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      1. **Signal Power and Energy**

The total energy over the time internal in a continuous-time signal is defined as

|  |  |
| --- | --- |
|  | (1.4) |

where denotes the magnitude of the (possibly complex) number .

The time averaged power is given by

|  |  |
| --- | --- |
|  | (1.4-1) |

Similarly, the total energy in a discrete-time signal over the time interval is defined as

|  |  |
| --- | --- |
|  | (1.5) |

The average power over the interval in this case is given by

|  |  |
| --- | --- |
|  | (1.5-1) |

It is interesting to find out the power and energy in a signal over an infinite time interval, i.e., for or for . In these cases the total energy and power carried out by the signal are given by:

For continuous-time (CT) case:

|  |  |  |
| --- | --- | --- |
| Total Energy |  | (1.6) |
| Total Average Power |  | (1.8) |

For discrete-time (DT) case:

|  |  |  |
| --- | --- | --- |
| Total Energy |  | (1.6) |
| Total Average Power |  | (1.8) |

Three important cases can be identified:

**Case 1: Signals with finite total energy, i.e., :**

Such a signal must have zero average power. For example, in continuous case, if , then

An example of a finite-energy signal is a signal that takes on the value of 1 for and 0 otherwise. In this case, and .

**Case 2: Signals with finite average power, i.e., :**

For example, consider the constant signal where This signal has infinite energy, as

However, the total average power is finite,

**Case 3: Signals with neither nor finite:**

A simple example of such a case could be . In this case both and are infinite.

* 1. **Transformations of the Independent Variable**

The ***transformation of a signal*** is one of the central concepts in the field of signals and systems.

We will focus on a very limited but important class of signal transformations that involves the *modifications of the independent variable, i.e., the time axis*.

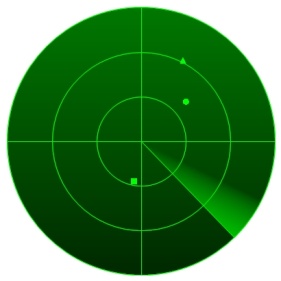
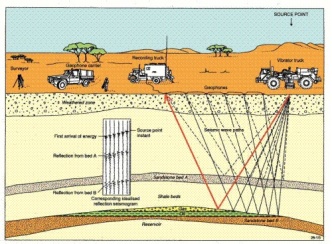
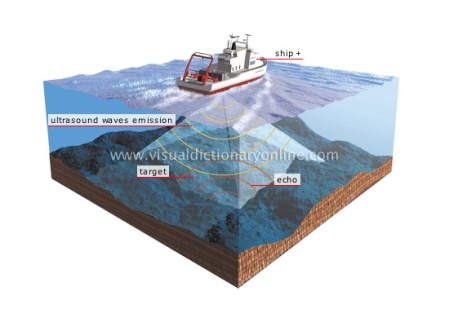
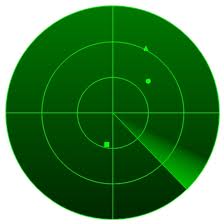
* + 1. **Examples of Transformations of the Independent Variable**

**Time Shift**

The original and the shifted signals are identical in shape, but are displaced or shifted along the time-axis with respect to each other. Signals could be termed as delayed or advanced in this case.

|  |  |  |
| --- | --- | --- |
| Signals | Discrete-Time (DT) | Continuous-Time (CT) |
| Original |  |  |
| Delayed |  |  |
| Advanced |  |  |

Such signals arise in applications such as radar, sonar and seismic signal processing. Several receivers placed at different locations receive the time shifted signals due to the transmission time they take while passing through a medium (air, water or rock etc.).

[](http://www.google.com/imgres?imgurl=http://www.codeproject.com/KB/grid/DrawingRadarDisplayWithCS/Radar1.jpg&imgrefurl=http://www.codeproject.com/KB/grid/DrawingRadarDisplayWithCS.aspx&usg=___QtZaiKQcbAW-GhpnO_S8jdAYic=&h=410&w=410&sz=57&hl=en&start=0&zoom=1&tbnid=ZemGDjRLXxCJvM:&tbnh=132&tbnw=132&ei=B2VmTbiyHsyWOvqF5JoL&prev=/images?q=radar&um=1&hl=en&safe=active&biw=1345&bih=583&tbs=isch:1&um=1&itbs=1&iact=hc&vpx=454&vpy=74&dur=6194&hovh=224&hovw=224&tx=133&ty=151&oei=B2VmTbiyHsyWOvqF5JoL&page=1&ndsp=21&ved=1t:429,r:2,s:0)

**Time Reversal (Reflection)**

In this case, the original signal is reflected about the time = 0. For example, if the original signal is some audio recording, then the time reversed signal would be the audio recording played backward.

|  |  |  |
| --- | --- | --- |
| Signals | Discrete-Time (DT) | Continuous-Time (CT) |
| Original |  |  |
| Time Reversed |  |  |

**Time Scaling**

In this case, if the original signal is , the time variable is multiplied with a constant to get a time-scaled signal, e.g., , or . If we think of the signal as audio recording, then is the audio recording played at twice the speed and is the recording played at half of the speed.

|  |  |
| --- | --- |
| Signals | Continuous-Time (CT) |
| Original |  |
| Stretching |  |
| Compressing |  |

**General Case of the Transformation of the Independent Variable**

A general case for the transformation of independent variable is the one in which for the original signal is changed to the form , where and are given numbers. It has the following effects on the original signal:

* The general shape of the signal is preserved.
* The signal is linearly stretched if .
* The signal is linearly compressed if .
* The signal is delayed (shifted in time) if .
* The signal is advanced (shifted in time) if .
* The signal is reversed in time (reflected) if .

**Example 1.1**

Given the signal as shown in the figure below, the signal can be obtained by shifting the given signal to the left by one unit (as depicted in the following figure).

Original Signal

Time Shifted Signal

The original signal can be written mathematically as:

To determine , the above definition of the original signal can be used. For example, to calculate the value of at , we determine , which according to the above definition is , therefore, at is equal to 1.

Similarly, the signal can be obtained using the mathematical definition or figure of the original signal . If we use the mathematical definition, then making the following table could be useful.

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  | .0 |  |

Plotting of this signal would give us the following figure:

Original Signal

**Time Reversed Signal**

In MATLAB®, the original signal can be written as an inline function. This function can then be used to plot the original signal, the shifted signal and the time-reversed signal using the following MATLAB® code.

**>> g = inline(' ((t>=0)&(t<1)) + (2-t).\*((t>=1) & (t<2))','t');**

**>> t = -3:0.001:3;**

**>> subplot(3,1,1), plot(t,g(t)), axis([-3 3 -0.1 1.1]), title('Original Signal')**

**>> subplot(3,1,2), plot(t,g(t+1)), axis([-3 3 -0.1 1.1]), title('Time-Shifted Signal')**

**>> subplot(3,1,3),plot(t,g(-t+1)),axis([-3 3 -0.1 1.1]), title('Time-Reversed Signal')**



* + 1. **Periodic Signals**

A periodic continuous-time signal is defined as

where is a positive number called the period.

A typical example is that of a sinusoidal signal for . This is shown in the figure below:

For the above signal, the period is **.** It can be noticed that for any time :

Also

where is an integer.

The ***fundamental period*** of is the smallest positive value of for which the equation holds.

Periodic signals are defined analogously in discrete-time.

A discrete-time signal is periodic with period , where is a positive integer, if it is unchanged by a time-shift of , i.e., if

for all values of .

The ***fundamental period*** of is the smallest positive value of for which the equation holds.

An example of a discrete-time periodic signal is given in the following diagram.

* + 1. **Even and Odd Signals**

A signal or is defined as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin.

|  |  |
| --- | --- |
| Even continuous-time Signal |  |
| Even Discrete-time Signal |  |

A signal or is defined as an odd signal if,

|  |  |
| --- | --- |
| Odd continuous-time Signal |  |
| Odd Discrete-time Signal |  |

As a special case, the odd signal must be zero at or . Examples of odd signals are given in the following figure.

An important fact is that any signal (continuous-time or discrete-time) can be broken into a sum of two signals, one of which is even and one of which is odd.

|  |  |  |
| --- | --- | --- |
| Signal | Component | Mathematical Form |
| Continuous-time Signal | Even Part |  |
| Odd Part |  |
| Discrete-time Signal | Even Part |  |
| Odd Part |  |

**A Discrete-Time Signal**

**Even Part**

**Odd Part**

* 1. **Exponential and Sinusoidal Signals**
     1. **Continuous-Time Complex Exponential and Sinusoidal Signals**

A continuous-time complex signal can be written as

|  |  |
| --- | --- |
|  | (1.20) |

where and are, in general, complex numbers.

***Real Exponential Signals***

In this case both and are real numbers, and is called a real exponential.



***Periodic Complex Exponential and Sinusoidal Signals***

Now we consider the case of complex exponentials where is purely imaginary. More, specifically, we consider:

|  |  |
| --- | --- |
|  | (1.21) |

An important property of this signal is that it is periodic. Let’s explore this.

By definition is periodic, if

for some real value of . Therefore,

Or,

|  |  |
| --- | --- |
|  | (1.23) |

This equation can be true,

1. If, , then , which is periodic for any value of .
2. If, , then the fundamental period of , i.e. the smallest value of for which equation (1.23) holds, is

|  |  |
| --- | --- |
|  | (1.24) |

Replacing the value of with this in equation (1.23), and using Euler’s formula, that is,

We get

Therefore, the signal as given by equation (1.21) is a periodic signal. Similarly, it can be shown that the signal has the same fundamental period.

A closely associated signal is the sinusoidal signal

|  |  |
| --- | --- |
|  | (1.25) |

Continuous-Time Sinusoidal Signal

x(t)=A cos(



0

t+



)

t

A

A cos(



)

Using Euler’s formula, the complex exponential of equation (1.21) can be written as

|  |  |
| --- | --- |
|  | (1.26) |

Similarly, the sinusoidal signal of equation (1.25) can be written as

|  |  |
| --- | --- |
|  | (1.27) |

It can noted that the two exponentials in equation 1.27 have complex amplitudes. If we define as the real component of a complex number , and as the imaginary part of , then

|  |  |
| --- | --- |
|  | (1.28) |
|  | (1.29) |

***Frequency and Period***

From equation 1.24, it is clear that the fundamental period of a continuous-time sinusoidal or a periodic complex exponential signal, is inversely proportional to the , which is called the *fundamental frequency*. If we decrease the value of the magnitude of , we slow down the rate of oscillations and hence increase the period . Alternatively, if we increase the value of the magnitude of , we increase the rate of oscillations and hence decrease the period .

***Energy and Power***

Over the one fundamental period of a continuous-time sinusoidal or a periodic complex exponential signal, the signal energy and power can be determined as:

|  |  |
| --- | --- |
|  | (1.30) |
|  | (1.31) |

As there are an infinite number of periods as ranges from to , the total energy integrated over all time is infinite. The total average power is however remains 1, as by definition,

|  |  |
| --- | --- |
|  | (1.32) |

***Harmonics of a Periodic Complex Exponential***

We have noted that,

|  |  |
| --- | --- |
|  | (1.33) |

which implies that is a multiple of , i.e.,

|  |  |
| --- | --- |
| , where | (1.34) |

This shows that must be an integer multiple of , i.e., the fundamental frequency. We can therefore, write

|  |  |
| --- | --- |
| , where | (1.36) |

This is called the k-harmonic of the complex exponential signal.

***General Complex Exponential Signals***

The general complex exponential signals are of the form

|  |  |
| --- | --- |
|  |  |

Where both and are complex numbers. Let us represent them as

Thus is written in polar form and is written in Cartesian form, then

|  |  |
| --- | --- |
|  | (1.42) |

Using Euler’s formula, it can be written as

|  |  |
| --- | --- |
|  | (1.43) |

1. For , the real and imaginary parts of a complex exponential are sinusoidal.
2. For , they correspond to sinusoidal signals multiplied with growing exponential.
3. For , they correspond to sinusoidal signals multiplied with decreasing exponentials.

This is shown graphically on the next page.



* + 1. **Periodic Signals (Recap)**

A periodic continuous-time signal is defined as

where is a positive number called the period.

A typical example is that of a sinusoidal signal for . This is shown in the figure below:

For the above signal, the period is **.** It can be noticed that for any time :

Also

where is an integer.

The ***fundamental period*** of is the smallest positive value of for which the equation holds.

Periodic signals are defined analogously in discrete-time.

A discrete-time signal is periodic with period , where is a positive integer, if it is unchanged by a time-shift of , i.e., if

for all values of .

The ***fundamental period*** of is the smallest positive value of for which the equation holds.

An example of a discrete-time periodic signal is given in the following diagram.

* + 1. **Even and Odd Signals (Recap)**

A signal or is defined as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin.

|  |  |
| --- | --- |
| Even continuous-time Signal |  |
| Even Discrete-time Signal |  |

A signal or is defined as an odd signal if,

|  |  |
| --- | --- |
| Odd continuous-time Signal |  |
| Odd Discrete-time Signal |  |

As a special case, the odd signal must be zero at or . Examples of odd signals are given in the following figures.

An important fact is that any signal (continuous-time or discrete-time) can be broken into a sum of two signals, one of which is even and one of which is odd.

|  |  |  |
| --- | --- | --- |
| Signal | Component | Mathematical Form |
| Continuous-time Signal | Even Part |  |
| Odd Part |  |
| Discrete-time Signal | Even Part |  |
| Odd Part |  |

**A Discrete-Time Signal**

**Even Part**

**Odd Part**

* + 1. **Discrete-Time Complex Exponential and Sinusoidal Signals**

A discrete-time complex exponential signal or sequence can be written as

|  |  |
| --- | --- |
|  | (1.44) |

where and are, in general, complex numbers. This could also be written as

|  |  |
| --- | --- |
|  | (1.45) |

where

|  |  |
| --- | --- |
|  |  |

The form given in equation (1.44) is more often used in discrete-time signal processing.

***Real Exponential Signals***

In this case both and are real numbers, and is called a real exponential. We see different types of behaviors, as shown below in the figure.





***Sinusoidal Signals***

Another important complex exponential is obtained by using the form given in equation 1.45, and by considering to be purely imaginary (so that ). More, specifically, we consider:

|  |  |
| --- | --- |
|  | (1.46) |

A closely associated signal is the sinusoidal signal

|  |  |
| --- | --- |
|  | (1.47) |

Using Euler’s formula, the complex exponential of equation (1.47) can be written as

|  |  |
| --- | --- |
|  | (1.48) |

Similarly, the sinusoidal signal of equation (1.25) can be written as

|  |  |
| --- | --- |
|  | (1.49) |

It can noted that the two exponentials in equation 1.27 have complex amplitudes. If we define as the real component of a complex number , and as the imaginary part of , then

|  |  |
| --- | --- |
|  |  |
|  |  |



***General Complex Exponential Signals***

The general discrete-time complex exponential signals are of the form

|  |  |
| --- | --- |
|  |  |

where both and are complex numbers. Let us represent them as

Thus and are written in polar form, then

|  |  |
| --- | --- |
|  |  |

Using Euler’s formula, it can be written as

|  |  |
| --- | --- |
|  | (1.50) |

1. For , the real and imaginary parts of a complex exponential are sinusoidal.
2. For , they correspond to sinusoidal signals / sequences multiplied with growing exponential.
3. For , they correspond to sinusoidal signals / sequences multiplied with decreasing exponentials.

This is shown graphically on the next page.



* + 1. **Periodicity Property of Discrete-Time Complex Exponential**

There are many similarities between continuous-time and discrete-time signals. But also there are many important differences. One of them is related with the discrete-time exponential signal .

The following properties were found with regard to the continuous-time exponential signal :

1. The larger the magnitude of , the higher is the rate of oscillations in the signal;
2. is periodic for any value of .

Let us now see how these properties are different in the discrete-time case:

1. To see the difference for the first property, consider the discrete-time complex exponential:

|  |  |
| --- | --- |
|  | (1.51) |

This shows that the exponential at is the same as that at frequency . This is very different when compared with the continuous-time exponential case, in which the signals are all distinct for distinct values of .

* In discrete-time, these signals are not distinct. In fact, the signal with frequency is identical to signals with frequencies , and so on. Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of size . The most commonly used intervals are or the interval .
* Due to equation 1.51, as is gradually increased, the rate of oscillations in the discrete-time signal does not keep on increasing. If is increased from 0 to , the rate of oscillations first increase and then decreases. This is shown in the figure 1.27 below.
* Note in particular that for or for any odd multiple of ,

|  |  |
| --- | --- |
|  | (1.52) |

so that the signal oscillates rapidly, changing sign at each point in time (see figure 1.27).



Figure 1.27: Discrete-time sinusoidal sequences for several different frequencies

1. The second property is the periodicity of the discrete-time complex exponential signals.

|  |  |
| --- | --- |
|  | (1.53) |

Or equivalently,

|  |  |
| --- | --- |
|  | (1.54) |

Equation 1.54 could only be true if is a multiple of

|  |  |
| --- | --- |
|  | (1.55) |

Or

|  |  |
| --- | --- |
|  | (1.56) |

This equation (1.56) means that the discrete-time signal is periodic only when is a rational number (i.e. it could written as a fraction).

**Table 1.1 Comparison of the signals and .**

|  |  |
| --- | --- |
|  |  |
| Distinct signals for distinct values of . | Identical signals for values of separated by multiples of . |
| Periodic for any choice of . | Periodic only if for some integer and . |
| Fundamental frequency . | Fundamental frequency . |
| Fundamental period  : undefined | Fundamental period  : undefined |