CHAPTER

5

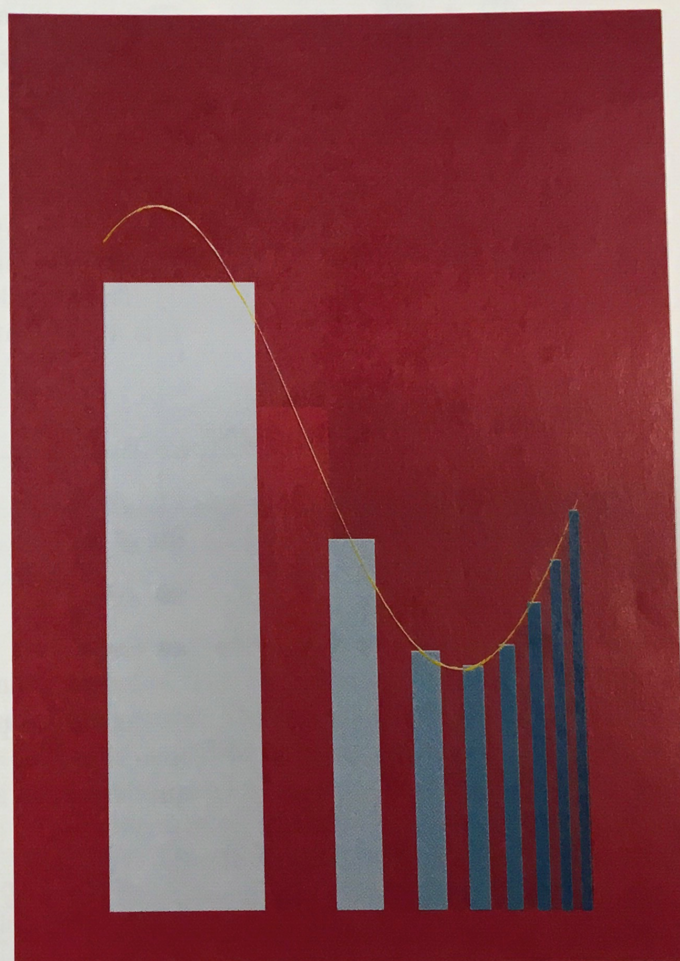
INTEGRALS

INTRODUCTION

The two most important tools in calculus are the derivative, considered in previous chapters, and the *definite integral*, defined in Section 5.4. The derivative was motivated by the problems of finding the slope of a tangent line and defining velocity. The definite integral arises naturally when we consider the problem of finding the area of a region in the xy-plane. However, this is merely one application. As we shall see in later chapters, the uses for definite integrals are as abundant and varied as those for derivatives.

The principal result in this chapter is the *fundamental theorem of calculus*, proved in Section 5.6. This outstanding theorem enables us to find exact values of definite integrals by using an *antiderivative* or *indefinite integral*. Each of these concepts is defined in Section 5.1; the procedure may be regarded as a reverse procedure to finding the derivative of a function. Thus, in addition to providing an important evaluation process, the fundamental theorem shows that there is a relationship between derivatives and integrals—a key result in calculus.

The chapter closes with a discussion of methods of *numerical integration*, used for approximating definite integrals that cannot be evaluated by means of the fundamental theorem. These methods are readily programmable for use with calculators and computers and are employed in a wide variety of applied fields.



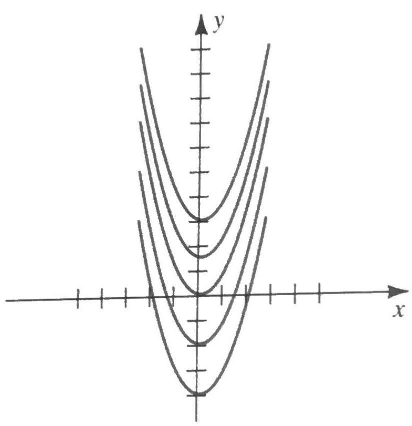
5.1 ANTIDERIVATIVES AND INDEFINITE INTEGRALS

In our previous work we solved problems of the following type: *Given a function f, find the derivative f'*. We shall now consider a related problem: *Given a function f, find a function F such that $F' = f$*. In the next definition we give F a special name.

Definition **(5.1)**

> A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for every x in I.

FIGURE 5.1



We shall also call $F(x)$ an antiderivative of $f(x)$. The process of finding F, or $F(x)$, is called **antidifferentiation**.

To illustrate, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, because

$$F'(x) = D_x(x^2) = 2x = f(x).$$

There are many other antiderivatives of $2x$, such as $x^2 + 2$, $x^2 - \frac{5}{3}$, and $x^2 + \sqrt{3}$. In general, if C is *any* constant, then $x^2 + C$ is an antiderivative of $2x$, because

$$D_x(x^2 + C) = 2x + 0 = 2x.$$

Thus, there is a *family of antiderivatives* of $2x$ of the form $F(x) = x^2 + C$, where C is any constant. Graphs of several members of this family are sketched in Figure 5.1.

The next illustration contains other examples of antiderivatives, where C is a constant.

ILLUSTRATION

$f(x)$	ANTIDERIVATIVES OF $f(x)$
x^2	$\frac{1}{3}x^3$, $\frac{1}{3}x^3 + 8$, $\frac{1}{3}x^3 + C$
$8x^3$	$2x^4$, $2x^4 - \sqrt[3]{7}$, $2x^4 + C$
$\cos x$	$\sin x$, $\sin x + \frac{4}{9}$, $\sin x + C$

As in the preceding illustration, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any constant C. The next theorem states that *every* antiderivative is of this form.

Theorem **(5.2)**

> Let F be an antiderivative of f on an interval I. If G is any antiderivative of f on I, then
>
> $$G(x) = F(x) + C$$
>
> for some constant C and every x in I.

PROOF If F and G are antiderivatives of f, let H be the function defined by

$$H(x) = G(x) - F(x)$$

for every x in I. We will show that H is a constant function on I; that is, $G(x) - F(x) = C$ for some C, or, equivalently, $G(x) = F(x) + C$.

Let a and b be any numbers in I such that $a < b$. To show that H is constant on I, it suffices to prove that $H(a) = H(b)$. Since F and G are antiderivatives of f,

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$$

for every x in I. Since $H(x)$ is differentiable, H is continuous, by Theorem (3.11). Applying the mean value theorem (4.12) to H on the interval $[a, b]$, there exists a number c in (a, b) such that

$$H'(c) = \frac{H(b) - H(a)}{b - a}.$$

Since c is in I, $H'(c) = 0$, and thus

$$H(b) - H(a) = 0, \quad \text{or} \quad H(a) = H(b),$$

which is what we wished to prove. ■

We refer to the constant C in Theorem (5.2) as an **arbitrary constant**. If $F(x)$ is an antiderivative of $f(x)$, then *all* antiderivatives of $f(x)$ can be obtained from $F(x) + C$ by letting C range through the set of real numbers. We shall employ the following notation for a family of antiderivatives of this type.

Definition **(5.3)**

The notation

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$ and C is an arbitrary constant, denotes the family of all antiderivatives of $f(x)$ on an interval I.

The symbol $\int$ used in Definition (5.3) is an **integral sign**. We call $\int f(x)\,dx$ the **indefinite integral** of $f(x)$. The expression $f(x)$ is the **integrand**, and C is the **constant of integration**. The process of finding $F(x) + C$, when given $\int f(x)\,dx$, is referred to as **indefinite integration**, **evaluating the integral**, or **integrating $f(x)$**. The adjective *indefinite* is used because $\int f(x)\,dx$ represents a *family* of antiderivatives, not any *specific* function. Later in the chapter, when we discuss definite integrals, we shall give reasons for using the integral sign and the differential dx that appears to the right of the integrand $f(x)$. At present we shall not interpret $f(x)\,dx$ as the product of $f(x)$ and the differential dx. We shall regard dx merely as a symbol that specifies the independent variable x, which we refer to as the **variable of integration**. If we use a different variable of integration, such as t, we write

$$\int f(t)\,dt = F(t) + C,$$

where $F'(t) = f(t)$.

ILLUSTRATION

- $\int x^4\,dx = \frac{1}{5}x^5 + C$ because $D_x\left(\frac{1}{5}x^5\right) = x^4$.
- $\int t^{-3}\,dt = -\frac{1}{2}t^{-2} + C$ because $D_t\left(-\frac{1}{2}t^{-2}\right) = t^{-3}$.
- $\int \cos u\,du = \sin u + C$ because $D_u \sin u = \cos u$.

Note that, in general,

$$\int [D_x f(x)]\,dx = f(x) + C$$

because $f'(x) = D_x f(x)$. This allows us to use any derivative formula to obtain a corresponding formula for an indefinite integral, as illustrated in the next table. As shown in Formula (1), it is customary to abbreviate $\int 1\,dx$ by $\int dx$.

Brief table of indefinite integrals **(5.4)**

DERIVATIVE $D_x[f(x)]$	INDEFINITE INTEGRAL $\int D_x[f(x)]\,dx = f(x) + C$
$D_x(x) = 1$	**(1)** $\int 1\,dx = \int dx = x + C$
$D_x\left(\dfrac{x^{r+1}}{r+1}\right) = x^r\ (r \neq -1)$	**(2)** $\int x^r\,dx = \dfrac{x^{r+1}}{r+1} + C\ (r \neq -1)$
$D_x(\sin x) = \cos x$	**(3)** $\int \cos x\,dx = \sin x + C$
$D_x(-\cos x) = \sin x$	**(4)** $\int \sin x\,dx = -\cos x + C$
$D_x(\tan x) = \sec^2 x$	**(5)** $\int \sec^2 x\,dx = \tan x + C$
$D_x(-\cot x) = \csc^2 x$	**(6)** $\int \csc^2 x\,dx = -\cot x + C$
$D_x(\sec x) = \sec x \tan x$	**(7)** $\int \sec x \tan x\,dx = \sec x + C$
$D_x(-\csc x) = \csc x \cot x$	**(8)** $\int \csc x \cot x\,dx = -\csc x + C$

Formula (2) is called the *power rule for indefinite integration*. As in the following illustration, it is often necessary to rewrite an integrand before applying the power rule or one of the trigonometric formulas.

ILLUSTRATION

- $\int x^3 \cdot x^5\,dx = \int x^8\,dx = \dfrac{x^{8+1}}{8+1} = \dfrac{1}{9}x^9 + C$
- $\int \dfrac{1}{x^3}\,dx = \int x^{-3}\,dx = \dfrac{x^{-3+1}}{-3+1} = -\dfrac{1}{2x^2} + C$
- $\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \dfrac{x^{2/3+1}}{\frac{2}{3}+1} = \dfrac{3}{5}x^{5/3} + C$
- $\int \dfrac{\tan x}{\sec x}\,dx = \int \cos x\,\dfrac{\sin x}{\cos x}\,dx = \int \sin x\,dx = -\cos x + C$

It is a good idea to check indefinite integrations (such as those in the preceding illustration) by differentiating the final expression to see if either the integrand or an equivalent form of the integrand is obtained.

The next theorem indicates that differentiation and indefinite integration are inverse processes, because each, in a sense, undoes the other. In statement (i) it is assumed that f is differentiable, and in (ii) that f has an antiderivative on some interval.

Theorem (5.5)

(i) $\int [D_x f(x)]\, dx = f(x) + C$

(ii) $D_x\left[\int f(x)\, dx\right] = f(x)$

PROOF We have already proved (i). To prove (ii), let F be an antiderivative of f and write

$$D_x\left[\int f(x)\, dx\right] = D_x\,[F(x) + C] = F'(x) + 0 = f(x). \quad \blacksquare$$

EXAMPLE 1 Verify Theorem (5.5) for the special case $f(x) = x^2$.

SOLUTION

(i) If we first differentiate x^2 and then integrate,

$$\int D_x\,(x^2)\, dx = \int 2x\, dx = x^2 + C.$$

(ii) If we first integrate x^2 and then differentiate,

$$D_x \int x^2\, dx = D_x\left(\frac{x^3}{3} + C\right) = x^2.$$

The next theorem is useful for evaluating many types of indefinite integrals. In the statements we assume that $f(x)$ and $g(x)$ have antiderivatives on an interval I.

Theorem (5.6)

(i) $\int cf(x)\, dx = c\int f(x)\, dx$ for any constant c

(ii) $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

(iii) $\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$

PROOF We shall prove (ii). The proofs of (i) and (iii) are similar. If F and G are antiderivatives of f and g, respectively,

$$D_x\,[F(x) + G(x)] = F'(x) + G'(x) = f(x) + g(x).$$

Hence, by Definition (5.3),

$$\int [f(x) + g(x)]\, dx = F(x) + G(x) + C,$$

where C is an arbitrary constant. Similarly,

$$\int f(x)\,dx + \int g(x)\,dx = F(x) + C_1 + G(x) + C_2$$

for arbitrary constants C_1 and C_2. These give us the same family of antiderivatives, since for any special case we can choose values of the constants such that $C = C_1 + C_2$. This proves (ii). ∎

EXAMPLE 2 Evaluate $\int (5x^3 + 2\cos x)\,dx$.

SOLUTION We first use (ii) and (i) of Theorem (5.6) and then formulas from (5.4):

$$\begin{aligned}\int (5x^3 + 2\cos x)\,dx &= \int 5x^3\,dx + \int 2\cos x\,dx\\ &= 5\int x^3\,dx + 2\int \cos x\,dx\\ &= 5\left(\frac{x^4}{4} + C_1\right) + 2(\sin x + C_2)\\ &= \tfrac{5}{4}x^4 + 5C_1 + 2\sin x + 2C_2\\ &= \tfrac{5}{4}x^4 + 2\sin x + C\end{aligned}$$

where $C = 5C_1 + 2C_2$.

In Example 2 we added the two constants $5C_1$ and $2C_2$ to obtain one arbitrary constant C. We can always manipulate arbitrary constants in this way, so it is not necessary to introduce a constant for each indefinite integration as we did in Example 2. Instead, if an integrand is a sum, *we integrate each term of the sum without introducing constants and then add one arbitrary constant C after the last integration.* We also often bypass the step $\int cf(x)\,dx = c\int f(x)\,dx$, as in the next example.

EXAMPLE 3 Evaluate $\int \left(8t^3 - 6\sqrt{t} + \dfrac{1}{t^3}\right) dt$.

SOLUTION First we find an antiderivative for each of the three terms in the integrand and then add an arbitrary constant C. We rewrite $\sqrt{t}$ as $t^{1/2}$ and $1/t^3$ as t^{-3} and then use the power rule for integration:

$$\begin{aligned}\int \left(8t^3 - 6\sqrt{t} + \frac{1}{t^3}\right) dt &= \int (8t^3 - 6t^{1/2} + t^{-3})\,dt\\ &= 8\cdot\frac{t^4}{4} - 6\cdot\frac{t^{3/2}}{\frac{3}{2}} + \frac{t^{-2}}{-2} + C\\ &= 2t^4 - 4t^{3/2} - \frac{1}{t^2} + C\end{aligned}$$

EXAMPLE 4 Evaluate $\int \dfrac{(x^2-1)^2}{x^2}\,dx$.

SOLUTION First we change the form of the integrand, because the degree of the numerator is greater than or equal to the degree of the denominator. We then find an antiderivative for each term, adding an arbitrary constant C after the last integration:

$$\begin{aligned}\int \frac{(x^2-1)^2}{x^2}\,dx &= \int \frac{x^4-2x^2+1}{x^2}\,dx\\ &= \int (x^2-2+x^{-2})\,dx\\ &= \frac{x^3}{3}-2x+\frac{x^{-1}}{-1}+C\\ &= \frac{1}{3}x^3-2x-\frac{1}{x}+C\end{aligned}$$

EXAMPLE 5 Evaluate $\int \frac{1}{\cos u \cot u}\,du$.

SOLUTION We use trigonometric identities to change the integrand and then apply formula (7) from Table (5.4):

$$\begin{aligned}\int \frac{1}{\cos u \cot u}\,du &= \int \sec u \tan u\,du\\ &= \sec u + C\end{aligned}$$

An applied problem may be stated in terms of a **differential equation**—that is, an equation that involves derivatives or differentials of an unknown function. A function f is a **solution** of a differential equation if it satisfies the equation—that is, if substitution of f for the unknown function produces a true statement. To **solve** a differential equation means to find all solutions. Sometimes, in addition to the differential equation, we may know certain values of f or f', called **initial conditions.**

Indefinite integrals are useful for solving certain differential equations, because if we are given a derivative $f'(x)$ we can integrate and use Theorem (5.5)(i) to obtain an equation involving the unknown function f:

$$\int f'(x)\,dx = f(x) + C$$

If we are also given an initial condition for f, it may be possible to find $f(x)$ explicitly, as in the next example.

EXAMPLE 6 Solve the differential equation

$$f'(x) = 6x^2 + x - 5$$

subject to the initial condition $f(0) = 2$.

SOLUTION We proceed as follows:

$$\begin{aligned}f'(x) &= 6x^2 + x - 5\\ \int f'(x)\,dx &= \int (6x^2+x-5)\,dx\\ f(x) &= 2x^3 + \tfrac{1}{2}x^2 - 5x + C\end{aligned}$$

Hence

$$\int C'(x)\,dx = \int (30 - 0.02x)\,dx$$

and

$$C(x) = 30x - 0.01x^2 + K$$

for some K. Letting $x = 1$ and using $C(1) = 35$, we obtain

$$35 = 30 - 0.01 + K, \quad \text{or} \quad K = 5.01.$$

Consequently,

$$C(x) = 30x - 0.01x^2 + 5.01.$$

In particular, the cost of producing 100 units is

$$C(100) = 3000 - 100 + 5.01 = \$2905.01.$$

EXERCISES 5.1

Exer. 1–42: Evaluate.

1 $\int (4x + 3)\,dx$

2 $\int (4x^2 - 8x + 1)\,dx$

3 $\int (9t^2 - 4t + 3)\,dt$

4 $\int (2t^3 - t^2 + 3t - 7)\,dt$

5 $\int \left(\frac{1}{z^3} - \frac{3}{z^2}\right) dz$

6 $\int \left(\frac{4}{z^7} - \frac{7}{z^4} + z\right) dz$

7 $\int \left(3\sqrt{u} + \frac{1}{\sqrt{u}}\right) du$

8 $\int (\sqrt{u^3} - \frac{1}{2}u^{-2} + 5)\,du$

9 $\int (2v^{5/4} + 6v^{1/4} + 3v^{-4})\,dv$

10 $\int (3v^5 - v^{5/3})\,dv$

11 $\int (3x - 1)^2\,dx$

12 $\int \left(x - \frac{1}{x}\right)^2 dx$

13 $\int x(2x + 3)\,dx$

14 $\int (2x - 5)(3x + 1)\,dx$

15 $\int \frac{8x - 5}{\sqrt[3]{x}}\,dx$

16 $\int \frac{2x^2 - x + 3}{\sqrt{x}}\,dx$

17 $\int \frac{x^3 - 1}{x - 1}\,dx, \quad x \neq 1$

18 $\int \frac{x^3 + 3x^2 - 9x - 2}{x - 2}\,dx, \quad x \neq 2$

19 $\int \frac{(t^2 + 3)^2}{t^6}\,dt$

20 $\int \frac{(\sqrt{t} + 2)^2}{t^3}\,dt$

21 $\int \frac{3}{4}\cos u\,du$

22 $\int -\frac{1}{5}\sin u\,du$

23 $\int \frac{7}{\csc x}\,dx$

24 $\int \frac{1}{4\sec x}\,dx$

25 $\int (\sqrt{t} + \cos t)\,dt$

26 $\int (\sqrt[3]{t^2} - \sin t)\,dt$

27 $\int \frac{\sec t}{\cos t}\,dt$

28 $\int \frac{1}{\sin^2 t}\,dt$

29 $\int (\csc v \cot v \sec v)\,dv$

30 $\int (4 + 4\tan^2 v)\,dv$

31 $\int \frac{\sec w \sin w}{\cos w}\,dw$

32 $\int \frac{\csc w \cos w}{\sin w}\,dw$

33 $\int \frac{(1 + \cot^2 z)\cot z}{\csc z}\,dz$

34 $\int \frac{\tan z}{\cos z}\,dz$

35 $\int D_x \sqrt{x^2 + 4}\,dx$

36 $\int D_x \sqrt[3]{x^3 - 8}\,dx$

37 $\int \frac{d}{dx}(\sin \sqrt[3]{x})\,dx$

38 $\int \frac{d}{dx}(\sqrt{\tan x})\,dx$

39 $D_x \int (x^3 \sqrt{x - 4})\,dx$

40 $D_x \int (x^4 \sqrt[3]{x^2 + 9})\,dx$

41 $\frac{d}{dx} \int \cot x^3\,dx$

42 $\frac{d}{dx} \int \cos \sqrt{x^2 + 1}\,dx$

Exer. 43–48: Evaluate the integral if a and b are constants.

43 $\int a^2\,dx$

44 $\int ab\,dx$

45 $\int (at + b)\,dt$

46 $\int \left(\frac{a}{b^2}t\right) dt$

47 $\int (a + b)\,du$

48 $\int (b - a^2)\,du$

Exer. 49–56: Solve the differential equation subject to the given conditions.

49 $f'(x) = 12x^2 - 6x + 1; \quad f(1) = 5$

50 $f'(x) = 9x^2 + x - 8; \quad f(-1) = 1$

51 $\frac{dy}{dx} = 4x^{1/2}; \quad y = 21 \text{ if } x = 4$