In some cases, we use the words "at random" or "randomly" to signify that the elements have the same probability.

Example: From a goup of 8 History books and 6 Maths books, one book is chosen at random.

Find the probability that:

a) this book is History.

b) ____________ is Math.

$|S| = 8 + 6 = 14$.

a) $A = \{\text{the book is History}\} = \{H\}$

$|A| = 8$.

$P(A) = \frac{8}{14}$

b) $B = \{M\}$, $|B| = 6$, $P(B) = \frac{6}{14}$.

Example: Suppose A and B are independent.
$P(A) = 1/4$, $P(B) = 1/6$.
Compute $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$$= \frac{1}{4} + \frac{1}{6} - \frac{1}{24} = \frac{6+4-1}{24} = \frac{9}{24}$$

$$P(A'|B) = \boxed{P(A')}$$
$$= 1 - P(A|B) = 1 - P(A) = P(A').$$

$$P(A' \cap B) = P(A') \cdot P(B) = 3/4 \cdot \frac{1}{6} = \frac{3}{24} = \frac{1}{8}.$$

1.5 Baye's formula:

* Total Probability formula:

Tree diagram:

$$\begin{array}{lll}
\xrightarrow{P(A_1)} A_1 & \xrightarrow{P(B|A_1)} & B \\
\xrightarrow{P(A_2)} A_2 & \xrightarrow{P(B|A_2)} & B \\
\vdots & \vdots & \\
\xrightarrow{P(A_n)} A_n & \xrightarrow{P(B|A_n)} & B
\end{array}$$

$$P(B) = P(A_1) \cdot P(B|A_1) + \cdots + P(A_n) \cdot P(B|A_n).$$

Example: Box: $7R + 6W + 4B$.
Choose 1^{st} ~~one~~ ball, keep it and choose a 2^{nd} ball.
Find $P(2^{nd}$ ball is blue$)$?

$$\begin{array}{lll}
\xrightarrow{7/17} R^1 & \xrightarrow{4/16} & B^2 \\
\xrightarrow{6/17} W^1 & \xrightarrow{4/16} & B^2 \\
\xrightarrow{4/17} B^1 & \xrightarrow{3/16} & B^2
\end{array}$$

$$P(2^{nd} \text{ ball is blue}) = \frac{7}{17} \cdot \frac{4}{16} + \frac{6}{17} \cdot \frac{4}{16} + \frac{4}{17} \cdot \frac{3}{16} =$$
$$= \frac{28+24+12}{16 \times 17} = \frac{64}{16 \times 17}.$$

Chapter ①: Probability.

① Introduction to Probability:

1.1 Terminology:

* Random experiment:
it is the experiment with unknow outcome.

Examples: * Tossing a coin. (Head, Tail).
* Tossing a die (6 faces, 1, 2, ..., 6).
* Choosing randomly one book from a list of books.

* Probability space:
it is the set of all possible outcomes of a random experiment.

Examples: Tossing a coin: $S = \{H, T\}$.
Tossing a die: $S = \{1, 2, 3, 4, 5, 6\}$;
Tossing Two Coins: $S = \{HH, HT, TH, TT\}$.
The score of a student in a future Final exam:
$$S = [0, 40].$$

* We use also the name of "Sample space".

* Baye's formula:

$$P(A^i|B) = \frac{P(B|A^i)\,P(A^i)}{P(B)}.$$

Example (Previous example):

Find P(1st ball is red | 2nd is blue)

$$= P(R^1|B^2) = \frac{P(B^2|R^1)\cdot P(R^1)}{P(B^2)}$$

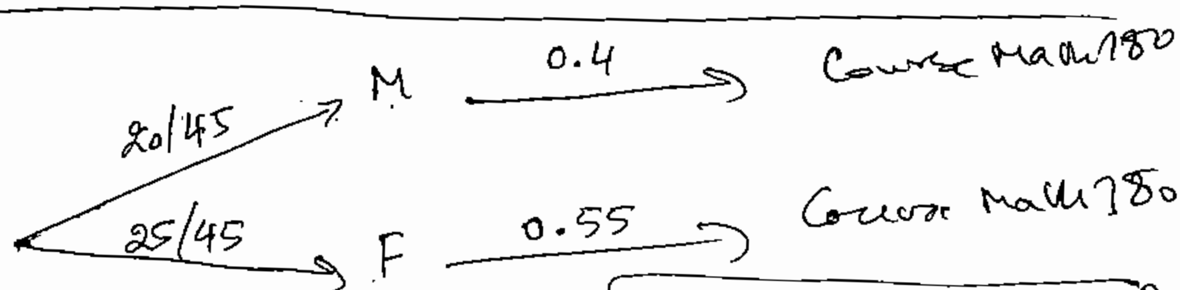
$$= \frac{4/16 \cdot 7/17}{\frac{64}{16\times 17}} = \frac{28}{64}.$$

Example:

A group of students is composed of 20 male students and 25 female students. 40% of male students ~~take~~ and 55% of female students take Math 380 course. one person is chosen randomly.

(2) Find P(this person is female | Math 380).

(1) Find P(this person take math 380 occure).

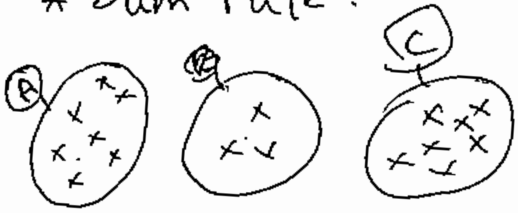


(1) $P(\text{Course Math 380}) = \boxed{\frac{20}{45} * 0.4 + \frac{25}{45} * 0.55} = x$

(2) $P(\cancel{\text{fema}}\ \text{female} \mid \text{Course}) = \frac{25/45 * 0.55}{x} =$

(1.2) Counting:

* Sum rule:



choose <u>one element</u> from A <u>or</u> B <u>or</u> C:

Number of ways $= |A| + |B| + |C|$.

* Product rule:

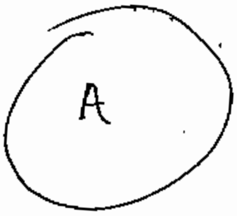
Choose one element, from each set:

Number of ways $= |A| \times |B| \times |C|$.

Example: In how many ways, one can choose one ball from Box1 (8 balls), <u>and</u> one ball from Box2 (10 balls).

$10 \times 8 = 80$.

* Combination:



Choose <u>at time</u>, r elements from A.

Number of ways $= \binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$

$A = \{a, b, c\}$

ab, ac, bc

~~ba~~, ~~ac~~

$\binom{3}{2} = \dfrac{3!}{2!\,1!} = 3$

1.4 Independent events:

We say that two events A and B are independent if:

$$P(A|B) = P(A).$$

$$\Updownarrow$$

$$P(A\cap B) = P(A)\cdot P(B).$$

$$\Updownarrow$$

$$P(B|A) = P(B).$$

Example: A fair coin is tossed twice.

$A = \{$head appears in the 1^{st} toss$\}$.

$B = \{$Tail appears in the 2^{nd} toss$\}$.

(1) Are A and B independent?

$S = \{HH, HT, TH, TT\}$.

$P(A) = P(HH, HT) = \frac{1}{4}+\frac{1}{4} = 1/2$.

$P(B) = P(HT, TT) = 1/4 + 1/4 = 1/2$.

$P(A\cap B) = P(HT) = \frac{1}{4}$.

$P(A)\cdot P(B) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = P(A\cap B)$.

A and B are independent.

---

(2) Compute $P(A\cap B') = P(A) - P(A\cap B)$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

Example: Box 1: 7 red + 8 white balls.
In how many ways, one can shoose:

a) Two balls,

$$\binom{15}{2} = \frac{15!}{2!\,13!} = \frac{14 \times 15}{2} = 105.$$

b) Two red balls,

$$\binom{7}{2} = \frac{7!}{2!\,5!} = \frac{6 \times 7}{2} = 21.$$

c) Two balls aith the same Color.

(2R) OR (2W): $\binom{7}{2} + \binom{8}{2} = 21 + \frac{8!}{2!\,6!} = 21 + \frac{7 \times 8}{2} = 49.$

d) Three balls Containing <u>at most</u>, one red.

$\boxed{(1R \text{ and } 2W)}$ OR $(0R \text{ and } 3W)$.

$$\binom{7}{1} \times \binom{8}{2} + \binom{8}{3} = 7 \times 28 + 56 = 252.$$

* Arrangement and Repetition:

Example: A car plate is composed of 4 letters fellowed by 3 <u>different</u> digits.

How many car plates, one can obtain:

a) without restriction

26	26	26	26	10	9	8

$26^4 \times 10 \times 9 \times 8.$

(7)

Example: let 3 events A, B and C:

$P(A) = 1/2$, $P(B) = 1/4$, $P(C) = 1/6$,

$P(A \cap B) = 1/5$, $P(A \cup C) = \frac{7}{12}$.

Compute $P(A|B)$, $P(A'|B)$, $P(A \cup B|B)$, $P(C|A)$, $P(C'|A')$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/5}{1/4} = 4/5.$$

$$P(A'|B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = \frac{1/4 - 1/5}{1/4}$$

$$= 1 - 4/5 = 1/5.$$

$$P(A \cup B|B) = \frac{P(\overbrace{(A \cup B)}^{E} \cap B)}{P(B)} = \frac{P(E) + P(B) - P(\boxed{E \cup B})}{P(B)}$$

$$\cancel{P(A) + P(B) - P(A \cap B) + P(B)}$$

$$= \frac{P(\cancel{E}) + P(B) - P(\cancel{E})}{P(B)} = 1.$$

$$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{P(C) + P(A) - P(C \cup A)}{P(A)}$$

$$= \frac{1/2 + 1/6 - 7/12}{1/2} = 1 + \frac{1}{3} - \frac{7}{6} = \frac{6+2-7}{6}$$

$$= \frac{1}{6}.$$

$$P(C'|A') = \frac{P(C' \cap A')}{P(A')} = \frac{P((C \cup A)')}{P(A')} = \frac{1 - P(C \cup A)}{1 - P(A)}$$

$$= \frac{1 - 7/12}{1/2}$$

$$= 2 - \frac{7}{6} = 5/6.$$

$(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$

$P(A'|B) = 1 - P(A|B)$.

$\cancel{P(A|B') = 1 - P(A|B)}$

* **Event:**

it is a subset of a Sample space.

Examples: * $S = \{H, T\}$, $A = \{H\}$, $B = \{T\}$

S is the Certain event.

$\emptyset$ is the impossible event.

* $S = \{1, 2, \dots, 6\}$, $A = \{1\}$,

$B = \{2, 4, 6\}$, $C = \{\text{the number is odd}\}$.

$= \{1, 3, 5\}$.

(1.2) Probability:

let S a sample space.

We call a Probability P on S, ~~by~~ any function from S to $\mathbb{R}$ (set of real number), Satisfying the following axioms:

* $P(A) \geq 0$, for any event A.
* $P(S) = 1$.
* let two disjoint events A and B:

$$P(A \cup B) = P(A) + P(B).$$

* let a sequence of disjoint events (A_n):

$$P\left(\bigcup_n A^n\right) = \sum_n P(A_n).$$

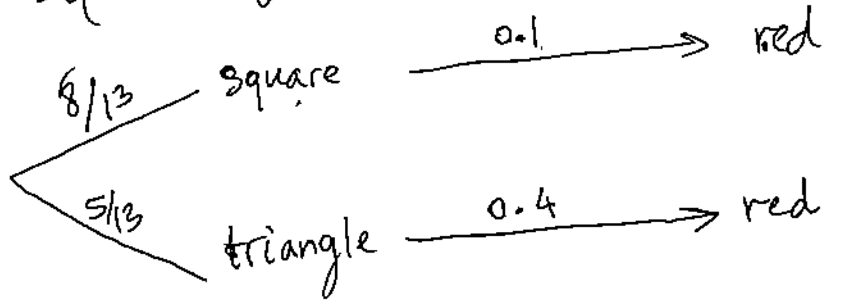
one element of the box is selected at <u>random</u>.

Example:

A box contains 8 squarres and 5 triangles.
10% of squarres and 40% of triangles are red.

a) Find the Probability that this $\boxed{\text{element is red}}$.

b) Find ______________________________ is a squarre given that it is red.



(a) $P(R) = \frac{8}{13} * 0.1 + \frac{5}{13} * 0.4 = x = \frac{14}{65}$

(b) $P(Sq \mid R) = \dfrac{P(R \mid Sq) \cdot P(Sq)}{P(R)}$

$= \dfrac{0.1 * \frac{8}{13}}{14/65} = \frac{4}{14} = \frac{2}{7}$.

Exercise: A die is rolled, ~~in~~ 20 times with each number is equally likely to appear.

(1) Find the average number of times, the number "5" will appear.

$X =$ number of time, "5" will appear.

$X \sim Bin\left(\frac{1}{6}, 20\right)$.

$E(X) = \left(\frac{1}{6}\right) * 20 = \frac{20}{6} = \frac{10}{3}$.

(2) Find the average number of times, "5" will not appear.

$Y =$ ____________

$Y = 20 - X$.

$Y \sim Bin(5/6, 20)$, $E(Y) = \left(\frac{5}{6}\right) \times 20 = \frac{100}{6}$.

$EY = 20 - E(X) = 20 - \frac{20}{6} = \frac{100}{6}$.

Exercise:

Let X with mass function:

x	-1	0	1	2
$f(x)$	$1/8$	$3/8$	$1/8$	$3/8$

Compute $E(X)$, $E(X^2)$, $E(2X+1)$,

* $E(X) = (-1)\frac{1}{8} + (0)\cdot\frac{3}{8} + (1)\frac{1}{8} + (2)\frac{3}{8}$

$= 6/8$

* $E(X^2) = (-1)^2\frac{1}{8} + (0)^2\frac{3}{8} + (1)^2\frac{1}{8} + (2)^2\frac{3}{8}$

$= \frac{14}{8}$.

* $E(2X+1) = 2E(X) + 1 = \frac{12}{8} + 1 = \frac{20}{8}$.

Exercise: $E(X) = \sum x f(x)$.

4 persons are selected at random from a group of 8 men and 10 women, to form a committee.

a) Find the probability that $\boxed{\text{the committee contains exactly 2 women.}}$ $\rightarrow$ A

b) Find the average number of women in this committee.

---

a) X = number of women in the committee.

$X \sim Bin(\frac{10}{18}, 4)$.

$P(X=2) = \binom{4}{2}\left(\frac{10}{18}\right)^2\left(\frac{8}{18}\right)^2$ | $P(A) = \frac{\binom{8}{2}\binom{10}{2}}{\binom{18}{4}}$

b) if the first letter is A or B.

2	26	26	26	10	9	8

$2 \times 26^3 \times 10 \times 9 \times 8$.

c) if the first two letters are identical.

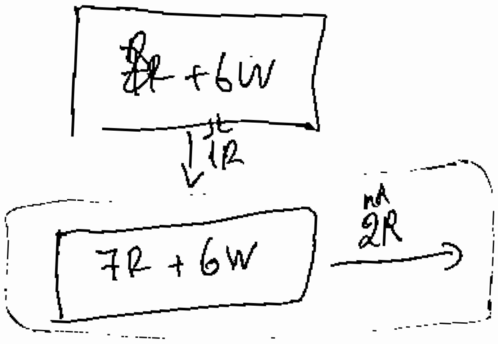
26	1	26	26	10	9	8

$26^3 \times 10 \times 9 \times 8$.

1.3 Conditional Probability:

Example: we choose ~~r~~randomly one ball from a box, contaning 8 red and 6 white balls. we keep it and choose a second ball.

① Find the Probability that the 2^{nd} ball is red, given (if) that the 1^{st} ball is red.



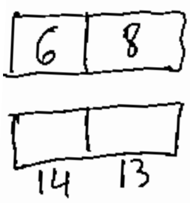
$$P(\overbrace{2^{nd} \text{ ball is red}}^{A} \text{ given that } 1^{st} \text{ is red}) = \frac{|A|}{|S|} = \frac{7}{13}.$$

Rule: Two event A and B.
The conditional Probability of A given B, is given by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

② Find $P(2^{nd} \text{ ball is red} \mid 1^{st} \text{ ball is white})$

$$= \frac{8}{13}.$$



$$= \frac{P(2^{nd} \text{ red}, 1^{st} \text{ white})}{P(1^{st} \text{ white})} = \frac{\frac{\cancel{6} \times 8}{\cancel{14} \times 13}}{\cancel{6}/\cancel{14}}$$

$$= \frac{8}{13}$$

Examples: (1) A fair coin is tossed 3 times. Let
X = numbers of heads.
Compute the Probability mass function of X?
$x \in \{3, 2, 1, 0\}$.
$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$f(0) = P(X=0) = P(TTT) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.
$f(1) = P(X=1) = P(\{HTT, THT, TTH\}) =$
$= P(HTT) + P(THT) + P(TTH) = \frac{3}{8}$.
$f(2) = P(X=2) = P(HHT, HTH, THH) = 3/8$.
$f(3) = P(X=3) = P(HHH) = \frac{1}{8}$.

x	0	1	2	3
$f(x)$	$1/8$	$3/8$	$3/8$	$\frac{1}{8}$

(2) An unweigthed die is tossed once. Let

unweigthed

$X = \begin{cases} 2 & \text{if the number is even.} \\ 0 & \text{if} \end{cases}$

Find the mass function of X?
$f(2) = P(X=2) = P(\{2,4,6\}) = 3/6 = 1/2$.
$f(0) = P(X=0) = P(\{1,3,5\}) = 3/6 = 1/2$.

x	0	2
$f(x)$	$1/2$	$1/2$

2 Random Variables:

Definition: let S the Sample space.
We call X a random variable if it is a function from S to $\mathbb{R}$.

Examples:

* Tossing a coin 4 times. $S = \{\boxed{HHHH}, \underline{HHHT}, HHTH, \dots, \underline{TTTT}\}$ HHTT
  X = number of heads $\in \{4, 3, 2, 1, 0\}$.
  $Y = \begin{cases} 1 & \text{if head appear in the fist toss.} \\ -1 & \text{if not.} \end{cases}$
* Rolling a die:
  $S = \{1, 2, 3, 4, 5, 6\}$.
  $X = \begin{cases} 2 & \text{if the number is even.} \\ 0 & \text{if not.} \end{cases}$
  $Y = \begin{cases} -1 & \text{if the number} = 3 \\ 3 & = 4 \text{ or } 5 \\ 4 & \text{if not.} \end{cases}$

2.1 Discrete random variables:

Definition: A discrete random variable is a random variable, taking a countable number of values.

* Probability mass function:
The Probability mass function f of a random variable X taking the values $\{x_1, x_2, x_3, \dots\}$, is the function defined by:

$$f(x) = P(X = x) \quad , \quad x = x_1, x_2, x_3, \dots$$

Example (2): An unweigthed die is tossed 20 times. Find the Probability that {the number 5 will appear exactly 7 times}.

Success = $\{5\}$.

X = number of times, number "5" will appear

$\sim \text{Bin}(\frac{1}{6}, 20)$..

$$P(A) = P(X=7) = \binom{20}{7}\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^{13}.$$

Mean:

$$E(X) = \sum_{k=0}^{n} k f(k) = \sum_{k=0}^{n} \underbrace{k\binom{n}{k}}\, p^k (1-p)^{n-k}.$$

$$k\binom{n}{k} = k\frac{n!}{k!\,(n-k)!} = \frac{n!}{(k-1)!\,(n-1-(k-1))!}$$

$$= \frac{n\,(n-1)!}{(k-1)!\,((n-1)-(k-1))!} = n\binom{n-1}{k-1}.$$

$$E(X) = \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$j = k-1$

$$= \sum_{j=0}^{n-1} n\binom{n-1}{j} p^{j+1}(1-p)^{n-j-1}.$$

$$= np \underbrace{\sum_{j=0}^{n-1}\binom{n-1}{j} p^j (1-p)^{(n-1)-j}}_{=1}$$

$$\boxed{E(X) = np}$$

$$\text{Var}(X) = np(1-p).$$

$$E(X^2) = \sum_{k=0}^{n} k^2\binom{n}{k} p^k (1-p)^{n-k}.$$

Remark: $X \sim \text{Bin}(p,n)$, $X = X^1 + X^2 + \dots + X^n$.

$X^i \sim \text{Ber}(p)$, $X^i = \begin{cases} 1 & \text{with } p. \\ 0 & \text{with } 1-p. \end{cases}$

* Examples of discrete random variables:

① Bernoulli random variable (distribution):

any random variable X taking only 2 values x_1 and x_2, with $f(x_1) = p$, $f(x_2) = 1-p$.

random experiment
- Success, p
- failure, $1-p$.

$$E(X) = x_1 p + x_2 (1-p).$$

② Binomial random variable (distribution).

- Success, p
- failure, $1-p$

we repeat this basic experiment $\underline{\underline{n}}$ times.

let X = number of successes.

X has a Binomial distribution.

The mass function of $X \in \{0, 1, 2, \dots, n\}$.

$$f(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Example: we toss a fair coin 10 times. Find the Probability that head appear exactly 4 times.

X = number of heads, $X \sim Bin(1/2, 10)$.

$$P(\text{head appear exactly 4 times}) = P(X=4)$$
$$= \binom{10}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^6.$$

* Mean and variance:

Mean or expectation of a random variable X is:

$$E(X) = \mu_X = \sum x f(x).$$

$$E(g(X)) = \sum g(x) f(x).$$

Variance of X:

$$\text{Var}(X) = \sigma_X^2 = E\left(X - E(X)\right)^2$$

$$= E(X^2) - (E(X))^2.$$

Examples:

let

x	-1	0	1	2
$f(x)$	$1/8$	$2/8$	$1/8$	$4/8$

Compute $E(X)$, $E(X^2)$, $E(X^3)$, $\text{Var}(X)$, $\text{Var}(X^2)$.

$$E(X) = (-1) * \frac{1}{8} + (0) * \frac{2}{8} + (1) * \frac{1}{8} + (2) * \frac{4}{8} = 1.$$

$$E(X^2) = (-1)^2 * \frac{1}{8} + (0)^2 * \frac{2}{8} + (1)^2 * \frac{1}{8} + (2)^2 * \frac{4}{8} = \frac{18}{8}.$$

$$E(X^3) = (-1)^3 * \frac{1}{8} + (0)^3 * \frac{2}{8} + (1)^3 * \frac{1}{8} + (2)^3 * \frac{4}{8} = 4.$$

$$\text{Var}(X) = \frac{18}{8} - (1)^2 = \frac{10}{8} = 5/4.$$

$$\text{Var}(X^2) = \underline{E(X^4)} - (E(X^2))^2 = \frac{66}{8} - \left(\frac{18}{8}\right)^2 = \frac{51}{16}.$$

* Cumulative distribution function:

$$F(x) = P(X \le x) = \sum_{y \le x} f(y) \;;\; x \in \mathbb{R}.$$

Examples:

(1) let f as follows:

x	-1	0	2	4
$f(x)$	$3/8$	$1/8$	$1/4$	$1/4$

Find the Cumulative distribution function?

0	$3/8$	$\frac{4}{8} = \frac{1}{2}$	$\frac{3}{4}$	1

$$F(x) = \begin{cases} 0 & x < -1 \\ 3/8 & -1 \le x < 0 \\ 1/2 & 0 \le x < 2 \\ 3/4 & 2 \le x < 4 \\ 1 & x \ge 4. \end{cases}$$

(2) let the Cum. dist. function:

$$F(x) = \begin{cases} 0 & x < -2 \\ 1/7 & -2 \le x < 1 \\ 3/7 & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Find the mass function f?

x	-2	1	3
$f(x)$	$1/7$	$2/7$	$4/7$

Example:

let $f(x) = \begin{cases} \frac{3}{2}(1-x^2) & ; \quad 0 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$

Compute F ?

$f=0 \qquad 0 \quad f=\frac{3}{2}(1-x^2) \quad 1 \qquad f=0$

$$F(x) = \boxed{\int_{-\infty}^{x} f(t)\, dt}.$$

$$\left[\cancel{\int_1^{\infty} 1\, dx = +\infty} \right]$$

$\underline{\underline{x < 0}}$:

$$F(x) = 0.$$

$\underline{\underline{0 \leq x < 1}}$: $F(x) = \cancel{\int_{-\infty}^{0} f(t)\, dt}^{=0} + \int_0^x \frac{3}{2}(1-t^2)\, dt$

$$= \frac{3}{2}\left(t - \frac{t^3}{3}\right)\Big|_0^x = \frac{3}{2}\left(x - x^3/3\right)$$

$\underline{\underline{x \geq 1}}$: $F(x) = \int_{-\infty}^{1} f(t)\, dt + \cancel{\int_1^x f(t)\, dt}^{=0}$

$$= F(1) = 1.$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 3/2\,(x - x^3/3) & 0 \leq x < 1 \\ 1 & x \geq 1. \end{cases}$$

Example: let the cumulative function:

$$F(x) = \begin{cases} 0 & x \leq 1 \\ (x-1)^2 & 1 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$

Compute $f(x)$?

$$f(x) = \begin{cases} 2(x-1) & 1 \leq x < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Discrete random variables : X

mass function, Cumulative distribution.

Mean, variance.

$$E\,g(X) = \sum_x g(x) f(x).$$

Bernoulli, Binomial, Geometric, Poisson.

(2.2) Continuous random variables :

① Probability density function :

we call a probability density function f of a continuous random variable, any function satisfying :

(1) $f \geq 0$.

(2) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Example : let the density function (P.d.f) :

$$f(x) = \begin{cases} m(1-x^2), & 0 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Compute m ?

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 m(1-x^2)\,dx = m\left(x - \frac{x^3}{3}\right)\Big|_0^1$$

$$= m\left(1-\frac{1}{3}\right) - m(0-0) = \frac{2}{3}m.$$

$$1 = \frac{2}{3}m \iff m = 3/2.$$

② Cumulative distribution function :

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt.$$

Example: Box = 8 red + 6 white + 4 green balls.
we choose a ball randomly and replace it, we repeat untill obtaining a red ball.

a) Find the probability that the red ball was in the 4^{th} trial?

b) Find the average (mean) number of trials?

X = number of trials to find a red ball.

$X \sim Geo\left(p = \frac{8}{18}\right)$.

a) $P(X=4) = (1-p)^3 p = \left(\frac{10}{18}\right)^3 \cdot \frac{8}{18}$.

b) $E(X) = \frac{1}{p} = \frac{18}{8} = 2.25$.

(4) Poisson random variable (distribution):

The Poisson distribution is used to count the number of times, an event can occur in a time interval.

let X a Poisson random variable.

$X \sim Poi(\lambda)$. $X \in \{0, 1, 2, 3, \dots\}$.

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

$E(X) = \lambda$; $Var(X) = \lambda$.

Example: In a call centre, the average of incoming calls in an hour is 4.
Find the probability that in the coming hour, 5 calls will arrive?

$X \sim Poi(\lambda)$; $\lambda = 4$, $P(X=5) = e^{-4} \frac{4^5}{5!}$.

(3) Geometric random variable (distribution).

we have
- Sucess
- failure.

we repeat untill ~~ob~~ obtaining sucess.

let X = number of trials.

X is called geometric random variable.

$$X \sim Geo(p).$$

$$X \in \{1, 2, 3, 4, \dots\}.$$

$$f(k) = P(\underbrace{X=k}) = P(\underbrace{FF\cdots F}_{(k-1)\text{ times}}S) = (1-p)^{k-1}p. \qquad \underline{q=1-p}.$$

$$E(X) = \sum_{k=1}^{\infty} kpq^{k-1} = p\frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

$$\left[\sum_{k=1}^{\infty} x^k = \frac{x}{1-x} \; ; \; \underline{\sum_{k=1}^{\infty} kx^{k-1}} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}\right]$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2pq^{k-1} = p\frac{1+q}{(1-q)^3} = p\frac{2-p}{p^3} = \frac{2-p}{p^2}.$$

$$\left[\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \; ; \; \sum_{k=1}^{\infty} k^2x^{k-1} = \frac{(1-x)^2+2x(1-x)}{(1-x)^4}\right.$$

$$\left. = \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}.\right]$$

$$Var(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}.$$

② Exponential distribution:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 . \\ 0 & x < 0 . \end{cases}$$

$* \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{\infty} = 1 .$

$* \quad F(x) = \begin{cases} 0 & x < 0 . \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$

$* \quad E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}\,dx}_{dv} .$

$u = x . \longrightarrow du = 1\,dx .$
$dv = \lambda e^{-\lambda x}\,dx \longrightarrow v = -e^{-\lambda x} .$

$E(X) = \underbrace{-x e^{-\lambda x}\Big|_0^{\infty}} + \int_0^{\infty} e^{-\lambda x}\,dx = -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = \frac{1}{\lambda} .$

$E(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx = \underbrace{-x^2 e^{-\lambda x}\Big|_0^{\infty}} + \int_0^{\infty} 2x e^{-\lambda x}\,dx .$

$= \frac{2}{\lambda} \int_0^{\infty} x \lambda e^{-\lambda x}\,dx = \frac{2}{\lambda^2} .$

$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} .$

Example: let $X \sim Exp(2)$.
Compute $P(X \leq 4)$, $P(X \geq 1)$, $P(1 \leq X \leq 3)$?

$P(X \leq 4) = F(4) = \int_0^4 2 e^{-2x}\,dx .$
$= 1 - e^{-8} .$

$P(X \geq 1) = 1 - F(1) = 1 - (1 - e^{-2}) = e^{-2} .$

$P(1 \leq X \leq 3) = F(3) - F(1) = (1 - e^{-6}) - (1 - e^{-2})$
$= e^{-2} - e^{-6} .$

Example:

let $f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

Compute $E(X) = ?$

$$\int_0^{\infty} e^{-x} dx = -e^{-x}\Big|_0^{\infty} = 1.$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} \underbrace{x}_{u} \underbrace{e^{-x} dx}_{dv}.$$

$u = x \longrightarrow du = 1\, dx.$

$dv = e^{-x} dx \longrightarrow v = -e^{-x}.$

$$E(X) = -x e^{-x}\Big|_0^{\infty} - \int_0^{\infty} -e^{-x} dx$$

$$0 + \int_0^{\infty} e^{-x} dx = 0 + (-e^{-x})\Big|_0^{\infty} = 1.$$

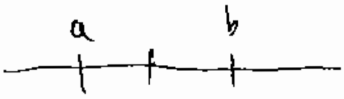
$$\lim_{x \to \infty} x e^{-x} = \lim \frac{x}{e^{x}} = \lim \frac{1}{e^{x}} = 0.$$

Examples of distributions:

① Uniform distribution:

$$f(x) = \begin{cases} m & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

$*$ $1 = \int_{-\infty}^{\infty} f(x) dx = \int_a^b m\, dx = m(b-a).$



$$\boxed{m = \frac{1}{b-a}}$$

$*$ $F(x) = \begin{cases} 0 & x < a \\ m(x-a) & a \leq x < b. \\ 1 & x \geq b \end{cases}$

$*$ $E(X) = \int_a^b m x\, dx = m \frac{x^2}{2}\Big|_a^b = \frac{m}{2}(b^2 - a^2).$

$= \frac{m}{2}(b-a)(b+a) = \frac{b+a}{2}.$

$*$ $E(X^2) = \int_a^b m x^2 dx = \frac{m}{3} x^3\Big|_a^b = \frac{m}{3}(b^3 - a^3)$

$= \frac{m}{3}(b-a)(b^2 + ab + a^2) = \frac{b^2 + ab + a^2}{3}$

$*$ $Var(X) = \frac{b^2 - 2ab + a^2}{12} = \frac{(a-b)^2}{12}.$

Mean:

$$E(X) = \mu_X = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

$$E\, g(X) = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx.$$

Variance:

$$\mathrm{Var}(X) = \sigma_X^2 = E\left(X - E(X)\right)^2 = E(X^2) - \left(E(X)\right)^2.$$

Example:

let $f(x) = \begin{cases} m(4 - x^2), & 0 \le x \le 2 \\ 0 & \text{elsewhere.} \end{cases}$

(1) Compute m.

(2) Compute $E(X)$, $E(X^2)$, $E(X^4)$, $\mathrm{Var}(X)$.

---

(1) $$1 = \int_{-\infty}^{\infty} f(t)\, dt = \int_0^2 m(4 - x^2)\, dx = m\left(4x - \frac{x^3}{3}\right)\Big|_0^2$$

$$= m\left(8 - \frac{8}{3}\right) - 0 = \frac{16}{3} m.$$

$$\boxed{m = 3/16}.$$

(2) $$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^2 \frac{3}{16} x (4 - x^2)\, dx.$$

$$= \frac{3}{16} \int_0^2 (4x - x^3)\, dx = \frac{3}{16}\left(2x^2 - \frac{x^4}{4}\right)\Big|_0^2$$

$$= \frac{3}{16}\left(8 - \frac{16}{4}\right) = 3/4.$$

$$E(X^2) = \int_0^2 \frac{3}{16} x^2 (4 - x^2)\, dx = \frac{3}{16}\int_0^2 (4x^2 - x^4)\, dx$$

$$= \frac{3}{16}\left(\frac{4}{3}x^3 - \frac{x^5}{5}\right)\Big|_0^2 = \frac{3}{16}\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{6}{\cancel{6}} \cdot \frac{2}{15}$$

$$= \frac{4}{5}.$$

$$\mathrm{Var}(X) = \frac{4}{5} - \left(3/4\right)^2 = \frac{4}{5} - \frac{9}{16} = \frac{19}{80}.$$

Probabilities:

Let X a cont. random variable with density function f:

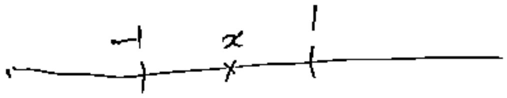
* $P(X \le a) = F(a) = \int_{-\infty}^{a} f(x)\,dx$.
* $P(X > a) = 1 - F(a) = \int_{a}^{\infty} f(x)\,dx$.
* $P(a \le X \le b) = F(b) - F(a) = \int_{a}^{b} f(x)\,dx$.
* $P(|X| \le a) = P(-a \le X \le a)$
  $= F(a) - F(-a) = \int_{-a}^{a} f(x)\,dx$.

Example: let the density function:

$$f(x) = \begin{cases} \frac{5}{2}x^4 & -1 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Compute: $P(X \ge 0)$, $P(X \le 3/4)$, $P(|X| \le \frac{1}{2})$.

$$F(x) = \begin{cases} 0 & x < -1 \\ x^5/2 + \frac{1}{2} & -1 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$



? $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} \cancel{f(t)}^{\,0}\,dt + \int_{-1}^{x} f(t)\,dt = \frac{\cancel{5}}{2}\frac{t^5}{\cancel{5}}\Big|_{-1}^{x}$

$= \frac{x^5}{2} - \left(\frac{-1}{2}\right) = \frac{x^5}{2} + \frac{1}{2}$.

* $P(X \ge 0) = 1 - F(0) = 1 - \frac{1}{2} = \frac{1}{2}$.
* $P(X \le 3/4) = F(3/4) = \frac{1}{2}(3/4)^5 + \frac{1}{2} = \frac{243}{2 \times 4^5} + \frac{1}{2}$.
* $P(|X| \le 1/2) = F(1/2) - F(-1/2)$
  $= \left(\frac{1}{2}\left(\frac{1}{2}\right)^5 + \cancel{\frac{1}{2}}\right) - \left(\frac{1}{2}\left(\frac{-1}{2}\right)^5 + \cancel{\frac{1}{2}}\right)$
  $= \frac{1}{2^5} = \frac{1}{32}$.

? $P(|X| \le -\frac{1}{2}) = 0$, $P(|X| \ge -\frac{1}{2}) = 1$
$\emptyset$

③ Normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} . \qquad (\mu, \sigma^2).$$

$$X \sim N(\mu, \sigma^2).$$

* X is called standard normal random variable if $X \sim N(0,1)$.

* $X \sim N(\mu, \sigma^2)$, $\frac{x-\mu}{\sigma} \sim N(0,1)$.

* $\boxed{X \sim N(0,1)}$ $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

* $E(X) = \mu$, $Var(X) = \sigma^2$.
