King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2016–2017
Academic Year (H) 1437–1438
Bachelor AFM: M. Eddahbi

Solution Second midterm Exam (two pages) : MATH. 380 (25%)

Sunday, April 30, 2017 / Sha'ban 4, 1438 (Two hours 12:45–2:45 PM)

Problem 1 (8 marks)

The dollar amount of damage involved in an automobile accident is an exponential random variable X with parameter λ. Of this, the insurance company only pays that amount exceeding (the deductible amount of) K. Set

$$Y = \begin{cases} 1, & \text{if } X > K \\ 0, & \text{if } X \leq K \end{cases}$$

This mean that $P(Y=1) = P(X > K)$ and $P(Y=0) = P(X \leq K)$.

Set

$$Z = \max(X-K, 0) = \begin{cases} X-K, & \text{if } X > K \\ 0, & \text{if } X \leq K. \end{cases}$$

1. **(1 mark)** Find $P(X > K)$ and $P(X \leq K)$
2. **(1 mark)** Find the distribution of Y.
3. **(1 mark)** Find $E[Z \mid Y]$,
4. **(1 mark)** Find $Var(Z \mid Y)$
5. **(1 mark)** Find $E[Z]$,
6. **(1 mark)** Recall the conditional variance relationship
7. **(1 mark)** Deduce the $Var(Z)$.
8. **(1 mark)** Find the expected value and the standard deviation of the amount the insurance company pays per accident when $E[X] = 1000$ and $K = 400$.

Solution:

1. The r.v. X is exponentially distributed with parameter λ and its c.d.f. is $F_X(x) := P(X \leq x) = 1 - e^{-\lambda x}$ then $P(X \leq K) = 1 - e^{-\lambda K}$ hence $P(X > K) = e^{-\lambda K}$.
2. We have $S_Y = \{0, 1\}$, and $P(Y=1) = P(X > K) = e^{-\lambda K}$ and $P(Y=0) = 1 - P(Y=1) = 1 - e^{-\lambda K}$ that is Y has a Bernoulli distribution with parameter $e^{-\lambda K}$ we write $Y \hookrightarrow \mathcal{B}\left(e^{-\lambda K}\right)$.
3. The r.v. $E[Z \mid Y]$ is given by $E[Z \mid Y=0]$, and $E[Z \mid Y=1]$, hence

$$E[Z \mid Y=0] = 0 \text{ because when } Y=0,\ Z=0$$

$$E[Z \mid Y=1] = E[X-K \mid Y=1] = E[(X-K)^+] = \frac{1}{\lambda} \text{ on the set } \{Y=1\}.$$

Consequently

$$E[Z \mid Y] = \frac{1}{\lambda} Y.$$

4. Similarly we have

$$E[Z^2 \mid Y] = E[(X-K)^+ \mid Y=1] = \frac{2}{\lambda^2} \text{ on the set } \{Y=1\}.$$

then

$$\mathrm{Var}(Z \mid Y) = \frac{1}{\lambda^2} Y$$

5. We know that

$$E[Z] = E\left[E[Z \mid Y]\right] = E\left[\frac{1}{\lambda} Y\right] = \frac{1}{\lambda} E\left[Y\right] = \frac{1}{\lambda} e^{-\lambda K}.$$

6. The conditional variance relationship says that

$$E\left[Var\left(Z \mid Y\right)\right] + Var\left(E[Z \mid Y]\right) = Var(Z).$$

7. First we have

$$E\left[Var\left(Z \mid Y\right)\right] = \frac{1}{\lambda^2} E\left[Y\right] = \frac{1}{\lambda^2} e^{-\lambda K}$$

and

$$Var\left(E[Z \mid Y]\right) = Var\left(\frac{1}{\lambda} Y\right) = \frac{1}{\lambda^2} Var\left(Y\right) = \frac{1}{\lambda^2} e^{-\lambda K}\left(1 - e^{-\lambda K}\right)$$

then

$$Var(Z) = \frac{1}{\lambda^2} e^{-\lambda K}\left(2 - e^{-\lambda K}\right).$$

8. Application $K = 400$ and $\lambda = \frac{1}{1000} = 10^{-3}$. The expected value of the amount the insurance company pays per accident is given by

$$E\left[Z\right] = \frac{1}{\lambda} e^{-\lambda K} = 10^3 e^{-0.4} = 670.32$$

and the standard deviation of the amount the insurance company pays per accident is given by

$$\sigma_Z = \sqrt{Var(Z)} = \frac{1}{\lambda}\sqrt{e^{-\lambda K}\left(2 - e^{-\lambda K}\right)} = 10^3 \sqrt{e^{-0.4}\left(2 - e^{-0.4}\right)} = 944.09$$

Problem 2 (9 marks)

Consider an homogenous Markov chain (M.C. for short) with state space $E = \{1,2,3,4\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1. **(1 mark)** Find: $P\left(X_7 = 3 \mid X_6 = 1\right) = \mathbf{P}_{1,3} = 0$

2. **(1 mark)** Find: $P\left(X_{1438} = 2 \mid X_{1436} = 4\right)$.

3. **(1 mark)** Assume that $X_0 = 4$, Find the expectation of X_1^2.

4. **(1 mark)** Draw the state transition diagram of the M.C.

5. **(1 mark)** List absorbing states if any.

6. **(1 mark)** Find communicating classes of this M.C.

7. **(1 mark)** Which classes are recurrent ?
8. **(1 mark)** Which classes are transient ?
9. **(1 mark)** Is this M.C. irreducible ? explain your answer.

Solution

1. We have $P(X_7 = 3 \mid X_6 = 1) = \mathbf{P}_{1,3} = 0$.
2. We have $P(X_{1438} = 2 \mid X_{1436} = 4) = \mathbf{P}^2_{42}$

$$\mathbf{P}^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ \frac{5}{9} & 0 & \frac{1}{9} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix}$$

   then $\mathbf{P}^2_{42} = 0$.
3. If $X_0 = 4$, then $\alpha_0 = (0, 0, 0, 1)$ and then

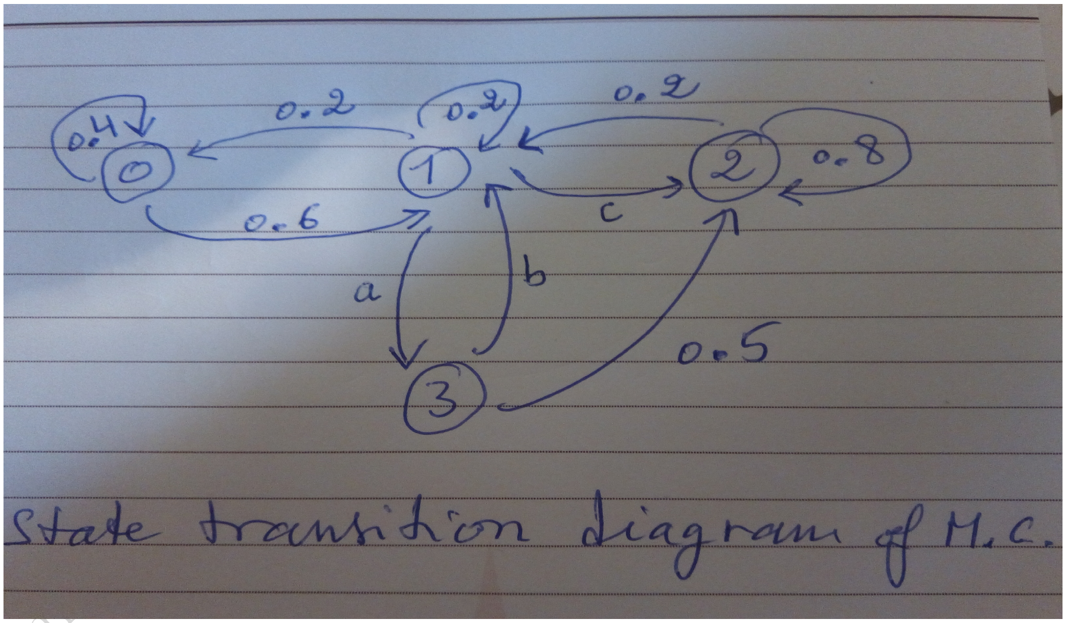
$$\alpha_1 = \alpha_0 \mathbf{P} = \alpha_0 = (0, 0, 0, 1) \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (0, 0, 1, 0)$$

   and $E[X_1^2] = \sum_{k=1}^{4} k^2 \alpha_1(k) = 3^2 \alpha_1(3) = 3^2$.
4. Easy
5. There is no absorbing state since $P_{i,i} \neq 1$ for all $i \in \{1, 2, 3, 4\}$.
6. We have $1 \longrightarrow 4 \longrightarrow 3 \longrightarrow 1$ this give the class $C_1 = \{1, 3, 4\}$ and 4 is accessible from 2 with probability that is $2 \xrightarrow{1} 4$ hence the state forms one single class $C_2 = \{2\}$.
7. The class C_1 is recurrent
8. The class C_2 is transient
9. This Markov chain is reducible since it has two communicating classes.

Problem 3 (8 marks)

Consider the following state transition diagram

1. **(1 mark)** Find the corresponding transition matrix P of this M.C.
2. **(1 mark)** Find a, b and c such that P is a transition probability matrix.
3. **(1 mark)** Specify communicating classes of the Markov chain.
4. **(1 mark)** Find recurrent and transient classes if any.
5. **(1 mark)** Find periodic and aperiodic classes
6. **(1 mark)** Is this Markov chain irreducible ? Explain your answer



7. **(1 mark)** Change the probabilities and the arrows of the previous diagram to make the Markov chain reducible ?

8. **(1 mark)** List your new communicating classes.

Solution

1. The transition matrix P of the transition diagram is

$$P = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.2 & 0.2 & c & a \\ 0 & 0.2 & 0.8 & 0 \\ 0 & b & 0.5 & 0 \end{pmatrix}$$

2. P is a transition probability matrix if and only if each line sums up to 1. That is $0.4 + c + a = 1$, $b + 0.5 = 1$, then $b = 0.5$ and $c = 0.6 - a$ provided that $0 < a < 0.6$. so the transition matrix P becomes

$$P = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.2 & 0.2 & 0.6 - a & a \\ 0 & 0.2 & 0.8 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix} \quad \text{for any} \quad 0 < a < 0.6.$$

3. We have $0 \longrightarrow 1 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0$ then all the state are communicating with each other. This leads to one communicating class $\{0, 1, 2, 3\}$.

4. The states are recurrent hence the $\{0, 1, 2, 3\}$ is recurrent. We say that the M.C. is recurrent.

5. The class communicating $\{0, 1, 2, 3\}$ is aperiodic.

6. The M.C. is irreducible because it has one communicating class.

7. Draw a new diagram such that at least two state are not communicating. Do not forget to assign probabilities in the new diagram.

8. Draw a diagram in such away that the probability to move from state i to j is 1 and $P_{i,i} = 0$.