(a) If the maximum flow rate is 0.6 L/sec, find a formula $dV/dt = a \sin bt$ that fits the given information.

(b) Use part (a) to estimate the amount of air inhaled during one cycle.

62 Many animal populations fluctuate over 10-year cycles. Suppose that the rate of growth of a rabbit population is given by $dN/dt = 1000 \cos(\frac{1}{5}\pi t)$ rabbits/yr, where N denotes the number in the population at time t (in years) and $t = 0$ corresponds to the beginning of a cycle. If the population after 5 years is estimated to be 3000 rabbits, find a formula for N at time t and estimate the maximum population.

63 Show, by evaluating in three different ways, that

$$\begin{aligned}\int \sin x \cos x \, dx &= \tfrac{1}{2}\sin^2 x + C\\ &= -\tfrac{1}{2}\cos^2 x + D\\ &= -\tfrac{1}{4}\cos 2x + E.\end{aligned}$$

How can all three answers be correct?

5.3 SUMMATION NOTATION AND AREA

In this section we shall lay the foundation for the definition of the *definite integral*. At the outset, it is virtually impossible to see any connection between definite integrals and indefinite integrals. In Section 5.6, however, we show that there is a very close relationship: *Indefinite integrals can be used to evaluate definite integrals.*

In our development of the definite integral we shall employ sums of many numbers. To express such sums compactly, it is convenient to use **summation notation**. Given a collection of numbers $\{a_1, a_2, \ldots, a_n\}$, the symbol $\sum_{k=1}^{n} a_k$ represents their sum as follows.

Summation notation **(5.9)**

$$\boxed{\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n}$$

The Greek capital letter Σ (sigma) indicates a sum, and a_k represents the kth term of the sum. The letter k is the **index of summation**, or the **summation variable**, and assumes successive integer values. The integers 1 and n indicate the extreme values of the summation variable.

EXAMPLE 1 Evaluate $\sum_{k=1}^{4} k^2(k-3)$.

SOLUTION Comparing the sum with (5.9), we see that $a_k = k^2(k-3)$ and $n = 4$. To find the sum, we substitute 1, 2, 3, and 4 for k and add the resulting terms. Thus,

$$\begin{aligned}\sum_{k=1}^{4} k^2(k-3) &= 1^2(1-3) + 2^2(2-3) + 3^2(3-3) + 4^2(4-3)\\ &= (-2) + (-4) + 0 + 16 = 10.\end{aligned}$$

Letters other than k can be used for the summation variable. To illustrate,

$$\sum_{k=1}^{4} k^2(k-3) = \sum_{i=1}^{4} i^2(i-3) = \sum_{j=1}^{4} j^2(j-3) = 10.$$

If $a_k = c$ for every k, then

$$\sum_{k=1}^{2} a_k = a_1 + a_2 = c + c = 2c = \sum_{k=1}^{2} c$$

$$\sum_{k=1}^{3} a_k = a_1 + a_2 + a_3 = c + c + c = 3c = \sum_{k=1}^{3} c.$$

In general, the following result is true for every positive integer n.

Theorem **(5.10)**

$$\boxed{\sum_{k=1}^{n} c = nc}$$

The domain of the summation variable does not have to begin at 1. For example,

$$\sum_{k=4}^{8} a_k = a_4 + a_5 + a_6 + a_7 + a_8.$$

EXAMPLE 2 Evaluate $\displaystyle\sum_{k=0}^{3} \frac{2^k}{(k+1)}$.

SOLUTION

$$\sum_{k=0}^{3} \frac{2^k}{(k+1)} = \frac{2^0}{(0+1)} + \frac{2^1}{(1+1)} + \frac{2^2}{(2+1)} + \frac{2^3}{(3+1)}$$
$$= 1 + 1 + \tfrac{4}{3} + 2 = \tfrac{16}{3}$$

Theorem **(5.11)**

> If n is any positive integer and $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are sets of real numbers, then
>
> **(i)** $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
>
> **(ii)** $\displaystyle\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$ for every real number c
>
> **(iii)** $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

PROOF To prove (i), we begin with

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n).$$

Rearranging terms on the right we obtain

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$$
$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

For (ii),

$$\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + ca_3 + \cdots + ca_n$$

$$= c(a_1 + a_2 + a_3 + \cdots + a_n) = c\left(\sum_{k=1}^{n} a_k\right).$$

To prove (iii), write $a_k - b_k = a_k + (-1)b_k$ and use (i) and (ii). ■

The formulas in the following theorem will be useful later in this section. They may be proved by mathematical induction (see Appendix I).

Theorem (5.12)

(i) $\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

(ii) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

EXAMPLE 3 Evaluate $\sum_{k=1}^{100} k$ and $\sum_{k=1}^{20} k^2$.

SOLUTION Using (i) and (ii) of Theorem (5.12), we obtain

$$\sum_{k=1}^{100} k = 1 + 2 + \cdots + 100 = \frac{100(101)}{2} = 5050$$

and

$$\sum_{k=1}^{20} k^2 = 1^2 + 2^2 + \cdots + 20^2 = \frac{20(21)(41)}{6} = 2870.$$

EXAMPLE 4 Express $\sum_{k=1}^{n} (k^2 - 4k + 3)$ in terms of n.

SOLUTION We use Theorems (5.11), (5.12), and (5.10):

$$\sum_{k=1}^{n} (k^2 - 4k + 3) = \sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3$$

$$= \frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + 3n$$

$$= \tfrac{1}{3}n^3 - \tfrac{3}{2}n^2 + \tfrac{7}{6}n$$

The definition of the definite integral (to be given in Section 5.4) is closely related to the areas of certain regions in a coordinate plane. We can easily calculate the area if the region is bounded by lines. For example, the area of a rectangle is the product of its length and width. The area

of a triangle is one-half the product of an altitude and the corresponding base. The area of any polygon can be found by subdividing it into triangles.

In order to find areas of regions whose boundaries involve graphs of functions, however, we utilize a limiting process and then use methods of calculus. In particular, let us consider a region R in a coordinate plane, bounded by the vertical lines $x = a$ and $x = b$, by the x-axis, and by the graph of a function f that is continuous and nonnegative on the closed interval $[a, b]$. A region of this type is illustrated in Figure 5.3. Since $f(x) \geq 0$ for every x in $[a, b]$, no part of the graph lies below the x-axis. For convenience, we shall refer to R as **the region under the graph of f from a to b**. We wish to define the area A of R.

FIGURE 5.3 Region under the graph of f

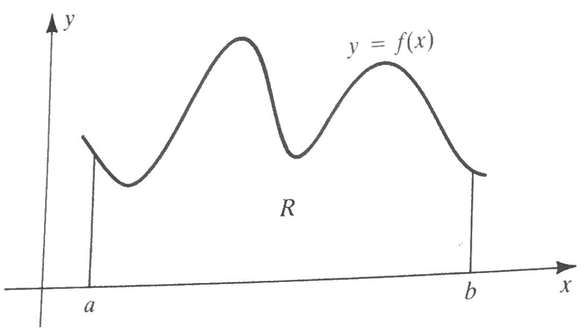
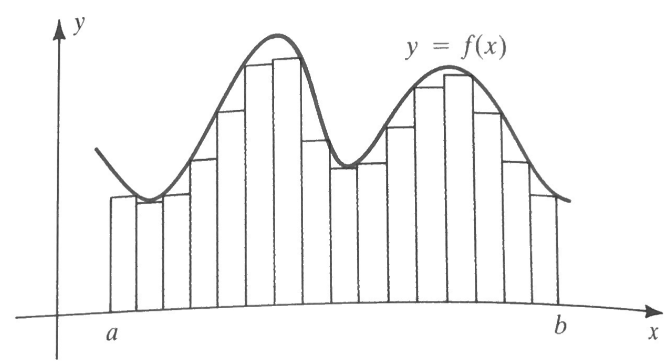


FIGURE 5.4 An inscribed rectangular polygon



To arrive at a satisfactory definition of A, we shall consider many rectangles of equal width such that each rectangle lies completely under the graph of f and intersects the graph in at least one point, as illustrated in Figure 5.4. The boundary of the region formed by the totality of these rectangles is called an **inscribed rectangular polygon**. We shall use the following notation:

$$A_{\text{IP}} = \text{area of an inscribed rectangular polygon}$$

If the width of the rectangles in Figure 5.4 is small, then it appears that

$$A_{\text{IP}} \approx A.$$

This suggests that we let the width of the rectangles approach zero and define A as a limiting value of the areas A_{IP} of the corresponding inscribed rectangular polygons. The notation discussed next will allow us to carry out this procedure rigorously.

If n is any positive integer, divide the interval $[a, b]$ into n subintervals, all having the same length $\Delta x = (b - a)/n$. We can do this by choosing numbers $x_0, x_1, x_2, \ldots, x_n$, with $a = x_0$, $b = x_n$, and

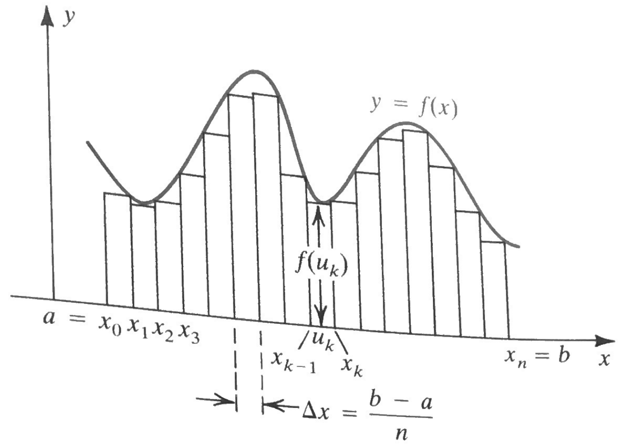
$$x_k - x_{k-1} = \frac{b-a}{n} = \Delta x$$

for $k = 1, 2, \ldots, n$, as indicated in Figure 5.5. Note that

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\,\Delta x, \quad x_3 = a + 3\,\Delta x, \quad \ldots$$

$$x_k = a + k\,\Delta x, \quad \ldots, \quad x_n = a + n\,\Delta x = b.$$

FIGURE 5.5



The function f is continuous on each subinterval $[x_{k-1}, x_k]$, and hence, by the extreme value theorem (4.3), f takes on a minimum value at some number u_k in $[x_{k-1}, x_k]$. For each k, let us construct a rectangle of width $\Delta x = x_k - x_{k-1}$ and height equal to the minimum distance $f(u_k)$ from the x-axis to the graph of f (see Figure 5.5). The area of the kth rectangle is $f(u_k)\,\Delta x$. The area A_{IP} of the resulting inscribed rectangular polygon is the sum of the areas of the n rectangles; that is,

$$A_{\text{IP}} = f(u_1)\,\Delta x + f(u_2)\,\Delta x + \cdots + f(u_n)\,\Delta x.$$

Using summation notation, we may write

$$A_{\text{IP}} = \sum_{k=1}^{n} f(u_k)\,\Delta x,$$

where $f(u_k)$ is the minimum value of f on $[x_{k-1}, x_k]$.

If n is very large, or, equivalently, if Δx is very small, then the sum A_{IP} of the rectangular areas should approximate the area of the region R. Intuitively we know that if there exists a number A such that $\sum_{k=1}^{n} f(u_k)\,\Delta x$ gets closer to A as Δx gets closer to 0 (but $\Delta x \neq 0$), we can call A the **area** of R and write

$$A = \lim_{\Delta x \to 0} A_{\text{IP}} = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(u_k)\,\Delta x.$$

The meaning of this *limit of sums* is not the same as that of the limit of a function, introduced in Chapter 2. To eliminate the word *closer* and arrive at a satisfactory definition of A, let us take a slightly different point of view. If A denotes the area of the region R, then the difference

$$A - \sum_{k=1}^{n} f(u_k)\,\Delta x$$

is the area of the portion in Figure 5.5 that lies *under* the graph of f and *over* the inscribed rectangular polygon. This number may be regarded as the error in using the area of the inscribed rectangular polygon to approximate A. We should be able to make this error as small as desired by choosing the width Δx of the rectangles sufficiently small. This is the motivation for the following definition of the area A of R. The notation is the same as that used in the preceding discussion.

Definition (5.13)

Let f be continuous and nonnegative on $[a, b]$. Let A be a real number, and let $f(u_k)$ be the minimum value of f on $[x_{k-1}, x_k]$. The notation

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(u_k)\,\Delta x$$

means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < \Delta x < \delta$, then

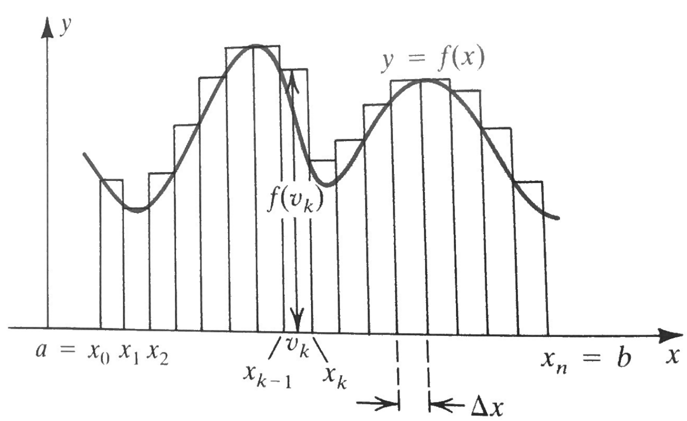
$$A - \sum_{k=1}^{n} f(u_k)\,\Delta x < \epsilon.$$

If A is the indicated limit and we let $\epsilon = 10^{-9}$, then Definition (5.13) states that by using rectangles of sufficiently small width Δx, we can make the difference between A and the area of the inscribed polygon less than one-billionth of a square unit. Similarly, if $\epsilon = 10^{-12}$, we can make this difference less than one-trillionth of a square unit. In general, the difference can be made less than *any* preassigned ϵ.

If f is continuous on $[a, b]$, it is shown in more advanced texts that a number A satisfying Definition (5.13) actually exists. We shall call A **the area under the graph of f from a to b**.

The area A may also be obtained by means of **circumscribed rectangular polygons** of the type illustrated in Figure 5.6. In this case we select the number v_k in each interval $[x_{k-1}, x_k]$ such that $f(v_k)$ is the *maximum* value of f on $[x_{k-1}, x_k]$.

FIGURE 5.6 A circumscribed rectangular polygon



Let

$$A_{\text{CP}} = \text{area of a circumscribed rectangular polygon.}$$

Using summation notation, we have

$$A_{\text{CP}} = \sum_{k=1}^{n} f(v_k)\,\Delta x,$$

where $f(v_k)$ is the maximum value of f on $[x_{k-1}, x_k]$. Note that

$$\sum_{k=1}^{n} f(u_k)\,\Delta x \le A \le \sum_{k=1}^{n} f(v_k)\,\Delta x.$$

The limit of A_{CP} as $\Delta x \to 0$ is defined as in (5.13). The only change is that we use

$$\sum_{k=1}^{n} f(v_k)\,\Delta x - A < \epsilon,$$

since we want this difference to be nonnegative. It can be proved that the same number A is obtained using either inscribed or circumscribed rectangles.

EXAMPLE 5 Let $f(x) = 16 - x^2$, and let R be the region under the graph of f from 0 to 3. Approximate the area A of R using

(a) an inscribed rectangular polygon with $\Delta x = \frac{1}{2}$

(b) a circumscribed rectangular polygon with $\Delta x = \frac{1}{2}$

FIGURE 5.7

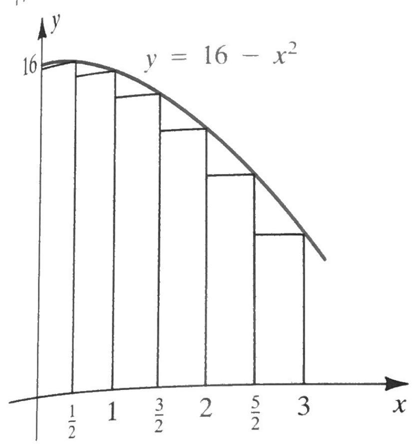
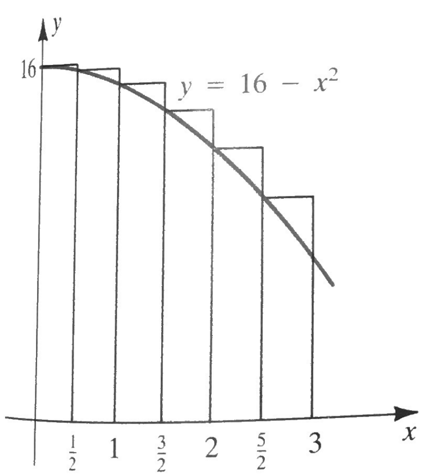


FIGURE 5.8



SOLUTION

(a) The graph of f and the inscribed rectangular polygon with $\Delta x = \frac{1}{2}$ are sketched in Figure 5.7 (with different scales on the x- and y-axes). Note that f is decreasing on $[0, 3]$, and hence the minimum value $f(u_k)$ on the kth subinterval occurs at the right-hand endpoint of the subinterval. Since there are six rectangles to consider, the formula for A_{IP} is

$$\begin{aligned}
A_{IP} &= \sum_{k=1}^{6} f(u_k)\,\Delta x \\
&= f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} + f(3)\cdot\tfrac{1}{2} \\
&= \tfrac{63}{4}\cdot\tfrac{1}{2} + 15\cdot\tfrac{1}{2} + \tfrac{55}{4}\cdot\tfrac{1}{2} + 12\cdot\tfrac{1}{2} + \tfrac{39}{4}\cdot\tfrac{1}{2} + 7\cdot\tfrac{1}{2} \\
&= \tfrac{293}{8} = 36.625.
\end{aligned}$$

(b) The graph of f and the circumscribed rectangular polygon are sketched in Figure 5.8. Since f is decreasing on $[0, 3]$, the maximum value $f(v_k)$ occurs at the left-hand endpoint of the kth subinterval. Hence

$$\begin{aligned}
A_{CP} &= \sum_{k=1}^{6} f(v_k)\,\Delta x \\
&= f(0)\cdot\tfrac{1}{2} + f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} \\
&= 16\cdot\tfrac{1}{2} + \tfrac{63}{4}\cdot\tfrac{1}{2} + 15\cdot\tfrac{1}{2} + \tfrac{55}{4}\cdot\tfrac{1}{2} + 12\cdot\tfrac{1}{2} + \tfrac{39}{4}\cdot\tfrac{1}{2} \\
&= \tfrac{329}{8} = 41.125.
\end{aligned}$$

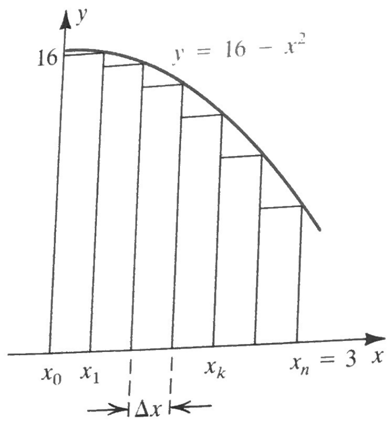
It follows that $36.625 < A < 41.125$. In the next example we prove that $A = 39$.

EXAMPLE 6 If $f(x) = 16 - x^2$, find the area of the region under the graph of f from 0 to 3.

SOLUTION The region was considered in Example 5 and is resketched in Figure 5.9, on the following page. If the interval $[0, 3]$ is divided into n equal subintervals, then the length Δx of each subinterval is $3/n$. Employing the notation used in Figure 5.5, with $a = 0$ and $b = 3$, we have

$$x_0 = 0, \quad x_1 = \Delta x, \quad x_2 = 2\,\Delta x, \quad \ldots, \quad x_k = k\,\Delta x, \quad \ldots, \quad x_n = n\,\Delta x = 3.$$

FIGURE 5.9



Since $\Delta x = 3/n$,

$$x_k = k\,\Delta x = k\frac{3}{n} = \frac{3k}{n}.$$

Since f is decreasing on $[0, 3]$, the number u_k in $[x_{k-1}, x_k]$ at which f takes on its minimum value is always the right-hand endpoint x_k of the subinterval; that is, $u_k = x_k = 3k/n$. Thus,

$$f(u_k) = f\left(\frac{3k}{n}\right) = 16 - \left(\frac{3k}{n}\right)^2 = 16 - \frac{9k^2}{n^2},$$

and the summation in Definition (5.13) is

$$\begin{aligned}\sum_{k=1}^{n} f(u_k)\,\Delta x &= \sum_{k=1}^{n}\left[\left(16 - \frac{9k^2}{n^2}\right)\cdot\frac{3}{n}\right]\\ &= \frac{3}{n}\sum_{k=1}^{n}\left(16 - \frac{9k^2}{n^2}\right),\end{aligned}$$

where the last equality follows from (ii) of Theorem (5.11). (Note that $3/n$ does not contain the summation variable k.) We next use Theorems (5.11), (5.10), and (5.12) to obtain

$$\begin{aligned}\sum_{k=1}^{n} f(u_k)\,\Delta x &= \frac{3}{n}\left(\sum_{k=1}^{n} 16 - \frac{9}{n^2}\sum_{k=1}^{n} k^2\right)\\ &= \frac{3}{n}\left[n \cdot 16 - \frac{9}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right]\\ &= 48 - \frac{9}{2}\,\frac{(n+1)(2n+1)}{n^2}.\end{aligned}$$

To find the area of the region, we let Δx approach 0. Since $\Delta x = 3/n$, we can accomplish this by letting n increase without bound. Although our discussion of limits involving infinity in Section 2.4 was concerned with a real variable x, a similar discussion can be given if the variable is an integer n. Assuming that this is true and that we can replace $\Delta x \to 0$ by $n \to \infty$, we obtain

$$\begin{aligned}\lim_{\Delta x \to 0}\sum_{k=1}^{n} f(u_k)\,\Delta x &= \lim_{n\to\infty}\left[48 - \frac{9}{2}\,\frac{(n+1)(2n+1)}{n^2}\right]\\ &= 48 - \tfrac{9}{2}\cdot 2 = 39.\end{aligned}$$

Thus, the area of the region is 39 square units.

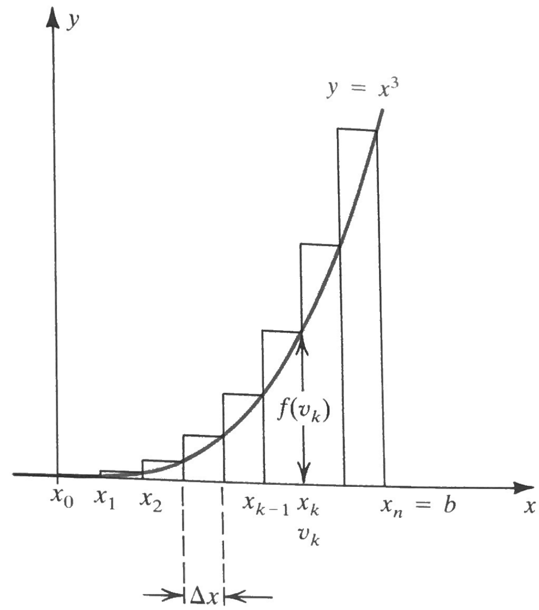
The area in the preceding example may also be found by using circumscribed rectangular polygons. In this case we select, in each subinterval $[x_{k-1}, x_k]$, the number $v_k = (k-1)(3/n)$ at which f takes on its maximum value.

The next example illustrates the use of circumscribed rectangles in finding an area.

EXAMPLE 7 If $f(x) = x^3$, find the area under the graph of f from 0 to b for any $b > 0$.

SOLUTION Subdividing the interval $[0, b]$ into n equal parts (see Figure 5.10), we obtain a circumscribed rectangular polygon such that $\Delta x = b/n$ and $x_k = k\,\Delta x$.

FIGURE 5.10



Since f is an increasing function, the maximum value $f(v_k)$ in the interval $[x_{k-1}, x_k]$ occurs at the right-hand endpoint; that is,

$$v_k = x_k = k\,\Delta x = k\frac{b}{n} = \frac{bk}{n}.$$

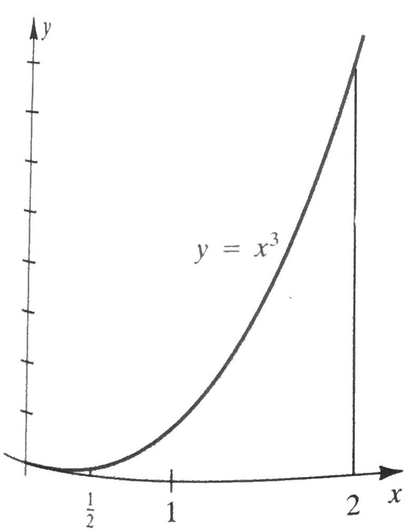
The sum of the areas of the circumscribed rectangles is

$$\begin{aligned}\sum_{k=1}^{n} f(v_k)\,\Delta x &= \sum_{k=1}^{n}\left[\left(\frac{bk}{n}\right)^3\cdot\frac{b}{n}\right] = \sum_{k=1}^{n}\frac{b^4}{n^4}k^3\\ &= \frac{b^4}{n^4}\sum_{k=1}^{n}k^3 = \frac{b^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2\\ &= \frac{b^4}{4}\cdot\frac{n^2(n+1)^2}{n^4},\end{aligned}$$

where we have used Theorem (5.12)(iii). If we let Δx approach 0, then n increases without bound and the expression involving n approaches 1. It follows that the area under the graph is

$$\lim_{\Delta x\to 0}\sum_{k=1}^{n} f(v_k)\,\Delta x = \frac{b^4}{4}.$$

FIGURE 5.11



EXAMPLE 8 If $f(x) = x^3$, find the area A of the region under the graph of f from $\frac{1}{2}$ to 2.

SOLUTION The region is sketched in Figure 5.11.

If we let A_1 = area under the graph of f from 0 to $\frac{1}{2}$

and A_2 = area under the graph of f from 0 to 2,

the area A can be found by subtracting A_1 from A_2:

$$A = A_2 - A_1$$

In Example 7 we found that the area under the graph of $y = x^3$ from 0 to b is $\frac{1}{4}b^4$. Hence, using $b = \frac{1}{2}$ for A_1 and $b = 2$ for A_2 yields

$$A = \frac{2^4}{4} - \frac{(\frac{1}{2})^4}{4} = 4 - \frac{1}{64} \approx 3.98.$$

EXERCISES 5.3

Exer. 1–8: Evaluate the sum.

1 $\sum_{j=1}^{4} (j^2 + 1)$

2 $\sum_{j=1}^{4} (2^j + 1)$

3 $\sum_{k=0}^{5} k(k - 1)$

4 $\sum_{k=0}^{4} (k - 2)(k - 3)$

5 $\sum_{n=1}^{10} [1 + (-1)^n]$

6 $\sum_{n=1}^{4} (-1)^n\left(\frac{1}{n}\right)$

7 $\sum_{i=1}^{50} 10$

8 $\sum_{k=1}^{1000} 2$

Exer. 9–12: Express the sum in terms of n (see Example 4).

9 $\sum_{k=1}^{n} (k^2 + 3k + 5)$

10 $\sum_{k=1}^{n} (3k^2 - 2k + 1)$

11 $\sum_{k=1}^{n} (k^3 + 2k^2 - k + 4)$

12 $\sum_{k=1}^{n} (3k^3 + k)$

Exer. 13–18: Express in summation notation.

13 $1 + 5 + 9 + 13 + 17$

14 $2 + 5 + 8 + 11 + 14$

15 $\frac{1}{2} + \frac{2}{5} + \frac{3}{8} + \frac{4}{11}$

16 $\frac{1}{4} + \frac{2}{9} + \frac{3}{14} + \frac{4}{19}$

17 $1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots + (-1)^n \frac{x^{2n}}{2n}$

18 $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}$

Exer. 19–22: Let A be the area under the graph of the given function f from a to b. Approximate A by dividing $[a, b]$ into subintervals of equal length Δx and using (a) A_{IP} and (b) A_{CP}.

19 $f(x) = 3 - x$; $a = -2$, $b = 2$; $\Delta x = 1$

20 $f(x) = x + 2$; $a = -1$, $b = 4$; $\Delta x = 1$

21 $f(x) = x^2 + 1$; $a = 1$, $b = 3$; $\Delta x = \frac{1}{2}$

22 $f(x) = 4 - x^2$; $a = 0$, $b = 2$; $\Delta x = \frac{1}{2}$

c 23 $f(x) = \sqrt{\sin x}$; $a = 0$, $b = 1.5$; $\Delta x = 0.15$

c 24 $f(x) = \frac{1}{\sqrt{x^3 + 1}}$; $a = 0$, $b = 3$; $\Delta x = 0.3$

Exer. 25–30: Refer to Examples 6 and 7. Find the area under the graph of the given function f from 0 to b using (a) inscribed rectangles and (b) circumscribed rectangles.

25 $f(x) = 2x + 3$; $b = 4$

26 $f(x) = 8 - 3x$; $b = 2$

27 $f(x) = 9 - x^2$; $b = 3$

28 $f(x) = x^2$; $b = 5$

29 $f(x) = x^3 + 1$; $b = 2$

30 $f(x) = 4x + x^3$; $b = 2$

Exer. 31–32: Refer to Example 7. Find the area under the graph of f corresponding to the interval (a) [1, 3] and (b) $[a, b]$.

31 $f(x) = x^3$

32 $f(x) = x^3 + 2$

5.4 THE DEFINITE INTEGRAL

In Section 5.3 we defined the area under the graph of a function f from a to b as a limit of the form

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(w_k)\, \Delta x.$$