King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2016–2017
Academic Year (H) 1437–1438
Bachelor AFM: M. Eddahbi

Solutions of homework 3 (joint, conditional distributions and conditional expectation)

Problem 1.

1. Let X be a r.v. taking values in $\{-b,-a,a,b\}$ (where a and b are real numbers such that $0<a<b$). Set $Y=X^2$.

   (a) Find the distribution of Y and the distribution of the couple (X,Y).
   Solution: We have $S_Y=\{a^2,b^2\}$ and

$$\begin{aligned} P\left(Y=a^2\right) &= P\left(X=a \text{ or } X=-a\right) \\ &= P\left(X=-a\right)+P\left(X=a\right) \\ &= \frac{1}{4}+\frac{1}{4}=\frac{1}{2}. \end{aligned}$$

   Consequently

$$P\left(Y=b^2\right)=1-P\left(Y=a^2\right)=\frac{1}{2}.$$

   We have $S_{(X,Y)}=(S_X,S_Y)=\{(-b,b^2);(b,b^2);(-a,a^2);(a,a^2)\}$ and

$$\begin{aligned} P\left((X,Y)=(-b,b^2)\right) &= P\left(X=-b,Y=b^2\right)=P\left(X=-b,X^2=b^2\right) \\ &= P\left(\{X=-b\}\cap\{X=b \text{ } or \text{ } X=-b\}\right)=P\left(X=-b\right)=\frac{1}{4} \end{aligned}$$

$$\begin{aligned} P\left((X,Y)=(b,b^2)\right) &= P\left(X=b,Y=b^2\right)=P\left(X=b,X^2=b^2\right) \\ &= P\left(\{X=b\}\cap\{X=b \text{ } or \text{ } X=-b\}\right)=P\left(X=b\right)=\frac{1}{4} \end{aligned}$$

$$\begin{aligned} P\left((X,Y)=(-a,a^2)\right) &= P\left(X=-a,Y=a^2\right)=P\left(X=-a,X^2=a^2\right) \\ &= P\left(\{X=-a\}\cap\{X=-a \text{ } or \text{ } X=a\}\right)=P\left(X=-a\right)=\frac{1}{4} \end{aligned}$$

$$\begin{aligned} P\left((X,Y)=(a,a^2)\right) &= P\left(X=a,Y=a^2\right)=P\left(X=a,X^2=a^2\right) \\ &= P\left(\{X=a\}\cap\{X=-a \text{ } or \text{ } X=a\}\right)=P\left(X=a\right)=\frac{1}{4} \end{aligned}$$

   (b) Show that $Cov(X,Y)=0$.
   Solution: By definition we have

$$\begin{aligned} Cov(X,Y) &= E\left[XY\right]-E\left[X\right]E\left[Y\right] \\ &= E\left[X^3\right]-E\left[X\right]E\left[X^2\right] \end{aligned}$$

   But

$$E\left[X\right]=\frac{1}{4}\left(-b-a+a+b\right)=0 \text{ and } E\left[X^3\right]=\frac{1}{4}\left(-b^3-a^3+a^3+b^3\right)=0$$

   Therefore $Cov(X,Y)=0$.

Prof. Mhamed Eddahbi
meddahbi@ksu.edu.sa
Random variables and p.d.f., c.d.f

(c) Are X and Y independent.
Solution: We have $Y = X^2$ (Y is a function of X) hence they are dependent.

2. Let X_1 and X_2 be two independent r.v. such that:

$$S_{X_1} = S_{X_2} = \{-1, 1\} \text{ and } P(\{X_1 = 1\}) = P(\{X_2 = 1\}) = \frac{1}{2}$$

(a) Set $X_3 = X_1 X_2$. Are the r.v. X_1, X_2 and X_3 mutually independent ?
Solution: We calculate

$$\begin{aligned} P(X_1 = 1, X_2 = -1, X_3 = -1) &= P(X_1 = 1, X_2 = -1) \\ &= P(X_1 = 1) P(X_2 = -1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

On the other hand

$$P(X_1 = 1) P(X_2 = -1) P(X_3 = -1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

because $S_{X_3} = \{-1, 1\}$

$$\begin{aligned} P(X_3 = 1) &= P(X_1 = -1, X_2 = -1) + P(X_1 = 1, X_2 = 1) \\ &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Therefore

$$P(X_1 = 1, X_2 = -1, X_3 = -1) \neq P(X_1 = 1) P(X_2 = -1) P(X_3 = -1).$$

X_1, X_2 and X_3 are not mutually independent.

3. Let X and Y be two r.v. with the following distribution:

$$S_X = \{-1, 1\} \text{ such that } P(X = -1) = \frac{1}{4} \text{ and } S_Y = \{1, 2\} \text{ such that } P(Y = 1) = \frac{1}{3}$$

Denote by p the probability of the event $\{X = -1\} \cap \{Y = 1\}$.

(a) Find the joint probability distribution of the couple (X, Y) in terms of p.
Solution: We have $S_{(X,Y)} = (S_X, S_Y) = \{(-1, 1); (-1, 2); (1, 1); (1, 2)\}$. Moreover we know that

$$P(X = -1, Y = 1) + P(X = -1, Y = 2) = P(X = -1) = \frac{1}{4}$$

and

$$P(X = -1, Y = 1) + P(X = 1, Y = 1) = P(Y = 1) = \frac{1}{3}.$$

and

$$P(X = -1, Y = 2) + P(X = 1, Y = 2) = P(Y = 2) = \frac{2}{3}$$

Then

$$P(X = -1, Y = 1) = p \quad \text{and} \quad P(X = -1, Y = 2) = \frac{1}{4} - p$$

and

$$P(X = 1, Y = 1) = \frac{1}{3} - p \quad \text{and} \quad P(X = 1, Y = 2) = \frac{2}{3} - \frac{1}{4} + p = \frac{5}{12} + p.$$

Prof. Mhamed Eddahbi
meddahbi@ksu.edu.sa
Random variables and p.d.f. ; c.d.f

(b) What the required conditions of p? The parameter p should satisfy $0 < p < 1$, $0 < \frac{1}{4} - p < 1$, $0 < \frac{1}{3} - p < 1$ and $\frac{5}{12} + p < 1$ that is $0 < p < \frac{1}{4}$

(c) Find the values of p in such away that X and Y become independent.
Solution: If X and Y are independent then we should have

$$P(X = 1, Y = 1) = \frac{1}{3} - p = P(X = 1) P(Y = 1) = \frac{3}{4}\frac{1}{3} = \frac{1}{4}$$

which implies that $p = \frac{1}{12}$. And

$$P(X = -1, Y = 2) = \frac{1}{4} - p = P(X = -1) P(Y = 2) = \frac{1}{4}\frac{2}{3} = \frac{1}{6}$$

which implies that $p = \frac{1}{12}$. And

$$P(X = -1, Y = 1) = p = P(X = -1) P(Y = 1) = \frac{1}{4}\frac{1}{3} = \frac{1}{12}.$$

And

$$P(X = 1, Y = 2) = \frac{5}{12} + p = P(X = 1) P(Y = 2) = \frac{3}{4}\frac{2}{3} = \frac{1}{2}$$

which implies that $p = \frac{1}{12}$. So X and Y are independent is and only if $p = \frac{1}{12}$.

(d) Find in this case the distributions of the following r.v.:

$$Z = XY \ ; \quad S = X + Y \ ; \quad D = X - Y \ ; \quad M = \max(X, Y) \ ; \quad I = \min(X, Y).$$

Solutions.

i. We have $p = 12$, i. $S_Z = \{-2, -1, 1, 2\}$ and ii. Probability mass function:

$$\begin{aligned} P(Z = -2) &= P(X = -1, Y = 2) = \frac{1}{6}, \quad P(Z = -1) = P(X = -1, Y = 1) = \frac{1}{12} \\ P(Z = 2) &= P(X = 1, Y = 2) = \frac{1}{2}, \quad P(Z = 1) = P(X = 1, Y = 1) = \frac{1}{4} \end{aligned}$$

ii. We have i. $S_S = \{0, 1, 2, 3\}$ and ii. Probability mass function:

$$\begin{aligned} P(S = 0) &= P(X = -1, Y = 1) = \frac{1}{12}, \quad P(S = 1) = P(X = -1, Y = 2) = \frac{1}{6} \\ P(S = 2) &= P(X = 1, Y = 1) = \frac{1}{4}, \quad P(S = 3) = P(X = 1, Y = 2) = \frac{1}{2} \end{aligned}$$

iii. We have i. $S_D = \{-3, -2, -1, 0, 1\}$ and ii. Probability mass function:

$$\begin{aligned} P(D = -3) &= P(X = -1, Y = 2) = \frac{1}{6}, \ P(D = -2) = P(X = -1, Y = 1) = \frac{1}{12}, \\ P(D = -1) &= P(X = 1, Y = 2) = \frac{1}{2}, \ P(D = 0) = P(X = 1, Y = 1) = \frac{1}{4}, \end{aligned}$$

iv. We have i. $S_M = \{1, 2\}$ and ii. Probability mass function:

$$\begin{aligned} P(M = 1) &= P(X = -1, Y = 1 \text{ or } X = 1, Y = 1) \\ &= P(X = -1, Y = 1) + P(X = 1, Y = 1) \\ &= \frac{1}{12} + \frac{1}{4} = \frac{1}{3} \\ P(M = 2) &= 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

Prof. Mhamed Eddahbi
medahbi@ksu.edu.sa
random variables and p.d.f.; c.d.f.

v. We have i. $S_I = \{-1, 1\}$ and ii. Probability mass function:

$$
\begin{aligned}
P(I=1) &= P(X=1, Y=1 \text{ or } X=1, Y=2) \\
&= P(X=1, Y=1) + P(X=1, Y=2) \\
&= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \\
P(I=-1) &= 1 - \frac{3}{4} = \frac{1}{4}.
\end{aligned}
$$

Problem 2.

(a) Two players are tossing fair coins. A tosses $(n+1)$ times the coin and B tosses n times the coin ($n \in \mathbb{N}^*$). Let X and Y be the number of "heads" got respectively by the player A and the player B.

i. Calculate the probability of the following events $\{X - Y = k\}$, $k \in \mathbb{Z}$, $\{X = Y\}$, $\{X > Y\}$.

(b) Let X_1 and X_2 be two i.i.d. r.v. with values in $\mathbb{N}$ such that :

$$\forall\, k \in \mathbb{N},\ P(X_1 = k) = \frac{1}{2^{k+1}}.$$

i. Find the distribution and calculate the expectation of $Y = \max(X_1, X_2)$.
Solution. i) $S_Y = \mathbb{N}$, ii) $\forall\, k \in \mathbb{N}\ p(k) = P(Y = k)$.
We have $S_Y = \mathbb{N}$, ii) $\forall\, k \in \mathbb{N}\ p(k) = P(Y = k)$. We have

$$
\begin{aligned}
P(Y=k) &= P(\max(X_1, X_2) = k) \\
&= P(X_2 = k, X_1 < X_2) + P(X_1 = k, X_1 \geq X_2) \\
&= P(X_2 = k, X_1 < k) + P(X_1 = k, X_2 \leq k) \\
&= P(X_2 = k)\, P(X_1 < k) + P(X_1 = k)\, P(X_2 \leq k) \\
&= P(X_2 = k)\,[P(X_1 < k) + P(X_2 \leq k)] \ \ (\text{since } P(X_1 = k) = P(X_2 = k)\,) \\
&= P(X_1 = k)\,[2P(X_1 \leq k) - P(X_1 = k)] \\
&= \frac{1}{2^{k+1}} \left(\sum_{i=0}^{k} \frac{2}{2^{i+1}} - \frac{1}{2^{k+1}} \right) = \frac{1}{2^{k+1}} \left(\frac{1 - 2^{-(k+1)}}{1 - 2^{-1}} \right) - \frac{1}{2^{2k+2}} \\
&= \frac{1}{2^k} \left(1 - \frac{1}{2^{k+1}} \right) - \frac{1}{2^{2k+2}} = \frac{1}{2^k} - \frac{3}{2^{2k+2}}
\end{aligned}
$$

where we have used the relation $P(X_1 < k) = P(X_1 \leq k) - P(X_1 = k)$).

(c) Let X and Y be two r.v. taking values in $\mathbb{N}$ such that:

$$\forall\, m \in \mathbb{N}^* \text{ and } \ \forall\, n \in \mathbb{N}\ \ P(\{X = m\} \cap \{Y = n\}) = \frac{e^{-1}}{n!} \times \frac{1}{2^m}.$$

i. Find the distributions of X and Y.
Solution: i) $S_X = \mathbb{N}^*$ and $S_Y = \mathbb{N}$, ii) $\forall\, m \in \mathbb{N}^*$ we have

$$
\begin{aligned}
P(X=m) &= \sum_{n=0}^{\infty} P(X=m, Y=n) \\
&= \sum_{n=0}^{\infty} \frac{e^{-1}}{n!} \frac{1}{2^m} = \frac{1}{2^m} \quad \left(\text{because } \sum_{n=0}^{\infty} \frac{e^{-1}}{n!} = 1\right).
\end{aligned}
$$

Prof. Mhamed Eddahbi
meddahbi@ksu.edu.sa
Random variables and p.d.f.; c.d.f

Hence $X \hookrightarrow \mathcal{G}(\frac{1}{2})$. And

$$\begin{aligned} P\left(Y=n\right) &= \sum_{m=1}^{\infty} P(X=m,Y=n) \\ &= \sum_{m=1}^{\infty} \frac{e^{-1}}{n!}\frac{1}{2^m} = \frac{e^{-1}}{n!} \quad \text{(because } \sum_{m=1}^{\infty}\frac{1}{2^m}=1). \end{aligned}$$

Then $Y \hookrightarrow \mathcal{P}(1)$.

ii. Are X and Y independent ?
Solution: We have $\forall\, m \in \mathbb{N}^*$ and $\forall\, n \in \mathbb{N}$

$$P\left(X=m,Y=n\right) = P\left(X=m\right)P\left(Y=n\right)$$

therefore X and Y are independent.

iii. Find their expectation and their variance.
Solution:

$$E\left[X\right] = \frac{1}{\frac{1}{2}} = 2 \quad \text{and} \quad Var(X) = \frac{1-\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2$$

and

$$E\left[Y\right] = 1 \;\; \text{and} \;\; Var(Y) = 1$$

(d) Let X and Z be two r.v. with integer values. Assume that Z is a Poisson r.v. with parameter λ such that

$$X \le Z \;\; \text{and} \;\; \forall\, n \ge 0, \;\; \forall\, k \le n, \quad P(X=k/Z=n) = C_n^k p^k (1-p)^{n-k} \; (0<p<1).$$

i. Show that X and $Y = Z - X$ are two independent Poisson r.v.
Solution: First $S_X = \mathbb{N}$ and $S_Y = \mathbb{N}$. Remark that

$$\begin{aligned} P\left(X=k,Y=j\right) &= P\left(X=k,Z=j+k\right) \\ &= P\left(X=k/Z=j+k\right)P\left(Z=j+k\right) \\ &= C_{j+k}^k p^k (1-p)^j e^{-\lambda}\frac{\lambda^{j+k}}{(j+k)!} \\ &= \frac{(p\lambda)^k}{k!}\frac{((1-p)\lambda)^j}{j!}e^{-\lambda} \\ &= e^{-\lambda p}\frac{(p\lambda)^k}{k!} \cdot e^{-\lambda(1-p)}\frac{((1-p)\lambda)^j}{j!}. \end{aligned}$$

Now, by taking the summation over j we get

$$\begin{aligned} P\left(X=k\right) &= \sum_{j=0}^{\infty} P\left(X=k,Y=j\right) \\ &= e^{-\lambda p}\frac{(p\lambda)^k}{k!} \cdot e^{-\lambda(1-p)}\sum_{j=0}^{\infty}\frac{((1-p)\lambda)^j}{j!} \\ &= e^{-\lambda p}\frac{(p\lambda)^k}{k!} \end{aligned}$$

Prof. Mhamed Eddahbi

meddahbi@ksu.edu.sa

Random variables and p.d.f. c.d.f.

and by taking the summation over k we get

$$\begin{aligned} P(Y=j) &= \sum_{k=0}^{\infty} P(X=k, Y=j) \\ &= e^{-\lambda(1-p)} \frac{((1-p)\lambda)^j}{j!} e^{-\lambda p} \sum_{j=0}^{\infty} \frac{(p\lambda)^k}{k!} \\ &= e^{-\lambda(1-p)} \frac{((1-p)\lambda)^j}{j!}. \end{aligned}$$

Consequently $X \hookrightarrow \mathcal{P}(p\lambda)$, $Y \hookrightarrow \mathcal{P}((1-p)\lambda)$ and

$$P(X=k, Y=j) = P(X=k)\, P(Y=j).$$

Which means that X and Y are independent Poisson random variable.

Problem 3.

(a) Consider the following joint probability density function p.d.f. of X and Y :

$$f(x,y) = \begin{cases} 4xy & \text{if } 0<y<1,\ 0<x<1, \\ 0 & \text{otherwise} \end{cases}$$

i. Find i. $P(0 < X \le 0.5, 0.25 \le Y \le 0.5)$, ii. $P(0 < Y < 1)$.
Solution: i) We have

$$\begin{aligned} P(0 < X \le 0.5, 0.25 \le Y \le 0.5) &= \int_0^{0.5} \int_{0.25}^{0.5} 4xy dx dy \\ &= \int_0^{0.5} 2x dx \int_{0.25}^{0.5} 2y dy \\ &= \left[x^2\right]_0^{0.5} \times \left[y^2\right]_{0.25}^{0.5} \\ &= \frac{5^2}{10^2}\left(\frac{5^2}{10^2} - \frac{(5^2)^2}{100^2}\right) \\ &= \frac{5^4}{10^4}\left(1 - \frac{5^2}{10^2}\right) = \frac{3}{64} \end{aligned}$$

ii) We have

$$\begin{aligned} P(0<Y<1) &= P(-\infty < X < +\infty, 0 < Y < 1) \\ &= \int_0^1 \int_0^1 4xy dx dy = \int_0^1 2x dx \int_0^1 2y dy \\ &= \left[x^2\right]_0^1 \times \left[y^2\right]_0^1 = 1 \end{aligned}$$

ii. Find the joint cumulative distribution function c.d.f. of X and Y , i.e.,

$$F_{(X,Y)}(x,y) = P(X \le x, Y \le y).$$

Solution: By definition of the c.d.f.

$$\begin{aligned} F_{(X,Y)}(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} 4uv du dv \\ &= \int_{-\infty}^{x} 2u du \int_{-\infty}^{y} 2v dv \end{aligned}$$

Prof. Mhamed Eddahbi
meddahbi@ksu.edu.sa
Random variables and p.d.f. c.d.f

A. if $x \le 0$ or $y \le 0$, then

$$F_{(X,Y)}(x, y) = 0,$$

B. if $0 < x < 1$ and $0 < y < 1$, then

$$\begin{aligned} F_{(X,Y)}(x, y) &= \int_0^x 2udu \int_0^y 2vdv \\ &= x^2 y^2 \end{aligned}$$

C. if $0 < x < 1$ and $y \ge 1$, then

$$\begin{aligned} F_{(X,Y)}(x, y) &= \int_0^x 2udu \int_0^1 2vdv \\ &= x^2 \end{aligned}$$

D. if $0 < y < 1$ and $x \ge 1$, then

$$\begin{aligned} F_{(X,Y)}(x, y) &= \int_0^1 2udu \int_0^y 2vdv \\ &= y^2 \end{aligned}$$

E. if $y \ge 1$ and $x \ge 1$, then

$$F_{(X,Y)}(x, y) = 1.$$

(b) Find i. $P(X = Y)$, ii. $P(X \le Y)$, iii. $P(X - Y > \frac{1}{2})$.

Solution: i. We have

$$\begin{aligned} P(X = Y) &= \int\int_{\{x=y\}} f(x, y)dxdy = \int\int_{\{x=y\}} f(x, x)dxdy \\ &= \int_0^1 \left(\int_y^y 4x^2 dx \right) dy = \int_0^1 \left(\left[\frac{4}{3} x^3 \right]_y^y \right) dy = \int_0^1 0dy = 0 \end{aligned}$$

ii. We have

$$\begin{aligned} P(X \le Y) &= \int\int_{\{x \le y\}} f(x, y)dxdy = \int_0^1 \left(\int_0^y 4xydx \right) dy = \int_0^1 4y \left(\int_0^y xdx \right) dy \\ &= \int_0^1 2y \left(\int_0^y 2xdx \right) dy = \int_0^1 2y \left(\left[x^2 \right]_0^y \right) dy = \int_0^1 2y^3 dy \\ &= \left[\frac{2}{4} x^4 \right]_0^1 = \frac{1}{2}. \end{aligned}$$

iii. We have

$$P\left(X - Y > \frac{1}{2}\right) = \int\int_{\{x > \frac{1}{2} + y\}} f(x, y)dxdy$$

Now we have $\frac{1}{2} + y < x$ and $0 < y < 1,\ 0 < x < 1$ then $0 < y < x - \frac{1}{2}$ and $\frac{1}{2} < x < 1$, hence

$$\begin{aligned} P\left(X - Y > \frac{1}{2}\right) &= \int\int_{\{x > \frac{1}{2} + y\}} f(x, y)dxdy = \int_{\frac{1}{2}}^1 \left(\int_0^{x - \frac{1}{2}} 4xydx \right) dy \\ &= \int_{\frac{1}{2}}^1 2x \left(\int_0^{x - \frac{1}{2}} 2ydy \right) dx = \int_{\frac{1}{2}}^1 2x \left(\left[y^2 \right]_0^{x - \frac{1}{2}} \right) dx \\ &= \int_{\frac{1}{2}}^1 2x \left(x - \frac{1}{2} \right)^2 dx = \left[\frac{1}{2} x^4 - \frac{2}{3} x^3 + \frac{1}{4} x^2 \right]_{\frac{1}{2}}^1 = \frac{7}{96}. \end{aligned}$$

Prof. Mhamed Eddahbi
meddahbi@ksu.edu.sa
Random variables and p.d.f., c.d.f

4. Consider the following joint p.d.f of X and Y :

$$f(x,y)=\begin{cases} c(x+3y) & \text{if } 0<y<1,\ 0<x<1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c
Solution: We have $\int_0^1\int_0^1(x+3y)dxdy=2$, hence $c=\frac{1}{2}$.

(b) Find the marginal density function of X
Solution: We have $f_X(x)=\int_0^1\frac{1}{2}(x+3y)dy=\frac{x}{2}+\frac{3}{4}$

(c) Find the joint c.d.f. of X and Y
Solution: The joint cumulative distribution function c.d.f. of X and Y is given by

$$F_{(X,Y)}(x,y)=P(X\le x,Y\le y).$$

Then

$$F_{(X,Y)}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}f(u,v)dudv.$$

We shall discuss for cases

i. if $x<0$ or $y<0$, then $F_{(X,Y)}(x,y)=0$.

ii. if $0<x<1,\ 0<y<1$, then

$$\begin{aligned} F_{(X,Y)}(x,y) &= \int_0^x\int_0^y f(u,v)dvdu=\int_0^x\int_0^y\frac{1}{2}(u+3v)dvdu \\ &= \int_0^x\left(\int_0^y\frac{1}{2}(u+3v)dv\right)du=\int_0^x\frac{1}{4}y\,(2u+3y)\,du=\frac{1}{4}xy\,(x+3y)\,. \end{aligned}$$

iii. if $1<x,\ 0<y<1$, then,

$$\begin{aligned} F_{(X,Y)}(x,y) &= \int_0^x\int_0^y f(u,v)dudv=\int_0^1\int_0^y\frac{1}{2}(u+3v)dvdu \\ &= \int_0^1\frac{1}{4}y\,(2u+3y)\,du=\frac{1}{4}y\,(1+3y) \end{aligned}$$

iv. if $0<x<1,\ 1<y$, then,

$$\begin{aligned} F_{(X,Y)}(x,y) &= \int_0^x\int_0^1 f(u,v)dudv=\int_0^x\int_0^1\frac{1}{2}(u+3v)dvdu \\ &= \int_0^x\left(\frac{1}{2}u+\frac{3}{4}\right)du=\frac{1}{4}x\,(x+3) \end{aligned}$$

v. if $1<x,\ 1<y$, then, $F_{(X,Y)}(x,y)=1$.

(d) Find $P(0<X<\frac{1}{2},0<Y<\frac{1}{2})$.
Solution: Frome the previous question we can write

$$P(0<X<\frac{1}{2},0<Y<\frac{1}{2})=F_{(X,Y)}(\frac{1}{2},\frac{1}{2})=\frac{1}{4}\frac{1}{2}\frac{1}{2}\left(\frac{1}{2}+3\frac{1}{2}\right)=\frac{1}{4}$$

(e) Find $P(X<Y)$ and $P(X+Y<\frac{1}{2})$
Solution: We have

$$\begin{aligned} P\,(X<Y) &= \int\int_{\{x<y\}}f(x,y)dxdy=\int_0^1dx\int_x^1\frac{1}{2}(x+3y)dy \\ &= \int_0^1\left(-\frac{5}{4}x^2+\frac{1}{2}x+\frac{3}{4}\right)dx=\frac{7}{12}. \end{aligned}$$

Prof. Mhamed Eddahbi
meddahbi@ksu.edu.sa
Random variables and p.d.f.; c.d.f

and

$$P\left(X+Y<\frac{1}{2}\right) = \int\int_{\{x+y<\frac{1}{2}\}} f(x,y)dxdy = \int_0^{\frac{1}{2}} dx \int_0^{\frac{1}{2}-x} \frac{1}{2}(x+3y)dy$$
$$= \int_0^{\frac{1}{2}} \left(\frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{16}\right) dx = \frac{1}{24}.$$

Problem 4.

1. Let X and Y be two random variables with the joint density

$$f(x,y) = \begin{cases} \frac{4}{3}(1-xy) & \text{if } \quad 0<x<1 \text{ and } 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Calculate $P(X+Y<1)$.
Solution: We have

$$P(X+Y<1) = \int\int_{\{x+y<1\}} f(x,y)dxdy = \int_0^1 dx \int_0^{1-x} \frac{4}{3}(1-xy)\,dy,$$

First we have

$$\int_0^{1-x} \frac{4}{3}(1-xy)\,dy = \frac{4}{3}(1-x) - x\frac{2}{3}\int_0^{1-x} 2ydy = \frac{4}{3}(1-x) - x\frac{2}{3}(1-x)^2$$

and

$$\int_0^1 \left(\frac{4}{3}(1-x) - x\frac{2}{3}(1-x)^2\right) dx = \frac{11}{18}$$

then

$$P(X+Y<1) = \frac{11}{18}.$$

(b) Find $E[X+Y]$ using the joint density function.
Solution: We have

$$E[X+Y] = \int_0^1 \int_0^1 (x+y)\frac{4}{3}(1-xy)\,dydx = \frac{8}{9}$$

(c) Calculate $E[X]$ and $E[Y]$.
Solution: First calculate $f_X(x)$ and $f_Y(y)$

$$f_X(x) = \int_0^1 \frac{4}{3}(1-xy)\,dy = \frac{4}{3} - \frac{2}{3}x$$

and

$$f_Y(y) = \int_0^1 \frac{4}{3}(1-xy)\,dx = \frac{4}{3} - \frac{2}{3}y$$

so

$$E[X] = \int_0^1 x\left(\frac{4}{3} - \frac{2}{3}x\right) dx = \frac{4}{9} = E[Y].$$

Prof. Mhamed Eddahbi meddahbi@ksu.edu.sa Random variables and p.d.f.; c.d.f

(d) Find $E[XY]$, $E[Y]E[Y]$
Solution: We have

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 (xy)\frac{4}{3}(1-xy)\,dydx = \frac{4}{3}\int_0^1 xdx \int_0^1 y(1-xy)\,dy \\ &= \frac{4}{3}\int_0^1 x\left(\frac{1}{2}-\frac{1}{3}x\right)dx = \frac{5}{27} \end{aligned}$$

and

$$E[Y]E[Y] = \left(\frac{4}{9}\right)^2 = \frac{16}{81}$$

(e) Deduce $Cov(X,Y)$
Solution: By definition

$$Cov(X,Y) = E[XY] - E[Y]E[Y] = \frac{5}{27} - \frac{16}{81} = -\frac{1}{81}$$

2. Suppose X and Y are continuous random variables with joint c.d.f. given by $F(x,y)$. For each of the following, find the answer in terms of $F(x,y)$.

(a) $P(X \le a, Y \le c)$
Solution: $P(X \le a, Y \le c) = F(a,c)$

(b) $P(X \le b)$
Solution: $P(X \le b) = \lim_{y\longrightarrow+\infty} F(b,y)$, $P(Y \le d) = \lim_{x\longrightarrow+\infty} F(x,d)$

(c) $P(a < X \le b, c < Y \le d)$
Solution: $P(a < X \le b, c < Y \le d) = F(b,d) - F(b,c) - F(a,d) + F(a,c)$

(d) $P(X > a, Y \le d)$
Solution: $P(X > a, Y \le d) = P(a < X < +\infty, -\infty < Y \le d) = F(+\infty,d) - F(+\infty,-\infty) - F(a,d) + F(a,-\infty) = F(+\infty,d) - F(a,d)$ (since $F(+\infty,-\infty) = F(a,-\infty) = 0$)

(e) $P(X > b, Y > d)$
Solution $P(X > b, Y > d) = P(b < X < +\infty, d < Y < +\infty) = 1 - F(+\infty,d) - F(b,+\infty) + F(b,d)$

3. Suppose the joint c.d.f. of two random variables X and Y is given by

$$F(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-x-y}, & \text{for } x,y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $P(X < 2, Y < 2)$,
Solution: By definition of the c.d.f.

$$\begin{aligned} P(X < 2, Y < 2) &= F_{(X,Y)}(2,2) = 1 - e^{-2} - e^{-2} + e^{-4} \\ &= 1 - 2e^{-2} + e^{-4} \end{aligned}$$

(b) Find $P(X < 5)$,
Solution:

$$\begin{aligned} P(X < 5) &= P(X < 5, Y < +\infty) = \lim_{y\longrightarrow\infty} F(5,y) \\ &= 1 - e^{-5} \end{aligned}$$

Prof. Mohamed Eddahbi meddahbi@ksu.edu.sa Random variables and p.d.f.; c.d.f

(c) Find $P(1 < X < 3, 2 < Y < 4)$:
Solution:

$$P(1 < X < 3, 2 < Y < 4) = F(3,4) - F(3,2) - F(1,4) + F(1,2)$$

(d) Find the joint p.d.f. $f_{(X,Y)}(x,y)$
Solution:

$$f_{(X,Y)}(x,y) = \frac{\partial^2 F}{\partial x \partial y}(x,y) = \begin{cases} e^{-x-y}, & \text{for } x,y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(e) Find the p.d.f $f_X(x)$ of X and the p.d.f $f_Y(y)$ of Y.
Solution:

$$\begin{aligned} f_X(x) &= \int_0^\infty f_{(X,Y)}(x,y)dy = e^{-x}\int_0^\infty e^{-y}dy \\ &= e^{-x} \end{aligned}$$

and

$$\begin{aligned} f_Y(x) &= \int_0^\infty f_{(X,Y)}(x,y)dx = e^{-y}\int_0^\infty e^{-x}dx \\ &= e^{-y} \end{aligned}$$

Prof. Mhamed Eddahbi

Problem 5.

1. Let X and Y be two r.v. with joint probability density $f(x,y) = e^{(-x/y)-y}/y$, for $x > 0$ and $y > 0$.

meddahbi@ksu.edu.sa

(a) Calculate the conditional density $f_{X|Y}(x|y)$.
Solution:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad \text{where } f_Y(y) = \int_0^{+\infty} f(x,y)dx$$

hence

$$f_Y(y) = \int_0^{+\infty} \frac{1}{y}e^{-\frac{x}{y}-y}dx = e^{-y}\left[-e^{-\frac{x}{y}}\right]_0^{+\infty} = e^{-y},$$

then

$$f_{X|Y}(x|y) = \frac{e^{-\frac{x}{y}-y}}{ye^{-y}} = \frac{e^{-\frac{x}{y}}}{y}.$$

Random variables and p.d.f.; c.d.f

(b) Calculate $\mathbb{E}[X|Y = y]$.
Solution: We have for each $y > 0$

$$\begin{aligned} \mathbb{E}[X|Y = y]. &= \int_0^{+\infty} x f_{X|Y}(x|y)dx = \int_0^{+\infty} x\frac{e^{-\frac{x}{y}}}{y}dx \\ &= \left[-xe^{-\frac{x}{y}}\right]_0^{+\infty} + \int_0^{+\infty} e^{-\frac{x}{y}}dx = \left[-ye^{-\frac{x}{y}}\right]_0^{+\infty} \\ &= y \end{aligned}$$

(c) Deduce $\mathbb{E}[X|Y]$.
Solution: We know that $\mathbb{E}[X|Y]$ is a random variable which take the same values as the r.v. Y hence $\mathbb{E}[X|Y] = Y$. In general if you find

$$\mathbb{E}[X|Y = y] = \Psi(y)$$

then

$$\mathbb{E}[X|Y] = \Psi(Y)$$

The conditional density of X given $\{Y = y\}$ is $f_{X|Y}(x|y) = xy^2e^{-yx}$, for $x > 0$ and $y \geq 1$. The probability density of Y is $f_Y(y) = \frac{1}{y^2}$, for $y \geq 1$.

(d) Calculate the conditional distribution of Y given $\{X = x\}$ and $\mathbb{E}[Y|X]$.
Solution: The conditional density of Y given $\{X = x\}$ is given

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \quad \text{and} \quad f(x,y) = f_{X|Y}(x|y) f_Y(y)$$

then

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y) f_Y(y)}{f_X(x)}$$

Now

$$f_X(x) = \int_1^{+\infty} x e^{-yx} dy = \left[-e^{-yx}\right]_1^{+\infty} = e^{-x}$$

Consequently

$$f_{Y|X}(y|x) = \frac{x e^{-yx}}{e^{-x}} = x e^{-(y-1)x}$$

First calculate $\mathbb{E}[Y|X = x]$ for $x > 0$. We have

$$\begin{aligned}
\mathbb{E}[Y|X = x] &= \int_1^{+\infty} y f_{Y|X}(y|x) dy = \int_1^{+\infty} y x e^{-(y-1)x} dy \\
&= x e^x \int_1^{+\infty} y e^{-yx} dy = x e^x \left[-y e^{-yx}\right]_1^{+\infty} + x e^x \int_1^{+\infty} e^{-yx} dy \\
&= x + x e^x \left[-\frac{1}{x} e^{-yx}\right]_1^{+\infty} = x + x e^x \left(-0 + \frac{e^{-x}}{x}\right) = x + 1
\end{aligned}$$

finally

$$\mathbb{E}[Y|X] = X + 1.$$

1. Consider the couple of random variable (X, Y) with a joint density given by

$$f(x,y) = \begin{cases} cxy & \text{if} \quad 0 < y \le x < 1, \\ 0 & \quad \text{otherwise.} \end{cases}$$

(a) Determine the value of c.
Solution: We should have $\int\int f(x,y) dx dy = 1$ that is

$$c \int_0^1 \int_0^x xy dx dy = c \int_0^1 dy \int_y^1 x dx = c \int_0^1 x dx \int_0^x y dx = \frac{c}{2} \int_0^1 x^3 dx = \frac{1}{8} c$$

then $c = 8$.

(b) Find the marginal density of X ($f_X(x)$).
Solution: We have $f_X(x) = 8 \int_0^x xy dy = 4x^3$ $0 \le x \le 1$ and $f_X(x) = 0$ otherwise.

(c) Find the conditional density of Y given $X = x$, that is, $f_{Y|X}(y|x)$. **Solution:** We have $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{x^2}$ for $0 < y \le x < 1$

(d) Compute the conditional expectations $E[Y|X = x]$ and $E[X|Y = y]$.
Solution: We have

$$E[Y \mid X = x] = \int y f_{Y|X}(y|x) dy = \int_0^x y \frac{2y}{x^2} dy = \frac{2}{3} x$$

but to compute $E[X|Y = y]$ we need to compute first $f_Y(y)$ which can be calculated by

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y) dx = 8 \int_y^1 xy dx = 4y\left(1 - y^2\right)$$

Prof. Mhamed Eddahbi meddahbi@ksu.edu.sa Random variables and pd.f.; c.d.f

then

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{8xy}{4y\left(1-y^2\right)} = \frac{2x}{1-y^2} \quad \text{for } 0 < y \le x < 1$$

therefore

$$\begin{aligned} E\left[X \mid Y = y\right] &= \int x f_{X|Y}(x|y)dx = \frac{2}{1-y^2}\int_y^1 x^2 dx = \frac{2}{3}\frac{y^3-1}{y^2-1} \\ &= \frac{2}{3}\frac{\left(y^2+y+1\right)(y-1)}{(y+1)(y-1)} = \frac{2}{3}\frac{y^2+y+1}{y+1}. \end{aligned}$$

(e) Identify $E\left[Y|X\right]$ and $E\left[X|Y\right]$
Solution: from the previous question we deduce that

$$E\left[Y|X\right] = \frac{2}{3}X$$

and

$$E\left[X|Y\right] = \frac{2}{3}\frac{1+Y+Y^2}{1+Y} = \frac{2}{3} + \frac{2}{3}\frac{Y^2}{1+Y}$$

(f) Find the conditional expectation $E[Y^4 - 5X \mid X = x]$.
Solution: By definition

$$E[Y^4 - 5X \mid X = x] = \int \left(y^4 - 5x\right) f_{Y|X}(y|x)dy = \int_0^x \left(y^4 - 5x\right)\frac{2y}{x^2}dy = \frac{1}{3}x\left(x^3 - 15\right)$$

(g) Identify $E[Y^4 \mid X]$.
Solution: We have

$$E[Y^4 - 5X \mid X = x] = \frac{1}{3}x\left(x^3 - 15\right)$$

then

$$\begin{aligned} E[Y^4 \mid X] &= E[Y^4 - 5X \mid X = x] + E[5X \mid X = x] \\ &= \frac{1}{3}X\left(X^3 - 15\right) + 5X = \frac{1}{3}X^4. \end{aligned}$$

Problem 6.

1. Let X and Y be two random variables with joint probability density functions

$$f(x,y) = 4xy \;\text{ for }\; 0 < x < 1 \text{ and } 0 < y < 1.$$

An insurance policy is written to cover the loss which can be modeled by the sum of the two variables X and Y. The policy has a deducible of 1.
Question: Calculate the expected payment under the policy.

Solution: Let Z denotes the payment under the policy, Z is given in terms for X and Y:

$$Z = \max\left(X+Y-1, 0\right) = \begin{matrix} X+Y-1 & \text{if} \quad X+Y \ge 1 \\ 0 & \text{otherwise.} \end{matrix}$$

You are asked to compute $E\left[Z\right]$. Then

$$\begin{aligned} E\left[Z\right] &= E\left[\max\left(X+Y-1,0\right)\right] = \int\int \left(\max\left(x+y-1,0\right)\right) f(x,y)dxdy \\ &= \int_0^1\int_0^1 \left(\max\left(x+y-1,0\right)\right) 4xydxdy, \end{aligned}$$

Prof. Mhamed Eddahbi meddahbi@ksu.edu.sa Random variables and p.d.f.; c.d.f

now, $\max(x+y-1,0)=x+y-1$ when $x+y\geq 1 \iff y>1-x$, for $0<x<1$. Hence

$$\begin{aligned}\int_0^1\int_0^1(\max(x+y-1,0))\,xydxdy &= \int_0^1 xdx\int_0^{1-x}(\max(x+y-1,0))\,ydy\\ &\quad+\int_0^1 xdx\int_{1-x}^1(\max(x+y-1,0))\,ydy\\ &= \int_0^1 xdx\int_{1-x}^1(x+y-1)\,ydy\\ &= \int_0^1 x\left(\frac{x^2}{6}(3-x)\right)dx=\frac{11}{120}.\end{aligned}$$

Therefore

$$E[Z]=4\frac{11}{120}=\frac{11}{30}.$$

2. Assume that an insurance company insures a large numbers of drivers. Let X be random variable representing the company's losses under collision insurance and let Y represents the company's losses under liability insurance. Assume that X and Y has a joint probability density functions

$$f(x,y)=\begin{cases}\frac{1}{4}(2x+2-y) & \text{for } 0<x<1 \text{ and } 0<y<2.\\ 0 & \text{otherwise}\end{cases}$$

Question: What is the probability that the total loss is at least 1.

Solution: The probability to compute is $P(X+Y>1)$. We can write

$$P(X+Y>1)=1-P(X+Y\leq 1).$$

Let us first compute $P(X+Y\leq 1)$. Using the joint density we have

$$\begin{aligned}P(X+Y\leq 1) &= \int\int_{\{x+y\leq 1\}} f(x,y)dxdy\\ &= \int\int_{\{x+y\leq 1,0<x<1,0<y<2\}}\frac{1}{4}(2x+2-y)\,dxdy\\ &= \int_0^1 dx\int_{\{y\leq 1-x,0<y<2\}}\frac{1}{4}(2x+2-y)\,dxdy\\ &= \int_0^1 dx\int_0^{1-x}\frac{1}{4}(2x+2-y)\,dxdy\\ &= \int_0^1\frac{x+1}{2}dx\left(\int_0^{1-x}dy-\frac{1}{4}\int_0^{1-x}ydy\right)\\ &= \int_0^1\left(\frac{x+1}{2}(1-x)-\frac{1}{8}(1-x)^2\right)dx\\ &= \frac{1}{2}\int_0^1\left(1-x^2-\frac{1}{4}(1-x)^2\right)dx=\frac{7}{24}\end{aligned}$$

Finally

$$P(X+Y>1)=1-P(X+Y\leq 1)=1-\frac{7}{24}=\frac{17}{24}.$$

Prof. Mhamed Eddahbi
meddahbi@ksu.edu.sa
Random variables and p.d.f.; c.d.f