System of Linear Equations

Method of Solving Systems of Linear Equations

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where $x_1, \ldots, x_n$ variables (unknown), $a_1, \ldots, a_n$ coeficient and b constant.

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Example

Linear

(a) $x + 3y = -15$

(b) $x + y + z = 0$

(b) $x = y$

(d) $3x + 2y = 7$

Non-Linear

$x = ay^2$ or $y = ax^2$

$x^2 + y + z^3 = 5$

$x^{-1} + yx + z^2 = 0$

$5x + y + z^{-5} = 3.$

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Examples

(a) $\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned}$ (b) $\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= 7 \end{aligned}$ (c) $\begin{aligned} 3x - y + 6z &= 6 \\ x + y + z &= 2 \\ 2x + y + 4z &= 3 \end{aligned}$

The systems $(a), (c)$ are consistent but the system (b) is inconsistent.

Suppose we have the following linear system:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
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We can write the linear system in the form of matrices product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We may write it in the form $AX = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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The augmented matrx of the linear system is

$$[A : b] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

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Write the matrix form and the augmented matrix for the following system:

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3x - y + 6z &= 6 \\
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Solution

The matrix form of the system is

$$\begin{bmatrix} 3 & -1 & 6 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

The augmented matrix is

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2. Find the solution by back subtitutions.

Example

Solve the following linear system by Guassian elimination method:

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= 8 \\
-x_1 + 2x_2 + 3x_3 &= 1 \\
3x_1 - 7x_2 + 4x_3 &= 10.
\end{aligned}
$$

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$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \xrightarrow{R_1+R_2, -R_1+R_3} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & 2 & -14 \end{bmatrix}$$

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Solution of the system is: $x_1 = 3, x_2 = 1, x_3 = 2$.

Exercise: Use Guassian elimination method to solve:

$$\begin{aligned} x + 8y + 2z &= 7 \\ 2x + 4y - 4z &= 3 \\ z + y + 2x &= 2. \end{aligned}$$

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Example

Suppose the points $(-2,1), (-1,2), (1,2)$ *lie on parabola*

$$y = a + bx + cx^2.$$

1. *Determine a linear system in 3 variables* a, b, c.
2. *Fined the equation of parabola by solving the linear system.*