infty} D_{m,n} = 1; \tag{37}$$

in fact, we have using some elementary estimations as in (38) below,

$$\begin{aligned} |D_{m,n} - 1| &\leqq t^2\mathbb{E}(S_{m,n}^2) + 2\big|\mathbb{E}\big(\cos(tS_{m,n})\big) - 1\big| \\ &\leqq 2t^2\mathbb{E}(S_{m,n}^2) \\ &\leqq 2t^2m^2\mathbb{E}(X_{0,n}^2), \end{aligned}$$

which tends to 0 from the asymptotic negligibility of $\mathbb{E}(X_{0,n}^2)$. Now the limit in (36) is deduced from (33), (37) and the trivial facts

$$\big|\cos(x) - 1\big| \leqq \frac{x^2}{2}, \quad \big|\sin(x) - x\big| \leqq \frac{x^2}{2}. \tag{38}$$

Hence, we prove (32) by collecting (36), (35) and (34) and by using the same method for the functions $h_2(x) = \sin(zx) - zx$ instead of h_1. □

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