(Riemann integrable)

[1] Suppose that $f \in R(a,b)$ and let $g$ be defined by $g(x) = f(x-c)$ for all $x \in [a+c, b+c]$, where $c$ is a real constant.

Prove that $g$ is Riemann integrable on $[a+c, b+c]$ and that

$$\int_{a+c}^{b+c} g(x)\,dx = \int_a^b f(x)\,dx$$

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[2] Suppose that $f \in R(a,b)$, and let $g$ be defined by $g(x) = f(x/c)$ for all $x \in [ac, bc]$, where $c > 0$.

Prove that $g \in R(a, b)$ and that $\int_a^b f = c \int_a^b g$

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[3] If $f$ is bounded on $[a,b]$ and $k > 0$.

Prove that $L(kf) = kL(f)$ and $U(kf) = kU(f)$

what if $k \leq 0$? Deduce that if $f \in R(a,b)$, then for any real number $k$, $kf \in R(a,b)$ and $\int_a^b kf = k\int_a^b f$

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[4] let $f$ and $g$ be bounded on $[a,b]$. If $f \leq g$

Prove that $L(f) \leq L(g)$

and $U(f) \leq U(g)$

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[5] If $f, g$ are differentiable on $[a,b]$ and if $f'$ and $g'$ Riemann integrable then

$$\int_a^b f'(x)\,g(x)\,dx + \int_a^b f(x)\,g'(x)\,dx = 0$$

if and only if $f(a)\,g(a) = f(b)\,g(b)$

6 Give an example of function $f \notin R(a,b)$ but $|f| \in R(a,b)$

7 Is $f(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$ Riemann integrable ?

Notation:

$f \in R(a,b) \xrightarrow{\text{i.e}} f$ is Riemann integrable on $[a,b]$

$U(f)$ is upper integral

$U(f) = \inf\{U(f,P) : P \in \mathcal{P}(a,b)\}$

upper sum

all partitions of $[a,b]$

$L(f) = \sup\{L(f,P) : P \in \mathcal{P}\}$

Lower sum

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