Testing Hypotheses

Case 1: Simple Hypotheses (Neyman-Pearson Lemma)

1. Let the random variable X have the pdf $f(x;\theta) = \left(1/_{\theta}\right)e^{-x/_{\theta}}, 0 < x < \infty$, zero elsewhere. Consider the simple hypothesis $H_0: \theta = 2$ and the alternative hypothesis $H_1: \theta = 4$. Let $X_1,\ X_2$ denote a random sample of size 2 from this distribution. Show that the best of H_0 against H_1 may be carried out by use of the statistic $X_1 +\ X_2$.
2. Let $X_1, X_2, \ldots\ldots X_{10}$ be a random sample of size 10 from a normal distribution $N(0, \sigma^2)$. Find a best critical region of size $\alpha = 0.05$ for testing $H_0: \sigma^2 = 1$ against $H_1: \sigma^2 = 2$.
3. If $X_1, X_2, \ldots\ldots X_n$ is a random sample from a distribution having pdf of the form $f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1$, zero elsewhere, show that a best critical region for testing $H_0: \theta = 1$ against $H_1: \theta = 2$ is $C = \{(x_1, x_2, \ldots\ldots x_n): c \le \prod_{i=1}^{n} x_i\}$.
4. Let $X_1, X_2, \ldots\ldots X_{10}$ be a random sample from a distribution that is $N(\theta_1, \theta_2)$. Find a best test of the simple hypothesis $H_0: \theta_1 = 0, \theta_2 = 1$ against the alternative simple hypothesis $H_1: \theta_1 = 1,\ \theta_2 = 4$.
5. Let $X_1, X_2, \ldots\ldots X_n$ denote a random sample from a normal distribution $N(\theta, 100)$. Show that $C = \{(x_1, x_2, \ldots\ldots x_n): c \le\ \bar{x}\}$ is a best critical region for testing $H_0: \theta = 75$ against $H_1: \theta = 78$. Find n and c so that

$$P[(X_1, X_2, \ldots., X_n) \in C;\ H_0] = P(\bar{X} \ge c;\ H_0) = 0.05 \text{ and}$$

$$P[(X_1, X_2, \ldots., X_n) \in C;\ H_1] = P(\bar{X} \ge c;\ H_1) = 0.90, \text{ approximately.}$$

6. If $X_1, X_2, \ldots\ldots X_n$ is a random sample from a beta distribution with parameters $\alpha = \beta = \theta > 0$, find a best critical region for testing $H_0: \theta = 1$ against $H_1: \theta = 2$.
7. Let $X_1, X_2, \ldots\ldots X_n$ be iid with pmf $f(x;p) = p^x(1-p)^{1-x}, x = 0, 1$, zero elsewhere. Show that $C = \{(x_1, x_2, \ldots\ldots x_n): \sum_1^n x_i \le c\}$ is a best critical region for testing $H_0: p = \frac{1}{2}$ against $H_1: p = \frac{1}{3}$. Use the Central Limit Theorem to find n and c so that approximately $P(\sum_1^n X_i \le c;\ H_0) = 0.10$ and $P(\sum_1^n X_i \le c;\ H_1) = 0.80$.
8. Let $X_1, X_2, \ldots\ldots X_{10}$ denote a random sample of size 10 from a Poisson distribution with mean θ. Show that the critical region C defined by $\sum_1^{10} x_i \ge c$ is a best critical region for testing $H_0: \theta = 0.1$ against $H_1: \theta = 0.5$. Determine, for this test, c and the power at $\theta = 0.5$ when the significance level $\alpha = 0.08$.
9. Let X be a random variable has the pdf

$$f(x) = \frac{\beta}{x^{\beta+1}}\ , \qquad x \ge 1, \qquad \beta > 0$$

Find the best critical region of size α for testing

$$H_0: \beta = \beta_0 \ \text{ versus } \ H_1: \beta = \beta_1\,,\ \beta_1 > \beta_0.$$