King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2016–2017
Academic Year (H) 1437–1438
Bachelor AFM: M. Eddahbi

Solution of Homework 2 (Cumulative distributions function, Expectation and Variance)

Problem 1.

1. The GPA (grade point average) random variable X assigns to the letter grades A, B, C, D and E the numerical values 4,3,2, 1 and 0. Find the expected value of X for a student selected at random from a class in which there were 15 A grades, 33 B grades, 51 C grades, 6 D grades, and 3 E grades. (This expected value can be thought of as the class average GPA for the course.).
Solution: The total number of students is $15 + 33 + 51 + 6 + 3 = 108$,
$$E[X] = 4 \times \frac{15}{108} + 3 \times \frac{33}{108} + 2 \times \frac{51}{108} + 1 \times \frac{6}{108} + 0 \times \frac{3}{108} = \frac{267}{108} \simeq 2.4722.$$

2. A construction company whose workers are used on high–risk projects insures its workers against injury or death on the job. One unit of insurance for an employee pays \$1000 for an injury and \$10000 for death. Studies have shown that in a year 7.3% of the workers suffer an injury and 0.41% are killed.
What is the expected unit claim amount (pure premium) for this insurance?
Solution: The expected unit claim amount (pure premium) is
$$1000 \times 0.073 + 10000 \times 0.0041 = \$114.$$

3. If the company has 10000 employees and exactly 7.3%, are injured and exactly 0.41% are killed. What is the average cost per unit of the insurance claims?
Solution: We have
$$10000 \times 0.073 = 730 \quad \text{injured}$$
and
$$10000 \times 0.0041 = 41 \quad \text{killed}$$
The average cost per unit of the insurance claims is
$$\frac{730 \times \$114 + 41 \times \$114}{730 + 41} = \$114$$

4. Suppose that in the above question the administrative costs are \$50 per person insured. The company purchases 10 units of insurance for each worker. Let X be the total of expected claim amount and administrative costs for each worker. Find the expectation of X.
Solution: $E[X] = 114 \times 10 + 50 = 1190.$

5. For the insurance policy that pays \$1000 for an injury and \$10000 for death, what is the standard deviation for the claim amount on 5 units of insurance? (Note: Some employees receive \$0 of claim payment. This value of the random variable must be included in your calculation).
Solution: The standard deviation for the claim amount on 5 units of insurance is
$$sd(claim) = \sqrt{5 \times \frac{730}{771} \times \frac{41}{771}} = \frac{5}{771}\sqrt{146 \times 41} = 0.5017$$

Problem 2.

An urn contains N bulls numbered from 1 to N. We pick a randomly a bull (all the bulls are equally likely to be extracted) and define the r.v. X by the number of the extracted bull.

1. Calculate the expectation and the variance of X.
   Solution: Remark fist that X is a uniform discrete random variable on $\{1, 2, 3, \ldots, N\}$.

$$E[X] = \sum_{k=1}^{N} kP(X = k) = \frac{1}{N} \sum_{k=1}^{N} k = \frac{(1+N)N}{2N} = \frac{1+N}{2}$$

   and

$$E\left[X^2\right] = \sum_{k=1}^{N} k^2 P(X = k) = \frac{1}{N} \sum_{k=1}^{N} k^2 = \frac{N(2N+1)(N+1)}{N6}.$$

   Hence

$$Var(X) = E\left[X^2\right] - (E[X])^2 = \frac{(2N+1)(N+1)}{6} - \frac{(1+N)^2}{4} = \frac{N^2-1}{12}.$$

2. We toss n times a fair coin and define the r.v. X to be the number of tails got after n tosses and define the r.v. $Y = \dfrac{a^X}{2^n}$, $(a \in \mathbb{R}_+^*)$. Calculate $E[Y]$.
   Solution: We know that X ha a binomiale distributuion $\mathcal{B}(n, \frac{1}{2})$ then

$$E[Y] = E\left[\frac{a^X}{2^n}\right] = \frac{1}{2^n} E\left[a^X\right] = \frac{1}{2^n} \sum_{k=0}^{n} a^k \frac{1}{2^n} \binom{n}{k} = \frac{1}{2^{2n}} \sum_{k=0}^{n} a^k \binom{n}{k} == \frac{1}{2^{2n}} (a+1)^n.$$

Problem 3.

1. Let X be a Poison r.v. with parameter λ and define the r.v. Y by

$$Y = \begin{cases} \dfrac{X}{2} & \text{if } X \text{ is even} \\ 0 & \text{if } X \text{ is odd.} \end{cases}$$

   (a) Find the distribution Y, and calculate its expectation and its variance.
   Solution: $S_Y = \mathbb{N}$, for $n \in \mathbb{N}$, the p.m.f. is defined by $f(n) = P(Y = n)$. So for any $n \geq 1$, we have

$$f(n) = P(Y = n) = P(X = 2n) = e^{-\lambda} \frac{\lambda^{2n}}{(2n)!}$$

   Details were done in the class room (see notes in the Bb: https://lms.ksu.edu.sa/).

2. Let X and Y be two independent r.v. taking values in $\mathbb{N}$: such that X follows the Bernoulli distribution with parameter p and Y follows a Poisson distribution of parameter λ. Now define the r.v. Z by $Z = XY$.

   (a) Calculate the distribution of Z.
   Solution: Since X and Y are r.v. then Z is also a r.v. defined on a probability space

$(\Omega, \mathcal{F}, P)$ Moreover $S_Z = \mathbb{N}$, for $n \in \mathbb{N}$, the p.m.f. is defined by $f(n) = P(Z = n)$. So for any $n \geq 1$, we have

$$\begin{aligned}
f(n) &= P(Z=n) = P(\{Z=n\} \cap (\{X=0\} \cup \{X=1\})) \text{ since } \{X=0\} \cup \{X=1\} = \Omega \\
&= P((\{Z=n\} \cap \{X=0\}) \cup (\{Z=n\} \cap \{X=1\})) \\
&= P((\{XY=n\} \cap \{X=0\}) \cup (\{XY=n\} \cap \{X=1\})) \\
&= P(\{XY=n\} \cap \{X=1\}) \text{ since } \{XY=n\} \cap \{X=0\} = \emptyset \text{ for all } n \geq 0 \\
&= P(\{Y=n\} \cap \{X=1\}) = P(\{Y=n\}) P(\{X=1\}) \text{ since } X \perp\!\!\!\perp Y \\
&= pe^{-\lambda} \frac{\lambda^n}{n!}.
\end{aligned}$$

and

$$\begin{aligned}
f(0) &= 1 - \sum_{n=1}^{\infty} pe^{-\lambda} \frac{\lambda^n}{n!} = pe^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} = 1 + pe^{-\lambda} - pe^{-\lambda} e^{\lambda}. \\
&= 1 - p + pe^{-\lambda}.
\end{aligned}$$

where we have used the fact that $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$. Hence the p.m.f. of Z is given by $(n, f(n))_{n \geq 0}$ where

$$f(n) = \begin{cases} 1 - p + pe^{-\lambda} & \text{for } n = 0 \\ pe^{-\lambda} \dfrac{\lambda^n}{n!} & \text{for } n \geq 1. \end{cases}$$

(b) Find the moment generating function (MGF:) of Z.
Solution: the MGF: of Z is given by

$$\begin{aligned}
M_Z(t) &= E\left[e^{tZ}\right] = \sum_{n=0}^{\infty} e^{tn} f(n) = e^{t \times 0} f(0) + \sum_{n=1}^{\infty} e^{tn} f(n) \\
&= 1 - p + pe^{-\lambda} + \sum_{n=1}^{\infty} e^{tn} pe^{-\lambda} \frac{\lambda^n}{n!} \\
&= 1 - p + pe^{-\lambda} + pe^{-\lambda} \sum_{n=1}^{\infty} \frac{(\lambda e^t)^n}{n!} \\
&= 1 - p + pe^{-\lambda} \left(1 + \sum_{n=1}^{\infty} \frac{(\lambda e^t)^n}{n!}\right) \\
&= 1 - p + pe^{-\lambda} e^{\lambda e^t} \text{ for all } t \in \mathbb{R}
\end{aligned}$$

(c) Deduce $E[Z]$ and $Var(Z)$.
Solution: We know that $E[Z] = M_Z'(0)$ and $Var(Z) = M_Z''(0) - (M_Z'(0))^2$. But we have

$$M_Z'(t) = pe^{-\lambda} \lambda e^t e^{\lambda e^t} \text{ and } M_Z''(t) = pe^{-\lambda} \lambda e^t e^{\lambda e^t} + pe^{-\lambda} (\lambda e^t)^2 e^{\lambda e^t}$$

hence for $t = 0$, we get

$$E[Z] = M_Z'(t) = pe^{-\lambda} \lambda e^{\lambda} = p\lambda \text{ and } M_Z''(0) = p\lambda + p\lambda^2$$

and

$$Var(Z) = p\lambda + p\lambda^2 - p^2\lambda^2 = p\lambda + p(1-p)\lambda^2.$$

(d) Calculate $P(X = 1 \mid Z = 0)$.
Solution: By definition of the conditional probability we have

$$\begin{aligned} P(X = 1 \mid Z = 0) &= \frac{P(X = 1\ ;\ Z = 0)}{P(Z = 0)} = \frac{P(X = 1\ ;\ XY = 0)}{1 - p + pe^{-\lambda}} \\ &= \frac{P(X = 1\ ;\ Y = 0)}{1 - p + pe^{-\lambda}} = \frac{P(X = 1)\,P(Y = 0)}{1 - p + pe^{-\lambda}} \quad \text{since } X \amalg Y \end{aligned}$$

then

$$P(X = 1 \mid Z = 0) = \frac{pe^{-\lambda}}{1 - p + pe^{-\lambda}}.$$

Problem 4.

1. Determine whether the random variable is discrete or continuous.

   (a) X is a randomly selected number in the interval $[0, 1]$. **Answer:** X is a continuous r.v. because it my take in value of the interval $[0, 1]$.

   (b) Y is the number of heart beats per minute. **Answer:** X is a discrete because it can take only finite integer values.

   (c) Z is the number of calls at a switchboard in a day. **Answer:** X is a discrete because it can take only finite integer values.

   (d) $U :]0, 1[\longmapsto \mathbb{R}$ defined by $U(s) = 2s - 1$. **Answer:** X is a continuous r.v. because it my take in value of the real line.

2. Let X be a random variable with values in $\{1, 2, \ldots, n\}$ such that for each $1 \leq k \leq n$

$$P[X = k] = \frac{2k}{n(n+1)}.$$

Find $E[X]$.
Answer: By definition

$$E[X] = \sum_{k=1}^{n} kP[X = k] = \sum_{k=1}^{n} k\frac{2k}{n(n+1)} = \frac{2}{n(n+1)}\sum_{k=1}^{n} k^2.$$

We know that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

therefore

$$E[X] = \frac{2}{n(n+1)} \times \frac{n(n+1)(2n+1)}{6} = \frac{2n+1}{3}.$$

3. Let X be a discrete random variable taking its values in the set $\mathbb{N}$: Suppose that

$$P(X = 0) = P(X = 1),$$

and for each $k \geq 1$

$$P(X = k+1) = \frac{1}{k}P(X = k).$$

(a) Find $P(X=0)$.
Answer: Remark first that

$$\begin{aligned} P(X=k+1) &= \frac{1}{k}P(X=k) = \frac{1}{k(k-1)}P(X=k-1) \\ &= \frac{1}{k(k-1)(k-2)\cdots\times 2\times 1}P(X=1) \\ &= \frac{1}{k!}P(X=1) = \frac{1}{k!}P(X=0). \end{aligned}$$

Moreover we have

$$\begin{aligned} & P(X=0) + P(X=1) + \sum_{k=1}^{\infty} P(X=k+1) = 1 \\ \Longleftrightarrow\quad & P(X=0) + P(X=0) + \sum_{k=1}^{\infty} \frac{P(X=0)}{k!} = 1 \\ \Longleftrightarrow\quad & P(X=0)\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}\right) = P(X=0)\left(\sum_{k=0}^{\infty}\frac{1}{k!}\right) = P(X=0)\,e = 1 \end{aligned}$$

where we have used $\sum_{k=0}^{\infty}\frac{1}{k!} = e$ therefore $P(X=0) = e^{-1}$. This means that X follows a Poisson distribution with parameter 1.

(b) Find the expectation and variance of X.
Solution: X has a Poisson distribution with parameter 1, hence $E[X] = Var(X) = 1$

4. The amount of a single loss X for an insurance policy is exponentially distributed with density function

$$f(x) = \begin{cases} \frac{2}{10^3}\exp\left(-\frac{2}{10^3}x\right) & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the expected value of a single loss
Solution: The expected value of a single loss $E[X]$ hence

$$\begin{aligned} E[X] &= \int x f(x)dx = \int_0^{\infty} \frac{2x}{10^3}\exp\left(-\frac{2}{10^3}x\right)dx \\ &= \left[-x\exp\left(-\frac{2}{10^3}x\right)\right]_0^{\infty} + \int_0^{\infty}\exp\left(-\frac{2}{10^3}x\right)dx = \frac{10^3}{2} \end{aligned}$$

(b) Find the standard deviation of a single loss.
Solution: The standard deviation of a single loss X is given by $sd(X) = \sqrt{Var(X)}$. First let us calculate $E[X^2]$

$$\begin{aligned} E\left[X^2\right] &= \int x^2 f(x)dx = \int_0^{\infty}\frac{2x^2}{10^3}\exp\left(-\frac{2}{10^3}x\right)dx \\ &= \left[-x^2\exp\left(-\frac{2}{10^3}x\right)\right]_0^{\infty} + \int_0^{\infty} 2x\exp\left(-\frac{2}{10^3}x\right)dx \\ &= 2\frac{10^3}{2}\int_0^{\infty}\frac{2x}{10^3}\exp\left(-\frac{2}{10^3}x\right)dx = 2\frac{10^3}{2}E[X] = 2\left(\frac{10^3}{2}\right)^2 \end{aligned}$$

Therefore

$$Var(X) = 2\left(\frac{10^3}{2}\right)^2 - \left(\frac{10^3}{2}\right)^2 = \left(\frac{10^3}{2}\right)^2,$$

consequently $sd(X) = \sigma_X = \frac{10^3}{2} = 500$.

5. (Insurance with a deductible) Suppose the insurance has a deductible of \$100 for each loss. Find the expected value of a single claim.
Solution: In this case the expectation of a single loss is given by $E[\max(X-100,0)]$. Remark first that $\max(x-100,0)=x-100$ if $x>100$ and $\max(x-100,0)=0$ if $x\le 100$, hence

$$\begin{aligned} E[\max(X-100,0)] &= \int \max(x-100,0)f(x)dx = \int_{100}^{\infty}(x-100)\frac{2}{10^3}\exp\left(-\frac{2}{10^3}x\right)dx \\ &= \int_{100}^{\infty}\frac{2x}{10^3}\exp\left(-\frac{2}{10^3}x\right)dx - 100\int_{100}^{\infty}\frac{2}{10^3}\exp\left(-\frac{2}{10^3}x\right)dx \end{aligned}$$

Now we have

$$\begin{aligned} \int_{100}^{\infty}\frac{2x}{10^3}\exp\left(-\frac{2}{10^3}x\right)dx &= \left[-x\exp\left(-\frac{2}{10^3}x\right)\right]_{100}^{\infty} + \int_{100}^{\infty}\exp\left(-\frac{2}{10^3}x\right)dx \\ &= 100e^{-\frac{1}{5}} + \left[-\frac{10^3}{2}\exp\left(-\frac{2}{10^3}x\right)\right]_{100}^{\infty} \\ &= 100e^{-\frac{1}{5}} + 500e^{-\frac{1}{5}} = 600e^{-\frac{1}{5}}. \end{aligned}$$

and

$$100\int_{100}^{\infty}\frac{2}{10^3}\exp\left(-\frac{2}{10^3}x\right)dx = \frac{2}{10^2}\left[-\frac{10^3}{2}\exp\left(-\frac{2}{10^3}x\right)\right]_{100}^{\infty} = 100e^{-\frac{1}{5}}.$$

Therefore

$$E[\max(X-100,0)] = 600e^{-\frac{1}{5}} - 100e^{-\frac{1}{5}} = 500e^{-\frac{1}{5}}$$

Problem 5.

1. Let $f(x) : a(e^{-2x}-e^{-3x})$, for $x>0$, and $f(x)=0$ elsewhere.

   (a) Find a so that f is a probability density function of a r.v. X.
   Solution: We have $\int_0^{\infty} a(e^{-2x}-e^{-3x})dx = \frac{a}{6} = 1$ then $a=6$.

   (b) Calculate $P(X<1)$?
   Solution: We have $P(X<1) = \int_0^1 6(e^{-2x}-e^{-3x})dx = 2e^{-3}-3e^{-2}+1 \simeq 0.6936$

2. Let X be a random variable with probability density function

$$f(x) = \begin{cases} 25x & \text{if } 0<x\le \frac{2}{10} \\ 1.5225\,(1-x) & \text{if } \frac{2}{10}<x<1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the c.d.f. of X
   Solution: The c.d.f. is given by

$$F_X(x) = \begin{cases} 0 & \text{if } x\le 0 \\ \int_0^x 25y\,dy & \text{if } 0<x\le\frac{2}{10} \\ \int_0^{\frac{2}{10}} 25y\,dy + \int_{\frac{2}{10}}^x 1.5225\,(1-y)\,dy & \text{if } \frac{2}{10}<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x\le 0 \\ \frac{25}{2}x^2 & \text{if } 0<x\le\frac{2}{10} \\ \frac{1}{2}+\frac{1.5225}{2}\left((1-\frac{2}{10})^2-(1-x)^2\right) = -\frac{609}{800}x^2+\frac{609}{400}x+\frac{4519}{20\,000} & \text{if } \frac{2}{10}<x<1 \\ 1 & \text{if } x\ge 1 \end{cases}$$

(b) Deduce the value of $P(\frac{1}{10} < X < \frac{6}{10})$.
Solution: We have $P(\frac{1}{10} < X < \frac{6}{10}) = F_X(\frac{6}{10}) - F_X(\frac{1}{10})$ and since $\frac{6}{10} \in]\frac{2}{10}; 1[$, we have

$$F_X\left(\frac{6}{10}\right) = -\frac{609}{800}\left(\frac{6}{10}\right)^2 + \frac{609}{400}\frac{6}{10} + \frac{4519}{20\,000} = \frac{4327}{5000}.$$

and $\frac{1}{10} \in]0; \frac{2}{10}]$, Then

$$P\left(\frac{1}{10} < X < \frac{6}{10}\right) = \frac{4327}{5000} - \frac{1}{8} = \frac{1851}{2500} \simeq 0.740\,4$$

since $F_X(\frac{1}{10}) = \frac{25}{2}(\frac{1}{10})^2 = \frac{1}{8}$.

Problem 6.

1. Let f be a function defined by:

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 0 \le x < 1. \\ 0 & \text{if } x \ge 1 \end{cases}$$

(a) show that f is a probability density function of a r.v. X.
Solution: f is a p.d.f. because $f(x) \ge 0$ foe all $x \in \mathbb{R}$, and $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = 1$.

(b) Find the c.d.f. of X.
Solution: The c.d.f. is given by

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if } x \le 0 \\ \int_0^x 1dt & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1. \end{cases} = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1. \end{cases}$$

(c) Calculate $E[X]$ and its variance.
Solution: We have

$$E[X] = \int_{-\infty}^{+\infty} tf(t)dt = \int_0^1 tdt = \frac{1}{2}$$

and

$$E\left[X^2\right] = \int_{-\infty}^{+\infty} t^2 f(t)dt = \int_0^1 t^2 dt = \frac{1}{3}$$

Then

$$Var(X) = E\left[X^2\right] - (E[X])^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

2. Let X be r.v. having a p.d.f. f given by:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ a(x+1) & \text{if } 0 \le x < 2 \\ a(x-1) & \text{if } 2 \le x < 4 \\ 0 & \text{if } x \ge 4 \end{cases}$$

(a) Find the value of the constant a.
Solution:

$$\int_0^4 f(x)dx = 1 \Longleftrightarrow \int_0^2 a(x+1)dx + \int_2^4 a(x-1)dx = 1 \Longleftrightarrow 8a = 1$$

hence $a = \frac{1}{8}$

(b) Give the c.d.f F_X of X

Solution: The c.d.f. of X is given by

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if} \quad x < 0 \\ \int_0^x \frac{1}{8}(t+1)dt & \text{if} \quad 0 \le x < 2 \\ \int_0^2 \frac{1}{8}(t+1)dt + \int_2^x \frac{1}{8}(t-1)dt & \text{if} \quad 2 \le x < 4 \\ 1 & \text{if} \quad x \ge 4 \end{cases}$$

but we have

$$\int_0^x \frac{1}{8}(t+1)dt = \frac{1}{16}x\,(x+2)$$

and

$$\int_0^2 \frac{1}{8}(t+1)dt + \int_2^x \frac{1}{8}(t-1)dt = \frac{1}{16}x\,(x-2) + \frac{1}{2}$$

Therefore

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{1}{16}x\,(x+2) & \text{if} \quad 0 \le x < 2 \\ \frac{1}{16}x\,(x-2) + \frac{1}{2} & \text{if} \quad 2 \le x < 4 \\ 1 & \text{if} \quad x \ge 4 \end{cases}$$

(c) Deduce the value $P\,[1 \le X < 3]$.

Solution: We have

$$\begin{aligned} P\,[1 \le X < 3] &= F_X(3) - F_X(1) \\ &= \frac{1}{16}3\,(3-2) + \frac{1}{2} - \frac{1}{16}1\,(1+2) = \frac{1}{2} \end{aligned}$$

(d) Calculate $E\,[X]$ and $Var\,[X]$

Solution: We have

$$E\,[X] = \int_{-\infty}^{+\infty} tf(t)dt = \int_0^2 t\frac{1}{8}(t+1)dt + \int_2^4 t\frac{1}{8}(t-1)dt = \frac{13}{6}$$

and

$$E\left[X^2\right] = \int_{-\infty}^{+\infty} t^2 f(t)dt = \int_0^2 t^2\frac{1}{8}(t+1)dt + \int_2^4 t^2\frac{1}{8}(t-1)dt = 6$$

Then

$$Var(X) = E\left[X^2\right] - (E\,[X])^2 = 6 - \left(\frac{13}{6}\right)^2 = \frac{47}{36}$$

3. Let X be a continuous r.v. with p.d.f. f such that

$$f(x) = \begin{cases} c\ln x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

(a) Find the true value of c.

Solution: $\int_0^1 c\ln(x)\,dx = [c(x\ln(x) - x)]_0^1 = -c = 1$ hence $c = -1$

(b) Give the c.d.f F_X of X

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ \displaystyle\int_0^x -\ln(t)dt & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad x \geq 1. \end{cases}$$

Hence the c.d.f. is given by

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x - x\ln(x) & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad x \geq 1. \end{cases}$$

(c) Calculate $E[X]$ and $Var[X]$

Solution:

$$E[X] = \int_0^1 -x\ln(x)\,dx = \left[\frac{1}{4}x^2 - \frac{1}{2}x^2\ln x\right]_0^1 = \frac{1}{4}$$

and

$$E\left[X^2\right] = \int_0^1 -x^2\ln(x)\,dx = \left[\frac{1}{9}x^3 - \frac{1}{3}x^3\ln x\right]_0^1 = \frac{1}{9}$$

and then

$$Var[X] = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

(d) Let X be a continuous r.v. with c.d.f. F given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \leq x_0 \\ 1 - \dfrac{K}{x^2} & \text{if} \quad x_0 < x < +\infty \end{cases}$$

(e) Find the p.d.f. f of X and find the value of K.

Solution: The p.d.f. f of X is given $f(x) = F'(x) = \dfrac{2K}{x^3}$. To find K we solve the equation

$$\int_{x_0}^{+\infty} f(x)dx = 1 \Longleftrightarrow K\int_{x_0}^{+\infty} \frac{2}{x^3}dx = 1 \Longleftrightarrow \frac{K}{x_0^2} = 1 \Longleftrightarrow K = x_0^2$$

Hence $f(x) = \dfrac{2x_0^2}{x^3}$

4. Let X be standard normal r.v. that is $X \hookrightarrow \mathcal{N}(0,1)$.

(a) Find the distribution of the r.v. $Y = \frac{X^2}{2}$.

Solution: We have $S_Y = \mathbb{R}^+$ then $F_Y(y) = 0$ and $f_Y(y) = 0$, for $y \leq 0$. For $y > 0$,

$$\begin{aligned} F_Y(y) &= P(X^2 \leq 2y) \\ &= P\left(-\sqrt{2y} \leq X \leq \sqrt{2y}\right) \\ &= F_X\left(\sqrt{2y}\right) - F_X\left(-\sqrt{2y}\right). \end{aligned}$$

Hence

$$f_Y(y) = F_Y'(y) = \frac{F_X'\left(\sqrt{2y}\right) + F_X'\left(-\sqrt{2y}\right)}{\sqrt{2y}}$$

But remember that

$$F'_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

hence

$$f_Y(y) = \frac{1}{\sqrt{4y\pi}} \left(\exp(-y) + \exp(-y)\right) = \frac{\exp(-y)}{\sqrt{y\pi}} \text{ for all } y > 0,$$

(b) Deduce $E[X^2]$ and $Var[X^2]$.
Solution:

$$E[X^2] = E[2Y] = 2\int_0^\infty \frac{ye^{-y}}{\sqrt{y\pi}}dy = \frac{2}{\sqrt{\pi}}\int_0^\infty \sqrt{y}e^{-y}dy = \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)$$

where $\Gamma(p) = \int_0^{+\infty} e^{-t}t^{p-1}dt$. And

$$E[X^4] = E[4Y^2] = 4\int_0^\infty \frac{y^2e^{-y}}{\sqrt{y\pi}}dy = \frac{4}{\sqrt{\pi}}\int_0^\infty y^{\frac{5}{2}}e^{-y}dy = \frac{4}{\sqrt{\pi}}\Gamma\left(\frac{7}{2}\right).$$

Consequently

$$Var(X^2) = E[X^4] - \left(E[X^2]\right)^2 = \frac{4}{\sqrt{\pi}}\left(\Gamma\left(\frac{7}{2}\right) - \frac{1}{\pi}\Gamma\left(\frac{3}{2}\right)^2\right).$$

5. Find the MGF: of the following distributions and deduce their expectation and their variance.

   (a) 1. Bernoulli distribution, 2. Binomial distribution, 3. Poisson distribution, 4. Geometric distribution, 5. Exponential distribution with parameter λ, 6. Normal distribution with mean μ and variance σ^2.

Discrete Distributions

1. Bernoulli p.m.f.: $f(k) = p^k(1-p)^{1-k}$, $k = 0, 1$, $\mathcal{B}(p)$ and $0 < p < 1$, MGF: $M(t) = 1 - p + pe^t$, $-\infty < t < \infty$, $\mu = p$, $\sigma^2 = p(1-p)$.

2. Uniform p.m.f.: $f(k) = \frac{1}{m}$, $k = 1, 2, \ldots, m$, $\mathcal{U}(m)$ $m > 0$, $\mu = \frac{m+1}{2}$, $\sigma^2 = \frac{m^2-1}{12}$.

3. Binomial p.m.f.: $f(k) = \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$, $k = 0, 1, 2, \ldots, n$ $\mathcal{B}(n, p)$, $0 < p < 1$, MGF: $M(t) = (1 - p + pe^t)^n$, $-\infty < t < \infty$, $\mu = np$, $\sigma^2 = np(1-p)$,

4. Geometric p.m.f.: $f(k) = (1-p)^{k-1}p$, $x = 1, 2, 3, \ldots$, $\mathcal{G}(p)$, $0 < p < 1$, $M(t) = \frac{pe^t}{1-(1-p)e^t}$, $t < -\ln(1-p)$ $\mu = \frac{1}{p}$, $\sigma^2 = \frac{1-p}{p^2}$.

5. Hypergeometric $f(k) = \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}}$, $k \le n$, $k \le N_1$, $n - k \le N_2$ for $N_1 > 0$, $N_2 > 0$ and $N = N_1 + N_2$, $\mu = n\frac{N_1}{N}$, $\sigma^2 = n\frac{N_1}{N}\frac{N_2}{N}\frac{N-n}{N-1}$.

6. Poisson p.m.f.: $f(k) = e^{-\lambda}\frac{\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$ $\mathcal{P}(\lambda)$; $\lambda > 0$, MGF: $M(t) = e^{\lambda(e^t-1)}$, $-\infty < t < \infty$ $\mu = \lambda$, $\sigma^2 = \lambda$

7. Negative Binomial p.m.f.: $f(k) = \binom{k-1}{r-1}p^r(1-p)^{k-r}$, $k = r, r+1, r+2, \ldots$ $0 < p < 1$, $r = 1, 2, 3, \ldots$, MGF: $M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}$, $t < -\ln(1-p)$, $\mu = \frac{r}{p}$, $\sigma^2 = \frac{r(1-p)}{p^2}$

Continuous Distributions:

1. Uniform p.d.f. $f(x) = \frac{1}{b-a}$, $a \leq x \leq b$, $\mathcal{U}(a, b)$ $-\infty < a < b < \infty$, MGF: $M(t) = \frac{e^{tb}-e^{ta}}{t(b-a)}$, $t = 0$; $M(0) = 1$, $\mu = \frac{a+b}{2}$, $\sigma^2 = \frac{(b-a)^2}{12}$.

2. Normal p.d.f. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2^2}\right)$, $-\infty < x < \infty$, $\mathcal{N}(\mu, \sigma^2)$ $-\infty < \mu < \infty$, MGF: $M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$, $-\infty < t < \infty$, $\sigma > 0$ $E(X) = \mu$, $Var(X) = \sigma^2$.

3. Exponential p.d.f. $f(x) = \frac{1}{\theta} e^{-x/\theta}$, $0 \leq x < \infty$, $\theta > 0$, MGF: $M(t) = \frac{1}{1-\theta t}, t < \frac{1}{\theta}$, $\mu = \theta$, $\sigma^2 = \theta^2$.

4. Chi–square p.d.f. $f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}$, $0 < x < \infty$ $\chi^2(r)$, $r = 1, 2, \ldots$, MGF: $M(t) = \frac{1}{(1-2t)^{r/2}}$, $t < \frac{1}{2}$ $\mu = r$, $\sigma^2 = 2r$

5. Gamma p.d.f. $f(x) = \frac{1}{\Gamma(\alpha)} \theta^\alpha x^{\alpha-1} e^{-x/\theta}$, $0 < x < \infty$ $\alpha > 0$, $\theta > 0$, MGF: $M(t) = \frac{1}{(1-\theta t)^\alpha}$, $t < \frac{1}{\theta}$ $\mu = \alpha\theta$, $\sigma^2 = \alpha\theta^2$

6. Beta p.d.f. $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$ $\alpha > 0$, $\beta > 0$, $\mu = \frac{\alpha}{\alpha+\beta}$, $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$