Question I Choose the correct answer:

(1)

$$\begin{aligned} 2\ln\frac{1}{4}-\ln 2 &= 2\left(\ln 1-\ln 4\right)-\ln 2 \\ &= 2\left(0-\ln 2^2\right)-\ln 2 \\ &= 2\left(-2\ln 2\right)-\ln 2 \\ &= -4\ln 2-\ln 2 \\ &= -5\ln 2 \end{aligned}$$

(2)

$$\begin{aligned} \frac{\cosh x-\sinh x}{e^x} &= \frac{\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}}{e^x} \\ &= \frac{\frac{e^x+e^{-x}-e^x+e^{-x}}{2}}{e^x} \\ &= \frac{e^{-x}}{e^x} \\ &= e^{-x-x} \\ &= e^{-2x} \end{aligned}$$

(3)

$$\frac{d}{dx}\left(\operatorname{sech} x^2\right)=2x\operatorname{sech} x^2\tanh x^2$$

Thus,

$$\begin{aligned} \left.\frac{d}{dx}\left(\operatorname{sech} x^2\right)\right|_{x=0} &= 20\operatorname{sech} 0^2\tanh 0^2 \\ &= 0 \end{aligned}$$

Question II B.

$$\begin{aligned} \frac{d}{dx}\left(\cosh^{-1}\left(x^3-1\right)\right) &= \frac{1}{\sqrt{\left(x^3-1\right)^2-1}}\frac{d}{dx}\left(x^3-1\right) \\ &= \frac{3x^2}{\sqrt{\left(x^3-1\right)^2-1}} \end{aligned}$$

Thus,

$$\begin{aligned} \left.\frac{d}{dx}\left(\cosh^{-1}\left(x^3-1\right)\right)\right|_{x=2} &= \frac{3\left(2^2\right)}{\sqrt{\left(2^3-1\right)^2-1}} \\ &= \frac{12}{\sqrt{48}} \end{aligned}$$

Question III Use the properties of the logarithm function to compute $f'(x)$

$$\begin{aligned} f(x) &= \ln\sqrt{\frac{e^{5x}}{\tanh^3 x}} \\ &= \ln\left(\frac{e^{5x}}{\tanh^3 x}\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\ln\left(\frac{e^{5x}}{\tanh^3 x}\right) \\ &= \frac{1}{2}\left(\ln e^{5x} - \ln\left(\tanh^3 x\right)\right) \\ &= \frac{1}{2}\left(5x - 3\ln(\tanh x)\right) \\ &= \frac{5}{2}x - \frac{3}{2}\ln(\tanh x) \end{aligned}$$

Thus,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{5}{2}x - \frac{3}{2}\ln(\tanh x)\right] \\ &= \frac{5}{2} - \frac{3}{2}\frac{1}{\tanh x}\operatorname{sech}^2 x \end{aligned}$$

Question IV Evaluate the following integrals:

(1)

$$I = \int \frac{1}{\sqrt{-5+6x-x^2}}dx$$

First, we complete the square

$$\begin{aligned} -5+6x-x^2 &= -\left(x^2-6x+5\right) \\ &= -\left(x^2-6x+9-9+5\right) \\ &= -\left((x-3)^2-4\right) \\ &= 4-(x-3)^2 \end{aligned}$$

The integral therefore becomes

$$I = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{x-3}{2}\right)^2}}dx$$

Using the substitution

$$\begin{aligned} u &= \frac{x-3}{2} \\ du &= \frac{1}{2}dx \end{aligned}$$

gives

$$\begin{aligned} I &= \int \frac{1}{\sqrt{1-u^2}} du \\ &= \sin^{-1}(u) + c \\ &= \sin^{-1}\left(\frac{x-3}{2}\right) + c \end{aligned}$$

(2)

$$\begin{aligned} I &= \int \tan^3 x \sec^3 x dx \\ &= \int \tan^2 x \sec^2 x \tan x \sec x dx \\ &= \int \left(\sec^2 x - 1\right) \sec^2 x \tan x \sec x dx \end{aligned}$$

Using the substitution

$$\begin{aligned} u &= \sec x \\ du &= \tan x \sec x dx \end{aligned}$$

gives

$$\begin{aligned} I &= \int \left(u^2 - 1\right) u^2 du \\ &= \int u^4 - u^2 du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + c \\ &= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c \end{aligned}$$

(3)

$$I = \int \frac{1}{x^2\sqrt{16-x^2}} dx$$

Using the substitution

$$\begin{aligned} x &= 4\sin\theta \\ dx &= 4\cos\theta d\theta \end{aligned}$$

gives

$$\begin{aligned}
I &= \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}} 4\cos\theta d\theta \\
&= \frac{1}{16}\int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}}\cos\theta d\theta \\
&= \frac{1}{16}\int \frac{1}{\sin^2\theta\sqrt{\cos^2\theta}}\cos\theta d\theta \\
&= \frac{1}{16}\int \frac{1}{\sin^2\theta\cos\theta}\cos\theta d\theta \\
&= \frac{1}{16}\int \frac{1}{\sin^2\theta} d\theta \\
&= \frac{1}{16}\int \csc^2\theta d\theta \\
&= \frac{1}{16}\int \csc^2\theta d\theta \\
&= \frac{-1}{16}\cot\theta + c
\end{aligned}$$

Since

$$x = 4\sin\theta$$

we have

$$\begin{aligned}
\sin\theta &= \frac{x}{4} \\
\cos\theta &= \frac{\sqrt{16-x^2}}{4}
\end{aligned}$$

from which we get

$$\cot\theta = \frac{\sqrt{16-x^2}}{x}$$

Therefore,

$$\begin{aligned}
I &= \frac{-1}{16}\cot\theta + c \\
&= \frac{d}{dx}\left(\frac{-\sqrt{16-x^2}}{16x}\right) + c
\end{aligned}$$