52 $\dfrac{dy}{dx} = 5x^{-1/3}$; $y = 70$ if $x = 27$

53 $f''(x) = 4x - 1$; $f'(2) = -2$; $f(1) = 3$

54 $f''(x) = 6x - 4$; $f'(2) = 5$; $f(2) = 4$

55 $\dfrac{d^2y}{dx^2} = 3 \sin x - 4 \cos x$; $y = 7$ and $y' = 2$ if $x = 0$

56 $\dfrac{d^2y}{dx^2} = 2 \cos x - 5 \sin x$; $y = 2 + 6\pi$ and $y' = 3$ if $x = \pi$

Exer. 57–58: If a point is moving on a coordinate line with the given acceleration $a(t)$ and initial conditions, find $s(t)$.

57 $a(t) = 2 - 6t$; $v(0) = -5$; $s(0) = 4$

58 $a(t) = 3t^2$; $v(0) = 20$; $s(0) = 5$

59 A projectile is fired vertically upward from ground level with a velocity of 1600 ft/sec. Disregarding air resistance, find

(a) its distance $s(t)$ above ground at time t

(b) its maximum height

60 An object is dropped from a height of 1000 feet. Disregarding air resistance, find

(a) the distance it falls in t seconds

(b) its velocity at the end of 3 seconds

(c) when it strikes the ground

61 A stone is thrown directly downward from a height of 96 feet with an initial velocity of 16 ft/sec. Find

(a) its distance above the ground after t seconds

(b) when it strikes the ground

(c) the velocity at which it strikes the ground

62 A gravitational constant for objects near the surface of the moon is 5.3 ft/sec^2.

(a) If an astronaut on the moon throws a stone directly upward with an initial velocity of 60 ft/sec, find its maximum altitude.

(b) If, after returning to Earth, the astronaut throws the same stone directly upward with the same initial velocity, find the maximum altitude.

63 If a projectile is fired vertically upward from a height of s_0 feet above the ground with a velocity of v_0 ft/sec, prove that if air resistance is disregarded, its distance $s(t)$ above the ground after t seconds is given by $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$, where g is a gravitational constant.

64 A ball rolls down an inclined plane with an acceleration of 2 ft/sec^2.

(a) If the ball is given no initial velocity, how far will it roll in t seconds?

(b) What initial velocity must be given for the ball to roll 100 feet in 5 seconds?

65 If an automobile starts from rest, what constant acceleration will enable it to travel 500 feet in 10 seconds?

66 If a car is traveling at a speed of 60 mi/hr, what constant (negative) acceleration will enable it to stop in 9 seconds?

67 A small country has natural gas reserves of 100 billion ft^3. If $A(t)$ denotes the total amount of natural gas consumed after t years, then dA/dt is the *rate of consumption*. If the rate of consumption is predicted to be $5 + 0.01t$ billion ft^3/year, in approximately how many years will the country's natural gas reserves be depleted?

68 Refer to Exercise 67. Based on U.S. Department of Energy statistics, the rate of consumption of gasoline in the United States (in billions of gallons per year) is approximated by $dA/dt = 2.74 - 0.11t - 0.01t^2$, with $t = 0$ corresponding to the year 1980. Estimate the number of gallons of gasoline consumed in the United States between 1980 and 1984.

69 A sportswear manufacturer determines that the marginal cost in dollars of producing x warm-up suits is given by $20 - 0.015x$. If the cost of producing one suit is \$25, find the cost function and the cost of producing 50 suits.

70 If the marginal cost function of a product is given by $2/x^{1/3}$ and if the cost of producing 8 units is \$20, find the cost function and the cost of producing 64 units.

5.2 CHANGE OF VARIABLES IN INDEFINITE INTEGRALS

The formulas for indefinite integrals in Table (5.4) are limited in scope, because we cannot use them directly to evaluate integrals such as

$$\int \sqrt{5x + 7}\, dx \quad \text{or} \quad \int \cos 4x\, dx.$$

In this section we shall develop a simple but powerful method for changing the variable of integration so that these integrals (and many others) can be evaluated by using the formulas in Table (5.4).

To justify this method, we shall apply formula (i) of Theorem (5.5) to a *composite* function. We intend to consider several functions f, g, and F, so it will simplify our work if we state the formula in terms of a function h as follows:

$$\int [D_x\, h(x)]\, dx = h(x) + C$$

Suppose that F is an antiderivative of a function f and that g is a differentiable function such that $g(x)$ is in the domain of F for every x in some interval. If we let h denote the composite function $F \circ g$, then $h(x) = F(g(x))$ and hence

$$\int [D_x\, F(g(x))]\, dx = F(g(x)) + C.$$

Applying the chain rule (3.33) to the integrand $D_x\, F(g(x))$ and using the fact that $F' = f$, we obtain

$$D_x\, F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Substitution in the preceding indefinite integral gives us

$$(*) \qquad \int f(g(x))g'(x)\, dx = F(g(x)) + C.$$

We can employ the following device to help remember this formula:

$$\text{Let} \quad u = g(x) \quad \text{and} \quad du = g'(x)\, dx.$$

Note that once we have introduced the variable $u = g(x)$, the differential du of u is determined by using (ii) of Definition (3.28). If we *formally substitute* into the last integration formula, we obtain

$$\int f(u)\, du = F(u) + C.$$

This has the same *form* as the integral in Definition (5.3); however, u represents a *function*, not an independent variable x, as before. This indicates that $g'(x)\, dx$ in $(*)$ may be regarded as the product of $g'(x)$ and dx. Since the variable x has been replaced by a new variable u, finding indefinite integrals in this way is referred to as a **change of variable**, or as the **method of substitution**. We may summarize our discussion as follows, where we assume that f and g have the properties described previously.

Method of substitution **(5.7)**

> If F is an antiderivative of f, then
>
> $$\int f(g(x))g'(x)\, dx = F(g(x)) + C.$$
>
> If $u = g(x)$ and $du = g'(x)\, dx$, then
>
> $$\int f(u)\, du = F(u) + C.$$

After making the substitution $u = g(x)$ as indicated in (5.7), it may be necessary to insert a constant factor k into the integrand in order to arrive at the proper form $\int f(u)\, du$. We must then also multiply by $1/k$ to maintain equality, as illustrated in the following examples.

EXAMPLE 1 Evaluate $\int \sqrt{5x+7}\,dx$.

SOLUTION We let $u = 5x + 7$ and calculate du:

$$u = 5x + 7, \quad du = 5\,dx$$

Since du contains the factor 5, the integral is not in the proper form $\int f(u)\,du$ required by (5.7). However, we can *introduce* the factor 5 into the integrand, provided we also multiply by $\frac{1}{5}$. Doing this and using (i) of Theorem (5.6) gives us

$$\begin{aligned}\int \sqrt{5x+7}\,dx &= \int \sqrt{5x+7}\,(\tfrac{1}{5})5\,dx\\ &= \tfrac{1}{5}\int \sqrt{5x+7}\,5\,dx.\end{aligned}$$

We now substitute and use the power rule for integration:

$$\begin{aligned}\int \sqrt{5x+7}\,dx &= \tfrac{1}{5}\int \sqrt{u}\,du\\ &= \tfrac{1}{5}\int u^{1/2}\,du\\ &= \tfrac{1}{5}\frac{u^{3/2}}{\frac{3}{2}} + C\\ &= \tfrac{2}{15}u^{3/2} + C\\ &= \tfrac{2}{15}(5x+7)^{3/2} + C\end{aligned}$$

In the future, after inserting a factor k into an integrand, as in Example 1, we shall simply multiply the integral by $1/k$, skipping the intermediate steps of first writing $(1/k)k$ and then bringing $1/k$ *outside*—that is, to the left of—the integral sign.

EXAMPLE 2 Evaluate $\int \cos 4x\,dx$.

SOLUTION We make the substitution

$$u = 4x, \quad du = 4\,dx.$$

Since du contains the factor 4, we adjust the integrand by multiplying by 4 and compensate by multiplying the integral by $\frac{1}{4}$ before substituting:

$$\begin{aligned}\int \cos 4x\,dx &= \tfrac{1}{4}\int (\cos 4x)4\,dx\\ &= \tfrac{1}{4}\int \cos u\,du\\ &= \tfrac{1}{4}\sin u + C\\ &= \tfrac{1}{4}\sin 4x + C\end{aligned}$$

It is not always easy to decide what substitution $u = g(x)$ is needed to transform an indefinite integral into a form that can be readily evaluated.

It may be necessary to try several different possibilities before finding a suitable substitution. In most cases *no* substitution will simplify the integrand properly. The following guidelines may be helpful.

Guidelines for changing variables in indefinite integrals **(5.8)**

1. Decide on a reasonable substitution $u = g(x)$.
2. Calculate $du = g'(x)\,dx$.
3. Using 1 and 2, try to transform the integral into a form that involves only the variable u. If necessary, introduce a *constant* factor k into the integrand and compensate by multiplying the integral by $1/k$. If any part of the resulting integrand contains the variable x, use a different substitution in 1.
4. Evaluate the integral obtained in 3, obtaining an antiderivative involving u.
5. Replace u in the antiderivative obtained in guideline 4 by $g(x)$. The final result should contain only the variable x.

The following examples illustrate the use of the guidelines.

EXAMPLE 3 Evaluate $\int (2x^3 + 1)^7 x^2\,dx$.

SOLUTION If an integrand involves an expression raised to a power, such as $(2x^3 + 1)^7$, we often substitute u for the expression. Thus, we let

$$u = 2x^3 + 1, \quad du = 6x^2\,dx.$$

Comparing $du = 6x^2\,dx$ with $x^2\,dx$ in the integral suggests that we introduce the factor 6 into the integrand. Doing this and compensating by multiplying the integral by $\frac{1}{6}$, we obtain the following:

$$\begin{aligned}\int (2x^3 + 1)^7 x^2\,dx &= \tfrac{1}{6}\int (2x^3 + 1)^7 6x^2\,dx \\ &= \tfrac{1}{6}\int u^7\,du \\ &= \frac{1}{6}\left(\frac{u^8}{8}\right) + C \\ &= \tfrac{1}{48}(2x^3 + 1)^8 + C\end{aligned}$$

A substitution in an indefinite integral can sometimes be made in several different ways. To illustrate, another method for evaluating the integral in Example 3 is to consider

$$u = 2x^3 + 1, \quad du = 6x^2\,dx, \quad \tfrac{1}{6}\,du = x^2\,dx.$$

We then substitute $\frac{1}{6}\,du$ for $x^2\,dx$,

$$\int (2x^3 + 1)^7 x^2\,dx = \int u^7 \tfrac{1}{6}\,du = \tfrac{1}{6}\int u^7\,du,$$

and integrate as before.

EXAMPLE 4 Evaluate $\int x\sqrt[3]{7-6x^2}\,dx$.

SOLUTION Note that the integrand contains the term $x\,dx$. If the factor x were missing or if x were raised to a higher power, the problem would be more complicated. For integrands that involve a radical, we often substitute for the expression under the radical sign. Thus, we let

$$u = 7 - 6x^2, \quad du = -12x\,dx.$$

Next we introduce the factor -12 into the integrand, compensate by multiplying the integral by $-\frac{1}{12}$, and proceed as follows:

$$\begin{aligned}\int x\sqrt[3]{7-6x^2}\,dx &= -\tfrac{1}{12}\int \sqrt[3]{7-6x^2}\,(-12)x\,dx\\ &= -\tfrac{1}{12}\int \sqrt[3]{u}\,du = -\tfrac{1}{12}\int u^{1/3}\,du\\ &= -\frac{1}{12}\left(\frac{u^{4/3}}{4/3}\right) + C = -\frac{1}{16}u^{4/3} + C\\ &= -\tfrac{1}{16}(7-6x^2)^{4/3} + C\end{aligned}$$

We could also have written

$$u = 7 - 6x^2 \quad du = -12x\,dx, \quad -\tfrac{1}{12}\,du = x\,dx$$

and substituted directly for $x\,dx$. Thus,

$$\int \sqrt[3]{7-6x^2}\,x\,dx = \int \sqrt[3]{u}\left(-\tfrac{1}{12}\right)du = -\tfrac{1}{12}\int \sqrt[3]{u}\,du.$$

The remainder of the solution would proceed exactly as before.

EXAMPLE 5 Evaluate $\displaystyle\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx$.

SOLUTION Let

$$u = x^3 - 3x + 1, \quad du = (3x^2 - 3)\,dx = 3(x^2-1)\,dx$$

and proceed as follows:

$$\begin{aligned}\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx &= \frac{1}{3}\int \frac{3(x^2-1)}{(x^3-3x+1)^6}\,dx\\ &= \frac{1}{3}\int \frac{1}{u^6}\,du = \frac{1}{3}\int u^{-6}\,du\\ &= \frac{1}{3}\left(\frac{u^{-5}}{-5}\right) + C = -\frac{1}{15}\left(\frac{1}{u^5}\right) + C\\ &= -\frac{1}{15}\,\frac{1}{(x^3-3x+1)^5} + C\end{aligned}$$

EXAMPLE 6 Evaluate $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.

SOLUTION We wish to use the formula $\int \cos u\, du = \sin u + C$, so let us make the substitution

$$u = \sqrt{x} = x^{1/2}, \quad du = \frac{1}{2}x^{-1/2}\, dx = \frac{1}{2\sqrt{x}}\, dx.$$

If we introduce the factor $\frac{1}{2}$ into the integrand and compensate by multiplying the integral by 2, we obtain

$$\begin{aligned}\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx &= 2\int \cos\sqrt{x}\left(\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\right) dx \\ &= 2\int \cos u\, du = 2\sin u + C \\ &= 2\sin\sqrt{x} + C.\end{aligned}$$

EXAMPLE 7 Evaluate $\int \cos^3 5x \sin 5x\, dx$.

SOLUTION The form of the integrand suggests that we use the power rule $\int u^3\, du = \frac{1}{4}u^4 + C$. Thus, we let

$$u = \cos 5x, \quad du = -5 \sin 5x\, dx.$$

The form of du indicates that we should introduce the factor -5 into the integrand, multiply the integral by $-\frac{1}{5}$, and then integrate as follows:

$$\begin{aligned}\int \cos^3 5x \sin 5x\, dx &= -\tfrac{1}{5}\int \cos^3 5x(-5\sin 5x)\, dx \\ &= -\tfrac{1}{5}\int u^3\, du \\ &= -\frac{1}{5}\left(\frac{u^4}{4}\right) + C \\ &= -\tfrac{1}{20}\cos^4 5x + C\end{aligned}$$

EXERCISES 5.2

Exer. 1–8: Evaluate the integral using the given substitution, and express the answer in terms of x.

1 $\int x(2x^2+3)^{10}\, dx; \quad u = 2x^2 + 3$

2 $\int \frac{x}{(x^2+5)^3}\, dx; \quad u = x^2 + 5$

3 $\int x^2\sqrt[3]{3x^3+7}\, dx; \quad u = 3x^3 + 7$

4 $\int \frac{5x}{\sqrt{x^2-3}}\, dx; \quad u = x^2 - 3$

5 $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\, dx; \quad u = 1 + \sqrt{x}$

6 $\int \frac{1}{(5x-4)^{10}}\, dx; \quad u = 5x - 4$

7 $\int \sqrt{x}\cos\sqrt{x^3}\, dx; \quad u = x^{3/2}$

8 $\int \tan x \sec^2 x\, dx; \quad u = \tan x$

Exer. 9–48: Evaluate the integral.

9 $\int \sqrt{3x-2}\, dx$

10 $\int \sqrt[4]{2x+5}\, dx$

11 $\int \sqrt[3]{8t+5}\, dt$

12 $\int \frac{1}{\sqrt{4-5t}}\, dt$

13 $\int (3z+1)^4\, dz$

14 $\int (2z^2-3)^5 z\, dz$

15 $\int v^2\sqrt{v^3-1}\, dv$

16 $\int v\sqrt{9-v^2}\, dv$

17 $\int \frac{x}{\sqrt[3]{1-2x^2}}\, dx$

18 $\int (3-x^4)^3 x^3\, dx$

19 $\int (s^2 + 1)^2 \, ds$

20 $\int (3 - s^3)^2 \, s \, ds$

21 $\int \frac{(\sqrt{x} + 3)^4}{\sqrt{x}} \, dx$

22 $\int \left(1 + \frac{1}{x}\right)^{-3} \left(\frac{1}{x^2}\right) dx$

23 $\int \frac{t - 2}{(t^2 - 4t + 3)^3} \, dt$

24 $\int \frac{t^2 + t}{(4 - 3t^2 - 2t^3)^4} \, dt$

25 $\int 3 \sin 4x \, dx$

26 $\int 4 \cos \frac{1}{2}x \, dx$

27 $\int \cos (4x - 3) \, dx$

28 $\int \sin (1 + 6x) \, dx$

29 $\int v \sin (v^2) \, dv$

30 $\int \frac{\cos \sqrt[3]{v}}{\sqrt[3]{v^2}} \, dv$

31 $\int \cos 3x \sqrt[3]{\sin 3x} \, dx$

32 $\int \frac{\sin 2x}{\sqrt{1 - \cos 2x}} \, dx$

33 $\int (\sin x + \cos x)^2 \, dx$ (*Hint:* $\sin 2\theta = 2 \sin \theta \cos \theta$)

34 $\int \frac{\sin 4x}{\cos 2x} \, dx$ (*Hint:* $\sin 2\theta = 2 \sin \theta \cos \theta$)

35 $\int \sin x(1 + \cos x)^2 \, dx$

36 $\int \sin^3 x \cos x \, dx$

37 $\int \frac{\sin x}{\cos^4 x} \, dx$

38 $\int \sin 2x \sec^5 2x \, dx$

39 $\int \frac{\cos t}{(1 - \sin t)^2} \, dt$

40 $\int (2 + 5 \cos t)^3 \sin t \, dt$

41 $\int \sec^2 (3x - 4) \, dx$

42 $\int \frac{\csc 2x}{\sin 2x} \, dx$

43 $\int \sec^2 3x \tan 3x \, dx$

44 $\int \frac{1}{\tan 4x \sin 4x} \, dx$

45 $\int \frac{1}{\sin^2 5x} \, dx$

46 $\int \frac{x}{\cos^2 (x^2)} \, dx$

47 $\int x \cot (x^2) \csc (x^2) \, dx$

48 $\int \sec \left(\frac{x}{3}\right) \tan \left(\frac{x}{3}\right) dx$

Exer. 49–52: Solve the differential equation subject to the given conditions.

49 $f'(x) = \sqrt[3]{3x + 2}$; $\quad f(2) = 9$

50 $\frac{dy}{dx} = x\sqrt{x^2 + 5}$; $\quad y = 12$ if $x = 2$

51 $f''(x) = 16 \cos 2x - 3 \sin x$; $\quad f(0) = -2$; $\quad f'(0) = 4$

52 $f''(x) = 4 \sin 2x + 16 \cos 4x$; $\quad f(0) = 6$; $\quad f'(0) = 1$

Exer. 53–56: Evaluate the integral by (a) the method of substitution and (b) expanding the integrand. In what way do the constants of integration differ?

53 $\int (x + 4)^2 \, dx$

54 $\int (x^2 + 4)^2 x \, dx$

55 $\int \frac{(\sqrt{x} + 3)^2}{\sqrt{x}} \, dx$

56 $\int \left(1 + \frac{1}{x}\right)^2 \frac{1}{x^2} \, dx$

57 A charged particle is moving on a coordinate line in a magnetic field such that its velocity (in cm/sec) at time t is given by $v(t) = \frac{1}{2} \sin (3t - \frac{1}{4}\pi)$. Show that the motion is simple harmonic (see page 223).

58 The acceleration of a particle that is moving on a coordinate line is given by $a(t) = k \cos (\omega t + \phi)$ for constants k, ω, and ϕ and time t (in seconds). Show that the motion is simple harmonic (see page 223).

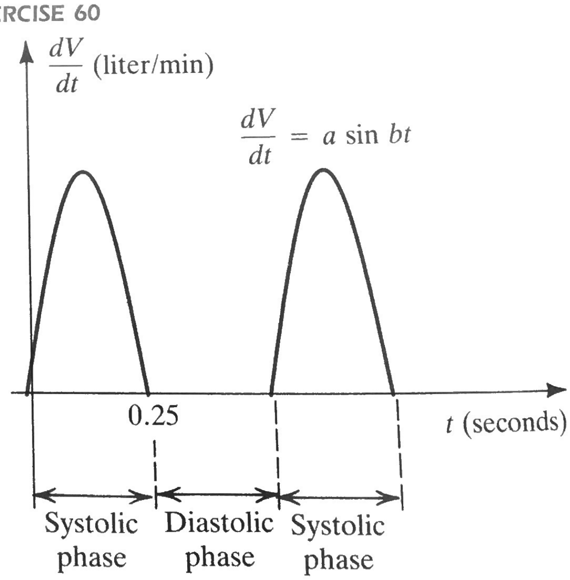
59 A reservoir supplies water to a community. In summer the demand A for water (in ft^3/day) changes according to the formula $dA/dt = 4000 + 2000 \sin (\frac{1}{90}\pi t)$ for time t (in days), with $t = 0$ corresponding to the beginning of summer. Estimate the total water consumption during 90 days of summer.

60 The pumping action of the heart consists of the systolic phase, in which blood rushes from the left ventricle into the aorta, and the diastolic phase, during which the heart muscle relaxes. The graph shown in the figure is sometimes used to model one complete cycle of the process. For a particular individual, the systolic phase lasts $\frac{1}{4}$ second and has a maximum flow rate dV/dt of 8 L/min, where V is the volume of blood in the heart at time t.

(a) Show that $dV/dt = 8 \sin (240\pi t)$ L/min.

(b) Estimate the total amount of blood pumped into the aorta during a systolic phase.

EXERCISE 60



61 The rhythmic process of breathing consists of alternating periods of inhaling and exhaling. For an adult, one complete cycle normally takes place every 5 seconds. If V denotes the volume of air in the lungs at time t, then dV/dt is the flow rate.