KING SAUD UNIVERSITY DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS: 40, SI /21/04/1437 MATH 204

Question 1. **[4,6]** **a)** Determine the largest region for which the following IVP admits a unique solution

$$\sqrt{\frac{x}{y}}y' = \cos(x+y), \; y \neq 0, \qquad y(1) = 1$$

b) Solve the differential equations

$$[\cos x \ln(2y-8) + \frac{1}{x}]dx + \frac{\sin x}{y-4}dy = 0, \;\; y > 4, \; x \neq 0.$$

$$[x\cos\left(\frac{y}{x}\right) - y]dx + xdy = 0, \;\; x > 0.$$

Question 2. **a)** **[3,4]**. Solve the initial value problem

$$(1-x)y' + xy = x(x-1)^2, \quad y(5) = 24.$$

b) Find the family of orthogonal trajectories for the family of curves

$$Cx^2 - y^2 = 1.$$

Question 3. **a)** **[4,4]**. Find the general solution of the differential equation

$$y'' - 2y' + y = \frac{e^x}{x}, \quad x > 0.$$

b) Write down the form of y_p for the solution of differential the equation

$$y^{(3)} + 4y' = 4 + xe^{-x} - e^x \sin x + 5\cos 2x.$$

Question 4 **[5]**. Find the power series solution about the ordinary point $x_0 = 0$ for the differential equation $y'' - 2xy' + 2y = 0$.

Question 5. **a)** **[5,5]**. Let f be 2π−periodic function defined by:

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 \leq x < \pi \end{cases}$$

Sketch the graph of f on $[-2\pi, 2\pi]$, find the Fourier Series of f, and deduce the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

b) Consider the function

$$f(x) = \begin{cases} 0, & x < -1 \\ 1-x, & -1 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$$

Sketch the graph of f, find its Fourier integral and deduce the value of $\int_0^{\infty} \frac{\sin\lambda}{\lambda} d\lambda$.

Complete Solutions of 204.M Final exam

First Semester. 2016-2017.

Question 1

1 a

$$\text{IVP: } \begin{cases} \sqrt{\dfrac{x}{y}}\, y' = \cos(x+y), \quad y \neq 0 \\ y(1) = 1 \end{cases}$$

$$y' = \frac{dy}{dx} = \cos(x+y)\left(\frac{x}{y}\right)^{-1/2} = f(x,y)$$

$$\frac{\partial f}{\partial y} = -\sin(x+y)\left(\frac{x}{y}\right)^{-1/2} - \frac{1}{2}\left(\frac{x}{y}\right)^{-3/2}\cdot\left(\frac{-x}{y^2}\right)\cos(x+y) \qquad 1$$

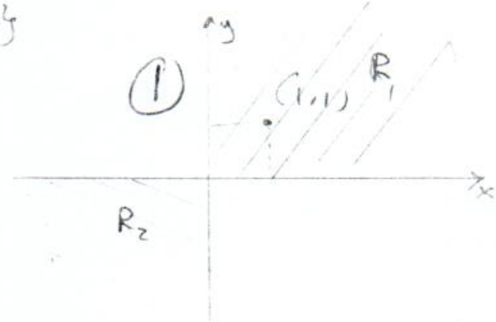
So it is clear that f and $\frac{\partial f}{\partial y}$ are continuous on the region

$$R = \{(x,y);\ \frac{x}{y} > 0\} \text{ or } \qquad 2$$

$$R = \{(x,y);\ x>0 \text{ and } y>0\} \cup \{(x,y);\ x<0 \text{ and } y<0\}$$

But $(x_0, y_0) = (1,1) \in R_1 = \{(x,y);\ x>0, \text{ and } y>0\}$

Then R_1 is the largest region s.t. the IVP admit a unique solution 1



1 b

i) $\overbrace{[(\cos x)\ln(2y-8) + \frac{1}{x}]}^{M}dx + \overbrace{\frac{\sin x}{y-4}}^{N}dy = 0, \quad x \neq 0, \ y > 4$

$$\frac{\partial M}{\partial y} = \cos x \frac{2}{2y-8} = \cos x \frac{1}{y-4}, \quad \frac{\partial N}{\partial x} = \frac{\cos x}{y-4}. \qquad 1$$

Then the D.E is exact, hence $\exists$ a function F of x and y s.t.

$$\frac{\partial F}{\partial x} = \cos x \ln(y-4) + \cos x \ln 2 + \frac{1}{x}$$

$$\frac{\partial F}{\partial y} = \frac{\sin x}{y-4}. \qquad 1$$

So $F(x,y) = \int \frac{\sin x}{y-4}dy = \sin x \ln(y-4) + \phi(x)$

$$\frac{\partial F}{\partial x} = \cos x \ln(y-4) + f'(x) = \cos x \ln(y-4) + \cos x \cdot \ln 2 + \frac{1}{x}$$

$$f(x) = \sin x \ln 2 + \ln x + c$$

Thus the Solution of the D.E is

$$\boxed{F(x,y) = \sin x \cdot \ln(y-4) + \sin(x) \cdot \ln 2 + \ln x + C = 0}$$

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2) $[x \cos(\frac{y}{x}) - y]dx + x\,dy = 0.$

This D.E is homogeneous. We can put $u = \frac{y}{x} \Rightarrow y = xu$

$$dy = x\,du + u\,dx$$

$$[\cos(\frac{y}{x}) - \frac{y}{x}]dx + dy = 0$$

$$(\cos u - u)\,dx + x\,du + u\,dx = 0$$

$$(\cos u)\,dx + x\,du = 0 \Rightarrow \frac{dx}{x} + \frac{du}{\cos u} = 0 \ ; \quad 0 < u = \frac{y}{x} < \frac{\pi}{2}$$

$$\ln x + \ln|\sec u + \tan u| = c$$

$$\boxed{\ln x + \ln|\sec(\frac{y}{x}) + \tan(\frac{y}{x})| = C}$$

Question ②

② ⓐ $\begin{cases} (1-x)y' + xy = x(x-1)^2 \ ; & x \neq 1 \ ; \ x > 1 \\ y(5) = 24 \end{cases}$

The D.E is linear: $y' + \frac{x}{1-x} y = x(1-x)$

I.F: $\mu(x) = e^{\int \frac{x}{1-x}dx} = e^{\int (-1 + \frac{1}{1-x})dx}$ 1

$$= e^{-x - \ln|1-x|} =$$

$$\boxed{\mu(x) = \frac{e^{-x}}{x-1}}$$

$$y\,\mu(x) = y\,\frac{e^{-x}}{x-1} = \int \frac{x(1-x)}{x-1} e^{-x} = -\int x e^{-x} dx$$

$$y\,\frac{e^{-x}}{x-1} = -[-xe^{-x} - e^{-x}] + c = +e^{-x}(x+1) + c$$

$$\boxed{y = +(x^2-1) + e^{x}(x-1)\,C}$$

1

But $y(5) = 24 \Rightarrow 24 = 24 + e^5(4)C \Rightarrow C = 0$, then the solution of the IVP is $y = x^2 - 1$

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$\boxed{2\,b}$ $cx^2-y^2=1 \Rightarrow c=\frac{y^2+1}{x^2}$, we take the derivative implicately two sides, $0=\frac{2yy'x^2-2x(y^2+1)}{x^4}=0$, hence

$$2yy'x^2=\cancel{2}x(y^2+1)$$

$$y'yx=y^2+1 \quad \text{or} \quad y'=\frac{y^2+1}{yx}=f(x,y).$$ (1)

Now we have to solve the D.E $y'=\frac{-1}{f(x,y)}=\frac{-(xy)}{y^2+1}$ (1)

$$(y^2+1)dy+xy\,dx=0$$

$$\frac{y^2+1}{y}dy+x\,dx=0 \quad \text{or} \quad \int\left(y+\frac{1}{y}\right)dy+\int x\,dx=0$$ (2)

$\boxed{\frac{1}{2}(y^2+x^2)+\ln|y|=c}$, this family of curves is orthogonal to $cx^2-y^2=1$

Question 3

$\boxed{3\,a}$ $y''-2y'+y=\frac{1}{x}e^x$; $x>0$

1) $y''-2y'+y=0$, $y=e^{mx} \Rightarrow (m^2-2m+1)=(m-1)^2=0$, $m=1,1$

$$y_c=c_1e^x+c_2xe^x \;/\; y_1=e^x,\; y_2=xe^x$$ (1)

2) $y_p=u_1y_1+u_2y_2$ s.t

$$\begin{cases} u_1'(e^x)+u_2'(xe^x)=0 \\ u_1'(e^x)+u_2'(e^x+xe^x)=\frac{1}{x}e^x \end{cases} \Rightarrow \begin{cases} u_1'+xu_2'=0 \\ u_1'+(1+x)=\frac{1}{x} \end{cases}$$

$$W=\begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix}=1, \quad u_1'=\frac{\begin{vmatrix} 0 & x \\ \frac{1}{x} & 1+x \end{vmatrix}}{1}=-1 \Rightarrow \boxed{u_1=-x}$$ (1)

$$u_2'=\frac{\begin{vmatrix} 1 & 0 \\ 1 & \frac{1}{x} \end{vmatrix}}{1}=\frac{1}{x} \Rightarrow \boxed{u_2=\ln x}$$ (1)

$$y_p=-xe^x+xe^x\ln x=\boxed{xe^x(\ln x-1)}$$ (1)

Thus the solution of the D.E is

$$\boxed{y = y_c + y_p = c_1 e^{x} + c_2 x e^{x} + x e^{x}(\ln x - 1)}$$

3 (b) $y^{(3)} + 4y' = 4 + xe^{-x} - e^{x}\sin x + 5\cos(2x)$

1) $y''' + 4y' = 0 \Rightarrow m^3 + 4m = 0 \quad \text{(1)}$ (1)

$m(m^2+4) = 0, \quad m = 0, \; m = \mp 2i$

0 is a root of (1) $\quad 4 = 4e^{0m} \longrightarrow Ax$

-1 is not root of (1) $\Rightarrow \quad xe^{-x} \rightarrow (Bx+c)e^{-x}$ (1)

$1 \mp i$ is not root of (1) $\Rightarrow e^{x}\sin x \longrightarrow De^{x}\sin x + Ee^{x}\cos x$

$0 \mp 2i$ is a root of (1) $\Rightarrow \; -5\cos(2x) \rightarrow Fx\cos(2x) + Gx\sin 2x$ (2)

Then
$$\boxed{y_p = Ax + (Bx+c)e^{-x} + De^{x}\sin x + Ee^{x}\cos x + Fx\cos(2x) + Gx\sin 2x}$$

is the general form of the D.E.

Question 4 $\quad y'' - 2xy' + 2y = 0,$

$\frac{a_1}{a_2} = -2x, \quad \frac{a_0}{a_2} = 2$ are two analytic function on $\mathbb{R}$

the solution of the D.E is the form $\quad y = \sum_{n=0}^{\infty} a_n x^n, \quad x \in \mathbb{R}$

then

$$\sum_{2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{1}^{\infty} 2na_n x^n + \sum_{0}^{\infty} 2a_n x^n = 0, \quad x \in \mathbb{R}$$

$n-2 = k \qquad n = k \qquad n = k$

$n = k+2$

$$\sum_{0}^{\infty} (k+2)(k+1)a_{k+2}x^{k} - \sum_{1}^{\infty} 2ka_k x^k + \sum_{0}^{\infty} 2a_k x^k = 0, \quad x \in \mathbb{R}$$

$$(2a_2 + 2a_0) + \sum_{0}^{\infty} [(k+2)(k+2)a_{k+2} + 2a_k(-k+1)]x^k = 0.$$

Hence $a_2 = -a_0$, $\quad \boxed{a_{k+2} = \frac{2(k-1)a_k}{(k+2)(k+1)}, \quad k \geq 1}$ (2)

$k=1, \quad a_3 = 0$

$k=2 \quad a_4 = \frac{\cancel{2}}{4.3} a_2 = \frac{-a_0}{6}$

$k=3 \quad a_5 = \frac{2.2\, a_3}{5.4} = 0 = a_3$

$k=4, \quad a_6 = \frac{2.3\, a_4}{6.5} = \frac{-a_0}{30}$, and so on.

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

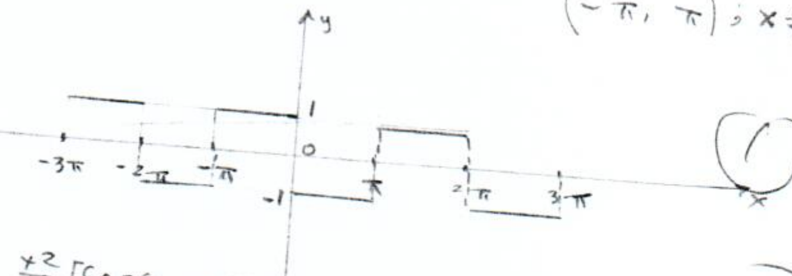
$= a_0 + a_1 x - a_0 x^2 + 0 - \frac{a_0}{6} x^4 + 0 - \frac{a_0}{30} x^6 + \cdots$

$$\boxed{y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 \cdots\right) = a_1 y_1 + a_0 y_2(x)}$$

where $\boxed{y_1 = x,}$ $\boxed{y_2 = 1 - x^2 - \frac{x^4}{6} - \frac{1}{30}x^6 \cdots}$ $\quad x \in \mathbb{R}$

Question ⑤

⑤, ⓐ $\quad f(x) = \begin{cases} 1 ; & -\pi < x < 0 \\ -1 & 0 \le x < \pi \end{cases}$, f is an odd function on $(-\pi, \pi) ; x \neq 0$



Then $a_n = 0, \; n = 0, 1, \ldots$

$b_n = \frac{-2}{\pi} \int_0^{\pi} (-1) \sin(nx)\, dx = \frac{+2}{\pi}\left[\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{\pi n}\left((-1)^n - 1\right)$, hence

$n = 1, 2, \ldots$

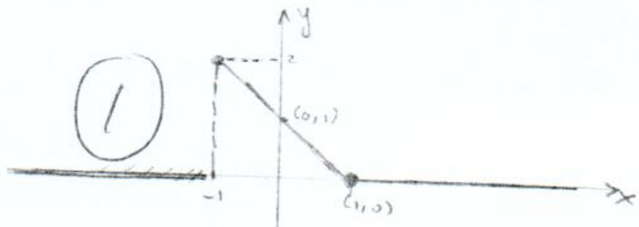
$$\boxed{\frac{f(x^+) + f(x^-)}{2} = \sum_{n=1}^{\infty} \frac{2}{\pi n}\left((-1)^n - 1\right) \sin(nx)} \quad -\pi < x < \pi$$

At $x = \frac{\pi}{2} \Rightarrow f(\frac{\pi}{2}) = -1 = \sum_1^{\infty} \frac{2}{\pi n}\left((-1)^n - 1\right) \sin\left(\frac{n\pi}{2}\right)$

$-1 = \sum_1^{\infty} \frac{-4}{\pi(2n-1)} \sin\left(\frac{2n-1}{2}\pi\right) = -\frac{4}{\pi}\sum_1^{\infty} \frac{(-1)^{n-1}}{2n-1}$

So $\boxed{\sum_1^{\infty} \frac{(-1)^{n-1}}{2n-1} = \sum_0^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}}$

5 b $\quad f(x) = \begin{cases} 0 ; & x < -1 \\ 1-x ; & -1 \le x < 1 \\ 0 , & x \ge 1 \end{cases}$ 1



1) $A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx = \int_{-1}^{1} (1-x)\cos(\alpha x) dx$

$= \left[(1-x) \frac{\sin \alpha x}{\alpha} \right]_{-1}^{1} + \int_{-1}^{1} \frac{\sin \alpha x}{\alpha} dx \qquad \alpha > 0$ 1

$= 0 - \frac{2 \sin(-\alpha)}{\alpha} - \left[\frac{\cos \alpha x}{\alpha^2} \right]_{-1}^{1}$

$= \frac{2}{\alpha} \sin \alpha - \frac{(\cos \alpha - \cos(-\alpha))}{\alpha^2} = \boxed{\frac{2}{\alpha} \sin \alpha}$

2) $B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin(\alpha x) d\alpha = \int_{-1}^{1} (1-x) \sin \alpha x dx$

$= \left[(1-x)\left(-\frac{\cos \alpha x}{\alpha} \right) \right]_{-1}^{1} - \int_{-1}^{1} \frac{\cos(\alpha x)}{\alpha} dx$

$= 0 + 2 \frac{\cos(-\alpha)}{\alpha} - \left[\frac{\sin \alpha x}{\alpha^2} \right]_{-1}^{1}$ 1

$= \frac{2 \cos \alpha}{\alpha} - \left(\frac{\sin \alpha}{\alpha^2} - \frac{\sin(-\alpha)}{\alpha^2} \right)$

$$\boxed{B(\alpha) = \frac{2 \cos \alpha}{\alpha} - \frac{2 \sin \alpha}{\alpha^2}}$$

$$\boxed{\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\pi} \int_0^{\infty} \left[\frac{2}{\alpha} \sin \alpha \cos \alpha x + \left(\frac{2 \cos \alpha}{\alpha} - \frac{2 \sin \alpha}{\alpha^2} \right) \sin \alpha x \right] d\alpha , \quad x \in \mathbb{R}}$$ 1

At $x = 0$, we have

$\frac{f(0^+) + f(0^-)}{2} = \frac{1+1}{2} = 1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha}{\alpha} d\alpha$ or 1

$$\boxed{\frac{\pi}{2} = \int_0^{\infty} \frac{\sin \alpha}{\alpha} d\alpha = \int_0^{\infty} \frac{\sin(\lambda)}{\lambda} d\lambda}$$