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      1. **Periodicity Property of Discrete-Time Complex Exponential (Recap)**

There are many similarities between continuous-time and discrete-time signals. But also there are many important differences. One of them is related with the discrete-time exponential signal .

The following properties were found with regard to the continuous-time exponential signal :

1. The larger the magnitude of , the higher is the rate of oscillations in the signal;
2. is periodic for any value of .

Let us now see how these properties are different in the discrete-time case:

1. To see the difference for the first property, consider the discrete-time complex exponential:

|  |  |
| --- | --- |
|  | (1.51) |

This shows that the exponential at is the same as that at frequency . This is very different when compared with the continuous-time exponential case, in which the signals are all distinct for distinct values of .

* In discrete-time, these signals are not distinct. In fact, the signal with frequency is identical to signals with frequencies , and so on. Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of size . The most commonly used intervals are or the interval .
* Due to equation 1.51, as is gradually increased, the rate of oscillations in the discrete-time signal does not keep on increasing. If is increased from 0 to , the rate of oscillations first increase and then decreases. This is shown in the figure 1.27 below.
* Note in particular that for or for any odd multiple of ,

|  |  |
| --- | --- |
|  | (1.52) |

so that the signal oscillates rapidly, changing sign at each point in time (see figure 1.27).



Figure 1.27: Discrete-time sinusoidal sequences for several different frequencies

1. The second property is the periodicity of the discrete-time complex exponential signals.

|  |  |
| --- | --- |
|  | (1.53) |

Or equivalently,

|  |  |
| --- | --- |
|  | (1.54) |

Equation 1.54 could only be true if is a multiple of

|  |  |
| --- | --- |
|  | (1.55) |

Or

|  |  |
| --- | --- |
|  | (1.56) |

This equation (1.56) means that the discrete-time signal is periodic only when is a rational number (i.e. it could written as a fraction).

**Table 1.1 Comparison of the signals and .**

|  |  |
| --- | --- |
|  |  |
| Distinct signals for distinct values of . | Identical signals for values of separated by multiples of . |
| Periodic for any choice of . | Periodic only if for some integer and . |
| Fundamental frequency . | Fundamental frequency . |
| Fundamental period  : undefined | Fundamental period  : undefined |

* 1. **The Unit Impulse and Unit Step Functions**
     1. **The Discrete-Time Unit Impulse and Unit-Step Sequences**

One of the simplest discrete-time function is the unit impulse or unit sample, denoted by . It is defined as,

|  |  |
| --- | --- |
|  | (1.63) |

This is shown in the figure below,



A second basic discrete-time signal is the discrete-time unit step, denoted by and defined by

|  |  |
| --- | --- |
|  | (1.64) |

It is shown in the following figure.



There is a close relationship between the discrete-time unit impulse and unit step signals. The discrete-time unit impulse can be written as the first-difference of the discrete-time unit step

|  |  |
| --- | --- |
|  | (1.65) |

Conversely, the discrete-time unit step is the sunning sum of the unit sample.

|  |  |
| --- | --- |
|  | (1.66) |

This is illustrated graphically in Figure 1.30 below. It is clear from the figure that the running sum of equation 1.66, is for and for .

By changing the variable (summation index) to , equation 1.66 can be written as

|  |  |
| --- | --- |
|  |  |

Or equivalently

|  |  |
| --- | --- |
|  | (1.67) |

This is illustrated in figure 1.31. In this case, the nonzero value of is at the value of equal to . Therefore, from equation 1.67, we see that the summation is for and for .



Another interpretation of equation 1.67 is that the unit step is the superposition of delayed unit-impulses, i.e., we can view equation 1.67 as the sum of unit impulse at , a unit impulse at , another, at , etc.



The unit impulse sequence can be used to sample the value of a signal at . In particular, since is non-zero (and equal to 1) only for , therefore

|  |  |
| --- | --- |
|  | (1.68) |

More generally, if we consider a unit impulse at , then

|  |  |
| --- | --- |
|  | (1.69) |

* + 1. **The Continuous-Time Unit Step and Unit Impulse Functions**

The continuous-time unit step function, denoted by is defined by

|  |  |
| --- | --- |
|  | (1.70) |

It is shown in the figure (1.322) below. It could be noticed that this function is discontinuous at .

u(t)

1

0 t

**Figure 1.32**: Continuous-time unit step function

The continuous-time unit impulse function is related to the unit step in a manner similar to the way the discrete-time unit impulse is related to the unit step. In discrete-time case, the unit step is the running sum of the unit impulse. In the continuous case, it is the running integral instead of the running sum. So the unit step can be written as the running integral of the unit impulse,

|  |  |
| --- | --- |
|  | (1.71) |

Also analogous to equation (1.65), where the discrete-time unit impulse is represented as the first difference of unit step and its delayed version, we write

|  |  |
| --- | --- |
|  | (1.72) |

that is, the unit impulse in the continuous-time can be written as the first derivative of the unit step in continuous time.

However, we notice that is discontinuous at (and consequently cannot be differentiated at ), therefore, there is some formal difficulty with this equation.

However, we interpret equation (1.72) by considering an approximation to the unit step (see figure 1.33 below), in which the function rises from to in a short time interval of length . The step function can be considered as an idealization of for so short that its duration doesn’t matter for any practical purpose. More formally, is the limit of as . Now the derivative of equation (1.72) can be written as

|  |  |
| --- | --- |
|  | (1.73) |

as shown in figure (1.34).

**Figure 1.33**: Continuous-time approximation to the unit step function,

1

0 t

0 t

**Figure 1.34**: Derivative of

It could be noticed that is short pulse of duration and with unit area for any value of . If we gradually decrease the value of , the pulse will become narrower and the height will increase (to maintain the area to unity). Therefore, in the limiting case, we can write

|  |  |
| --- | --- |
|  | (1.73) |

Since, represents a pulse with no duration and unit area, we adopt the graphical representation as given in figure 1.35.

**Figure 1.35**: Continuous-time unit impulse

1

0 t

**Figure 1.36**: Continuous-time scaled impulse

k

k

0 t

In general, a scaled impulse will have an area , , and

|  |  |
| --- | --- |
|  |  |

Also, the continuous-time step function can be written as the running integral

|  |  |
| --- | --- |
|  | (1.75) |

1. **(b)**

**Figure 1.37**: Running Integral given in equation 1.71. (a) ; (b) .

1

0

Interval of integration

1

0

Interval of integration

Also for any continuous-time signal , we can write

|  |  |
| --- | --- |
|  | (1.76) |

This is called the sampling property of the continuous-time impulse function.

* 1. **Continuous-Time and Discrete-Time Systems**

A continuous-time system is a system in which continuous-time input signals are applied and result in continuous-time output signals. The input-output relation of such systems can be represented by the notation:

|  |  |
| --- | --- |
|  | (1.78) |

This is shown in the figure below,

**Continuous-time**

**system**

A discrete-time system is a system in which discrete-time input signals are applied and result in discrete-time output signals. The input-output relation of such systems can be represented by the notation:

|  |  |
| --- | --- |
|  | (1.79) |

This is shown in the figure below,

**Discrete-time**

**system**

* + 1. **Simple Examples of Systems**

It has been noticed that systems from many different applications have very similar mathematical descriptions. This serves as an important motivation for developing general tools for systems analysis and design. We illustrate this point with some simple examples.

**Example 1.8**

Consider the RC circuit of Figure 1.1 (shown in the following).

**Figure 1.1**: A simple ***RC*** circuit with source ***VS*** and capacitor voltage ***VC***.

***R***

***C VC***

***VS***

According to Ohm’s law, the current through the resistor is proportional (with proportionality constant ) to the voltage drop () across the resistor, i.e.,

|  |  |
| --- | --- |
|  | (1.80) |

Similarly, the current is related with the rate of change with time of the voltage across the capacitor, i.e.,

|  |  |
| --- | --- |
|  | (1.81) |

Equating the right-hand sides of equations (1.80) and (1.81), we obtain the differential equation giving a relationship between the input and the output ,

|  |  |
| --- | --- |
|  | (1.82) |

**Example 1.9**

Consider the RC circuit of Figure 1.1 (shown in the following).

**Figure 1.2**: An automobile responding to an applied force ***f*** from the engine and to a retarding force ***v*** proportional to the automobile’s velocity ***v***.



***f***

***v***

Here we regard the force as the input and velocity as the output. If we let denote the mass of the automobile and the resistance due to friction, then equating acceleration i.e. the time derivative of velocity, with net force divided by mass, we get

|  |  |
| --- | --- |
|  | (1.83) |
| i.e., |  |
|  | (1.84) |

Examining and comparing equations (1.82) and (1.84) in the above examples, we notice that the input-output relationship of these entirely different systems are basically the same. In fact, both of them are example of the first-order linear differential equations of the form

|  |  |
| --- | --- |
|  | (1.85) |

where is the input, is the output, and and are constants. Therefore, if we could develop methods for analyzing general classes of systems (such as represented by equation 1.85), we will be able to use them in a wide variety of applications.

**Example 1.10**

As a simple example of a discrete-time system, consider a simple model for the balance in a bank account from month to month. Specifically, let denote the balance at the end of the n-th month, and suppose that evolves from month to month according to the equation,

|  |  |
| --- | --- |
|  | (1.86) |

Or equivalently

|  |  |
| --- | --- |
|  | (1.87) |

Where represents the net deposit (i.e., deposits minus withdrawals) during the month and the term is the 1% profit each month.

**Example 1.11**

As a second example, consider the digital simulation of the differential equation in equation (1.84) in which we resolve the time into discrete intervals of length and approximate the derivative at by the first backward difference, i.e.,

In this case, if we let

and

We obtain the following discrete-time model relating the sampled signals and :

|  |  |
| --- | --- |
|  | (1.88) |

Comparing equations (1.87) and (1.88) we see that they both are examples of the same first-order linear difference equation, namely,

|  |  |
| --- | --- |
|  | (1.89) |

* + 1. **Interconnections of Systems**

Many real systems are built as interconnections of several subsystems.



For example, a modern digital telephone system involves the interconnections of a microphone receiver, audio to digital converter, a transmitter, a receiver, a digital to audio convertor and one or more speakers (apart from several other sub-systems).

One can construct a variety of system interconnections. However, there are several basic system interconnections that are encountered more frequently. They are shown in the following figures.

**Series (cascade) interconnection**

**System 1**

**Input**

**Output**

**System 2**

**Parallel Interconnection**

**Input**

**Output**

**System 1**

**System 2**

**Series-Parallel Interconnection**

**Output**

**System 4**

**System 2**

**System 3**

**System 1**

**Input**

**Feedback Interconnection**

**System 1**

**System 2**

**Input**

**Output**

* 1. **Basic System Properties**
     1. **Systems with and without Memory**

**A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent on the input at only that same time.**

For example, the system specified by the relationship

|  |  |
| --- | --- |
|  | (1.90) |

is memoryless, as the value of at any particular time depends on the value of only at that time, i.e. .

* As a particular case, a resistor can be considered as a memoryless system: with the input taken as the current and with voltage taken as the output , the input-output relationship for a resistor is,

|  |  |
| --- | --- |
|  | (1.91) |

where is the resistance.

* Another particular, simple memoryless system is the ***identity system***, whose output is identical with the input. That is, the input-output relationship for the continuous-time identity system is

|  |  |
| --- | --- |
|  |  |

and the corresponding relationship in discrete-time is

|  |  |
| --- | --- |
|  |  |

* An example of a discrete-time system with memory is an ***accumulator*** or ***summer***

|  |  |
| --- | --- |
|  | (1.92) |

and a second example is a delay

|  |  |
| --- | --- |
|  | (1.93) |

* A capacitor is an example of a continuous-time system with memory; since if the input is taken to be the current and the output is the voltage, then

|  |  |
| --- | --- |
|  | (1.94) |

where is the capacitance.

The concept of memory in a system corresponds to the presence of a mechanism in the system that retains or stores information about the input values at times other than the current time.

* The delay of equation (1.93), retains or stores the preceding value of the input.
* The accumulator of equation (1.92), “remembers” or stores information about the past inputs.

Equation (1.92) can also be written as:

|  |  |
| --- | --- |
|  | (1.95) |

or equivalently,

|  |  |
| --- | --- |
|  | (1.96) |

This shows that to obtain the output at the current time n, the accumulator must remember the running sum of previous input values (which is exactly the preceding value of the accumulator output).

In many physical systems, the memory is directly associated with the storage of energy.

* The capacitor of equation (1.94) stores energy by accumulating electrical charge (represented in the form of integral of the current).
* In discrete-time systems implemented with computers or digital microprocessors, memory is typically directly associated with the storage registers that retain values between clock ticks.

Memory dependent on the future values of the input and the output:

* Our formal definition of memory-less systems leads us to the conclusion that ***a system can be considered to have memory, even if the current value of the output is dependent upon the future values of the input and output***.
  + 1. **Invertibility and Inverse Systems**

**A system is said to be invertible if distinct inputs lead to distinct outputs.**

**Inverse**

**system**

**System**

**Figure 1.45 (a)**: Concept of an inverse system for a general invertible system.

As depicted in figure 1.45(a) above, if a system is invertible, than an inverse system exists that, when cascaded with the original system, yields an output equal to the input to the first system. The overall input-output relationship of the system shown in figure 1.45(a) is that of an identity system.

* An example of an invertible continuous-time system is

|  |  |
| --- | --- |
|  | (1.97) |

For which the inverse system is,

|  |  |
| --- | --- |
|  | (1.98) |

This system is shown in figure 1.45(b) below.

**Figure 1.45 (b)**: Concept of an inverse system for the system described by equation (1.98)

* Another example of an invertible system is the accumulator of equation (1.92). The inverse system in this case is shown in figure 1.45(c).

**Figure 1.45 (c)**: Concept of an inverse system for the system described by equation (1.92)

Examples of noninvertible systems are:

|  |  |
| --- | --- |
| A system that produces a zero output sequence for any input sequence. | (1.100) |
| A system where the output is the square of the input  Here we cannot find out about the sign of the input from the knowledge of the output. | (1.101) |

The concept of invertibility is important in many applications. One particular example is that of systems used for encoding in a variety of communication systems.

* 1. **Basic System Properties**
     1. **Systems with and without Memory**

**A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent on the input at only that same time.**

* + 1. **Invertibility and Inverse Systems**

**A system is said to be invertible if distinct inputs lead to distinct outputs.**

* + 1. **Causality**

**A system is causal if the output at any time depends on values of the input at only the present and past times.**

Such systems are also referred to as ***non-anticipative***, as the system output does not anticipate future values of the input.

* The RC circuit of figure 1.1, is causal, since the capacitor voltage depends only on the present and past values of the source voltage.
* The motion of an automobile is causal, as it does not anticipate future actions of the driver.
* The systems described by equations (1.92) – (1.94) are also causal.
* All memoryless systems are causal, since the output responds only to the current value of the input.

The following systems are not causal:

|  |  |
| --- | --- |
|  | (1.102) |
| And |  |
|  | (1.103) |

**Figure 1.1**: A simple ***RC*** circuit with source ***VS*** and capacitor voltage ***VC***.

***R***

***C VC***

***VS***

Causality is not an essential consideration in applications where the independent variable is not time, such as in image processing.

* In processing data that have been collected previously, as often is the case with speech, geophysical or meteorological signals etc., we are by no means constrained to causal processing.
* An example of noncausal averaging system is

|  |  |
| --- | --- |
|  | (1.104) |

**Example 1.12**

It is important to carefully analyze the input-output relationship of a system, while checking the causality. The following two systems can be used as examples.

1. Consider the following system first

|  |  |
| --- | --- |
|  | (1.105) |

We note that the output at a positive time depends only on the value of the input signal at time (), which is negative and therefore in the past of . This tempts us to infer that the system is causal. But, if we check carefully, we come to know that the system is not causal. Checking for the negative time, e.g. , we see that , so that the output at this time depends on a future value of the input. Hence the system is not causal.

1. It is also important to distinguish carefully the effects of the input from those of any other functions used in the definition of the system. For example

|  |  |
| --- | --- |
|  | (1.106) |

In this system, the output at any time equals the input at that same time multiplied with a number that fluctuate with time. Specifically, we can re-write

|  |  |
| --- | --- |
|  | (1.107) |

where is a time-varying function, namely . Thus, only the current value of the input influences the current value of the output, and we conclude that this system is causal (and, also memoryless).

* + 1. **Stability**

**A system is said to be *stable* if a small input leads to a response that does not diverge.**

There are several examples of stable systems. “Stability of physical systems generally results from the presence of mechanisms that dissipate energy”. For example, in the ***RC*** circuit of Example 1.8, the resistor dissipates energy and this circuit is a stable system. The system in Example 1.9 is also stable, because of the dissipation of energy through friction.

More specifically,

**If the input to a *stable system* is bounded (i.e., if its magnitude does not grow without bounds), then the output must also be bounded, and therefore cannot diverge.**

This definition of stability, we will use throughout this course.

As an example, consider a system whose input-output relationship is given by Equation (1.104), i.e.,

|  |  |
| --- | --- |
|  | (1.104) |

If the input to the system is bounded (say by a number ), for all values of , then according to Equation (1.104) above, the output of the system is also bounded by . This is because the output is the average of a finite set of values of the input. Therefore, the output is bounded and the system is stable.

On the other hand, consider the accumulator described by Equation (1.92).

|  |  |
| --- | --- |
|  | (1.92) |

This systems sums all of the past values of the input rather than just a finite set of values, and the system is unstable, since the sun can grow even if the input is bounded.

**Example 1.13**

Suppose we suspect that a particular system is unstable, then a useful strategy is to look for a specific bounded input that leads to an unbounded output for that system. To illustrate, let us consider the following two systems

|  |  |
| --- | --- |
|  | (1.109) |

and

|  |  |
| --- | --- |
|  | (1.110) |

Now, for system , a constant input yields , which is unbounded: since no matter what finite constant input we pick, will exceed that constant for some . Therefore, the system is unstable.

For system , let us the input be bounded by a positive number , i.e.

|  |  |
| --- | --- |
|  | (1.111) |

or

|  |  |
| --- | --- |
|  | (1.112) |

for all . Using the definition of from Equation (1.110), we can write

|  |  |
| --- | --- |
|  | (1.113) |

The system is therefore, stable.

* + 1. **Time Invariance**

**A system is said to be *time invariant* if a time shift in the input signal leads to an identical time shift in the output signal.**

That is, if is the output of a discrete-time, time-invariant system when is the input, then is the output when is applied as an input. In continuous-time when is the output corresponding to the input , a time-invariant system will have as the output when is the input.

**Example 1.14**

Consider the continuous-time system defined by,

|  |  |
| --- | --- |
|  | (1.114) |

To check this system is time invariant, we must determine whether the time-invariance property holds for any input and any time shift . Thus, let be an arbitrary input to this system, and let

|  |  |
| --- | --- |
|  | (1.115) |

to be the corresponding output. Then, consider a second input obtained by shifting in time:

|  |  |
| --- | --- |
|  | (1.116) |

The corresponding output to this new input (according to input-output relationship given by Equation (1.114)) is:

|  |  |
| --- | --- |
|  | (1.117) |

Similarly, from Equation (1.115),

|  |  |
| --- | --- |
|  | (1.118) |

From Equations (1.117) and (1.118), we see that , and therefore, the system is time invariant.

**Example 1.15**

Consider now the discrete-time system defined by,

|  |  |
| --- | --- |
|  | (1.119) |

Suppose, we consider the input signal , which yields an output (since . However, the input yields the output . Thus, while is a shifted version of , is not a shifted version of .

* + 1. **Linearity**

**A linear system, in continuous-time or discrete-time, is a system that possesses the important property of superposition: If an input consists of the weighted sum of several signals, then the output is the superposition – that is, the weighted sum – of the responses of the system to each of those signals.**

Let be the response of a continuous-time system to an input, and let be the response of a continuous-time system to an input. Then the system is linear if,

1. The response to is .
2. The response to is , where is any complex constant.

The first of these two properties is called the ***additivity*** property and the second is known as the ***scaling*** or ***homogeneity*** property.

The two properties defining a linear system can be combined into a single statement:

|  |  |
| --- | --- |
| Continuous-time: | (1.121) |
| Discrete-time: | (1.122) |

Here and are complex constants.

Furthermore, it is straightforward to show from the definition of linearity that if , are a set of inputs to a discrete-time linear system with corresponding outputs , then the response to a linear combination of these inputs given by

|  |  |
| --- | --- |
|  | (1.123) |

is

|  |  |
| --- | --- |
|  | (1.124) |

This very important fact is known as the ***superposition property***, which holds for linear systems in both continuous and discrete time.

**Example 1.17**

Consider a system whose input and output are related by

To determine whether or not is linear, we consider two arbitrary inputs and,

Let be a linear combination of and. That is,

where and are arbitrary scalars. If is the input to, then the corresponding output may be expressed as:

We conclude that the system is linear.

**Example 1.18**

Let now consider another system whose input and output are related by

Like the previous example, to determine whether or not is linear, we consider two arbitrary inputs and,

Let be a linear combination of and. That is,

where and are arbitrary scalars. If is the input to, then the corresponding output may be expressed as:

which clearly shows that the system is not linear.

**Example 1.19**

While checking the linearity of a system, it is important to keep in mind that the system must satisfy both the additivity and homogeneity properties and that the signals as well the scaling constants are allowed to be complex.

To illustrate these points, let us consider a system whose input and output are related by

We can show that this system is additive; however, it does not satisfy the homogeneity condition. Let us assume

where and are the real and imaginary parts of the complex signal , and that the corresponding output is given by

Now we consider the scaling of the complex input with a complex number say , i.e.

Therefore, the corresponding output is given by

which is not equal to the scaled version of the

We conclude that the system violates the homogeneity property, therefore it is not linear.