Ex. 8

① $* X \sim unif(0,1) \Rightarrow f_X(x) = \frac{1}{1} = 1 \ , \quad 0<x<1$

$* Y = -2\ln X \Rightarrow \underset{\sim\sim\sim}{X = e^{-\frac{y}{2}}} \Rightarrow \frac{d}{dy}X = -\frac{1}{2}e^{-\frac{y}{2}} \Rightarrow \underset{\sim\sim\sim}{\left|\frac{d}{dy}X\right| = \frac{1}{2}e^{-\frac{y}{2}}}$

$* 0<X<1 \Rightarrow \ln 0 < \ln x < \ln 1 \Rightarrow -\infty < \ln x < 0 \Rightarrow \infty > -2\ln x > 0 \Rightarrow \quad \underset{\sim\sim\sim}{0<y<\infty}$

$* f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = \frac{1}{2}e^{-\frac{y}{2}}$

$\therefore Y \sim exp(\lambda = \frac{1}{2})$

②

③

* $X \sim N(\mu, \sigma^2) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$

* $Y = e^X \Rightarrow \underline{X = \ln Y} \Rightarrow \frac{d}{dy}X = \frac{1}{y} \Rightarrow \underline{\left|\frac{d}{dy}X\right| = \frac{1}{y}}$

* $-\infty < x < \infty \Rightarrow 0 < e^x < \infty \Rightarrow \underline{0 < y < \infty}$

* $f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma}\right)^2} \frac{1}{y}$

④

* $X \sim \exp(1) \Rightarrow f_X(x) = e^{-x}, \quad \infty > x > 0$

* $Y = -\ln X \Rightarrow \underline{X = e^{-y}} \Rightarrow \frac{d}{dy}X = -e^{-y} \Rightarrow \underline{\left|\frac{d}{dy}X\right| = e^{-y}}$

* $x > 0 \Rightarrow \infty > \ln x > -\infty \Rightarrow -\infty < -\ln x < \infty \Rightarrow \underline{-\infty < y < \infty}$

* $f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = e^{-e^{-y}} e^{-y} = e^{-(y+e^{-y})}$

⑤

* $X \sim \text{unif}(0,1) \Rightarrow f_X(x) = 1, \quad 0 < x < 1$

* $Y = \sqrt{X} \Rightarrow \underline{X = Y^2} \Rightarrow \frac{d}{dy}X = 2Y \Rightarrow \underline{\left|\frac{d}{dy}X\right| = 2Y}$

* $0 < x < 1 \Rightarrow 0 < \sqrt{x} < 1 \Rightarrow \underline{0 < y < 1}$

* $f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = 2y$

⑥

* $X \sim$ dis. $f_X(x) = \frac{1}{2}x, \ 0<x<2$

* $Y = X^3 \Rightarrow X = Y^{\frac{1}{3}} \Rightarrow \frac{d}{dy}X = \frac{1}{3}Y^{-\frac{2}{3}} \Rightarrow \left|\frac{d}{dy}X\right| = \frac{1}{3}Y^{-\frac{2}{3}}$

* $0<x<2 \Rightarrow 0<x^3<8 \Rightarrow 0<y<8$

* $f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = \frac{1}{2}y^{\frac{1}{3}}\,\frac{1}{3}y^{-\frac{2}{3}} = \frac{1}{6}y^{-\frac{1}{3}}$

* $P(\frac{1}{2}<X<1) = \frac{1}{2}\int_{1/2}^{1} x\,dx = \frac{1}{4}x^2\Big|_{1/2}^{1} = \frac{1}{4}(-\frac{1}{4}+1) = \frac{1}{4}(\frac{3}{4}) = \frac{3}{16}$

$P(\frac{1}{8}<Y<1) = \frac{1}{6}\int_{1/8}^{1} y^{-\frac{1}{3}}\,dy = \frac{1}{6}(\frac{3}{2})\,y^{2/3}\Big|_{1/8}^{1} = \frac{1}{4}(1-\frac{1}{4}) = \frac{1}{4}(\frac{3}{4}) = \frac{3}{16}$

we can see that are the same because

$\frac{1}{2}<X<1 \Rightarrow \frac{1}{8}<X^3<1 \Rightarrow \frac{1}{8}<Y<1$ and so $P(\frac{1}{2}<X<1) = P(\frac{1}{8}<Y<1)$

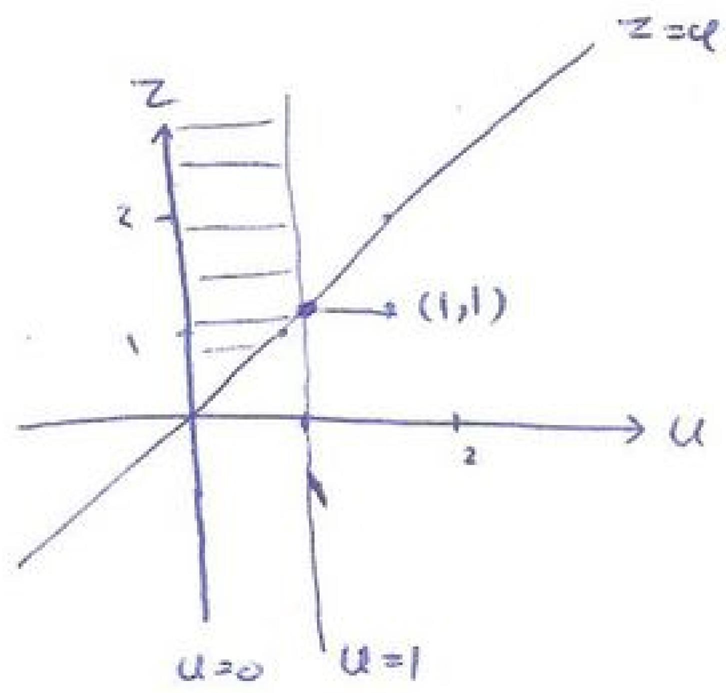
⑧

* $X \sim unif(0,1) \Rightarrow f(x) = 1, \ 0<x<1$
$Y \sim exp(1) \Rightarrow f(y) = e^{-y}, \ y>0$
$\Bigg\}\ f(x,y) = e^{-y}, \ 0<x<1, \ y>0$

* $Z = X+Y \Rightarrow Y = Z-X$
$U = X$
$\therefore Y = Z-U$

*
$\left.\begin{matrix} 0<x<1 \\ 0<y<\infty \end{matrix}\right\} \Rightarrow \begin{matrix} 0<x+y<\infty \Rightarrow 0<z<\infty \\ 0<x<1 \Rightarrow 0<u<1 \end{matrix}$

$\left.\begin{matrix} 0<x<1 \\ 0<y<\infty \end{matrix}\right\} \Rightarrow \begin{matrix} 0<u<1 \\ 0<z-u<\infty \Rightarrow u<z<\infty \end{matrix}$

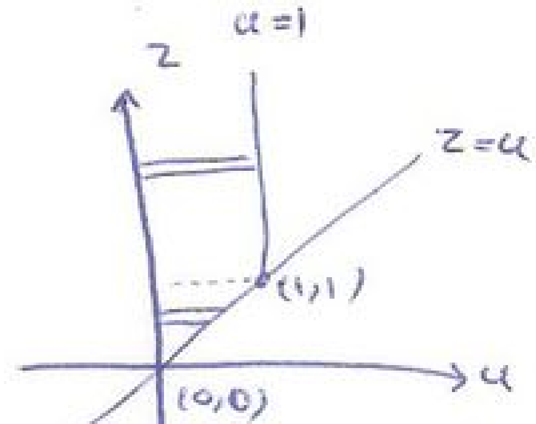


* $J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x}z & \frac{\partial}{\partial y}z \\ \frac{\partial}{\partial x}u & \frac{\partial}{\partial y}u \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0-1 = -1 \Rightarrow |J(x,y)|^{-1} = 1$

* $f(z,u) = f(x,y)\,|J(x,y)|^{-1} = e^{-(z-u)}$

* $f(z) = \int f(z,u)\,du = \begin{cases} \int_0^z e^{-(z-u)}\,du, & 0<z<1 \\ \int_0^1 e^{-(z-u)}\,du, & 1<z<\infty \end{cases}$

$= \begin{cases} 1-e^{-z}, & 0<z<1 \\ (e-1)e^{-z}, & 1<z<\infty \end{cases}$



Exercise 8

(12) X, Y have $f(X,Y) = \frac{2}{5}(x+4y)$
$0<X<1,\ 0<Y<1$

(a) Conditional Pdf of $Y \mid X = x$
(b) $P(Y < \frac{1}{3} \mid X = \frac{1}{2})$

(a) $$f(Y \mid X=x) = \frac{f(x,y)}{f(x)}$$

$$f(x) = \int_{y} f(x,y)\,dy = \int_{y=0}^{1} \frac{2}{5}(x+4y)\,dy = \frac{2}{5}(x+2)$$

$$\therefore f(Y \mid X=x) = \frac{\frac{2}{5}(x+4y)}{\frac{2}{5}(x+2)} = \frac{x+4y}{x+2}$$

(b) $$f(Y \mid X=\tfrac{1}{2}) = \frac{\frac{1}{2}+4y}{\frac{1}{2}+2} = \frac{1+8y}{5}$$

$$\therefore P(Y < \tfrac{1}{3} \mid X=\tfrac{1}{2}) = \int_{0}^{\frac{1}{3}} \frac{1+8y}{5}\,dy = \frac{1}{5}\left(y+4y^2\right)\Big]_{0}^{\frac{1}{3}}$$

$$= \frac{1}{5} \times \left(\frac{1}{3} + \frac{4}{9}\right) = \frac{7}{45}$$

(13) a)

* $X \sim \text{unif}(0,1) \Rightarrow f_X(x) = 1, \quad 0<x<1$

$Y \sim \exp(1) \Rightarrow f_y(y) = e^{-y}, \quad 0<y<\infty$

$\Big\}$ X, Y indep. $\Rightarrow f(x,y) = f_X(x) f_y(y) = e^{-y}, \quad 0<x<1, \quad 0<y<\infty$

* $Z = X+Y \Rightarrow Y = Z - X$

$U = \frac{X}{Y} \Rightarrow X = UY = U(Z-X) = UZ - UX \Rightarrow X + UX = UZ \Rightarrow X = \frac{UZ}{(1+U)}$

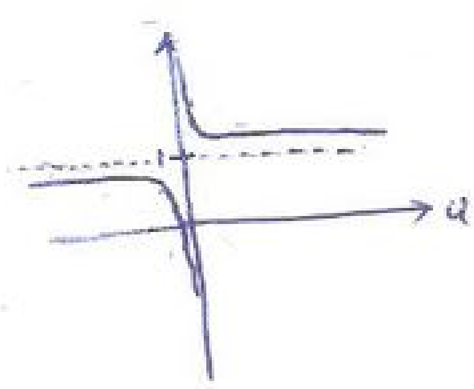
$\therefore Y = Z - X = Z - \frac{UZ}{1+U} = \frac{Z+UZ-UZ}{1+U} = \frac{Z}{1+U} \Rightarrow Y = \frac{Z}{1+U}$

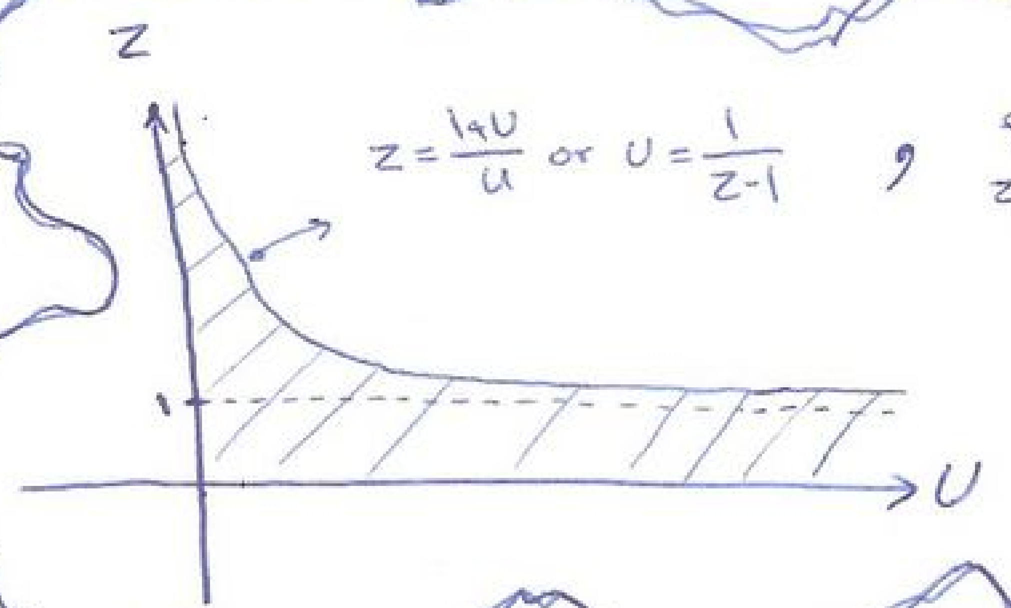
* $\left.\begin{array}{l} 0<x<1 \\ 0<y<\infty \end{array}\right\} \Rightarrow$ $0 < x+y < 1+\infty \Rightarrow 0<Z<\infty$

$\frac{0}{0} < \frac{x}{y} < \frac{1}{\infty}$ (X) but $0<U<\infty$

$\left.\begin{array}{l} 0<x<1 \\ 0<y<\infty \end{array}\right\} \Rightarrow$ $0 < \frac{UZ}{1+U} < 1 \Rightarrow 0 < UZ < 1+U \Rightarrow 0 < Z < \frac{1+U}{U}$

$0 < \frac{Z}{1+U} < \infty \Rightarrow 0 < Z < \infty$





$Z = \frac{1+U}{U}$ or $U = \frac{1}{Z-1}$, $\lim_{Z\to\infty} \frac{1}{Z-1} = 0$, $\lim_{U\to\infty} \frac{1+U}{U} = 1$

$Z>1 \Rightarrow 0<U<\frac{1}{Z-1}$

$0<Z<1 \Rightarrow 0<U<\infty$

$0<U<\infty \Rightarrow 0<Z<\frac{U+1}{U}$

* $$J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x} z & \frac{\partial}{\partial y} z \\ \frac{\partial}{\partial x} u & \frac{\partial}{\partial y} u \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1/y & \frac{-x}{y^2} \end{vmatrix} = \frac{-x}{y^2} - \frac{1}{y} = \frac{-x-y}{y^2} = \frac{-(x+y)}{y^2} = \frac{-Z}{\left(\frac{Z}{1+U}\right)^2} = \frac{-(1+U)^2}{Z}$$

$$\Rightarrow |J(x,y)|^{-1} = \frac{Z}{(1+U)^2}$$

* $$f(z,u) = f(x,y)\,|J(x,y)|^{-1} = e^{-\left(\frac{z}{1+U}\right)} \frac{z}{(1+U)^2}$$

b) $$f(z) = \begin{cases} \displaystyle\int_0^{\frac{1}{z-1}} \frac{z}{(1+U)^2} e^{-\left(\frac{z}{1+U}\right)} du = \int_0^{1-z} e^{w} dw = e^{1-z} - e^{-z} = e^{-z}(e-1), & z>1 \\[2ex] \displaystyle\int_0^{\infty} \frac{z}{(1+U)^2} e^{-\left(\frac{z}{1+U}\right)} du = \int_{-z}^{0} e^{w} dw = 1 - e^{-z}, & 0<z<1 \end{cases}$$

using that $w = -\frac{z}{1+U} \Rightarrow dw = \frac{z}{(1+U)^2} du$

c) $f(u) = \int_0^{\frac{u+1}{u}} \frac{z}{(1+u)^2} e^{-\frac{z}{u+1}} \, dz = \frac{1}{(1+u)^2} \int_0^{\frac{u+1}{u}} z e^{-\frac{z}{u+1}} \, dz$

$= \frac{-1}{(u+1)^2}\left[\frac{z}{1/(u+1)} + \frac{1}{1/(u+1)^2}\right] e^{-\frac{z}{1+u}} \Big|_0^{\frac{u+1}{u}} = \frac{-1}{(u+1)^2}\left[z(u+1) + (u+1)^2\right] e^{-\frac{z}{1+u}} \Big|_0^{\frac{u+1}{u}}$

$= \frac{-1}{(u+1)^2} \left[\left[\frac{(u+1)^2}{u} + (u+1)^2\right] e^{-\frac{1}{u}} - \left[(u+1)^2\right] \right]$

$= -\left[\left[\frac{1}{u} + 1\right] e^{-\frac{1}{u}} - 1 \right] = 1 - \left(\frac{1}{u} + 1\right) e^{-1/u}, \quad u > 0$

Using that $\int_a^b x^n e^{-kx} \, dx = -\left[\frac{x^n}{k} + \frac{\frac{d}{dx}(x^n)}{k^2} + \cdots + \frac{\frac{d^n}{dx^n}(x^n)}{k^{n+1}} \right] e^{-kx} \Big|_a^b$

14 a)

* $f(x,y) = \frac{1}{x^2 y^2}, \quad x > 1, y > 1$

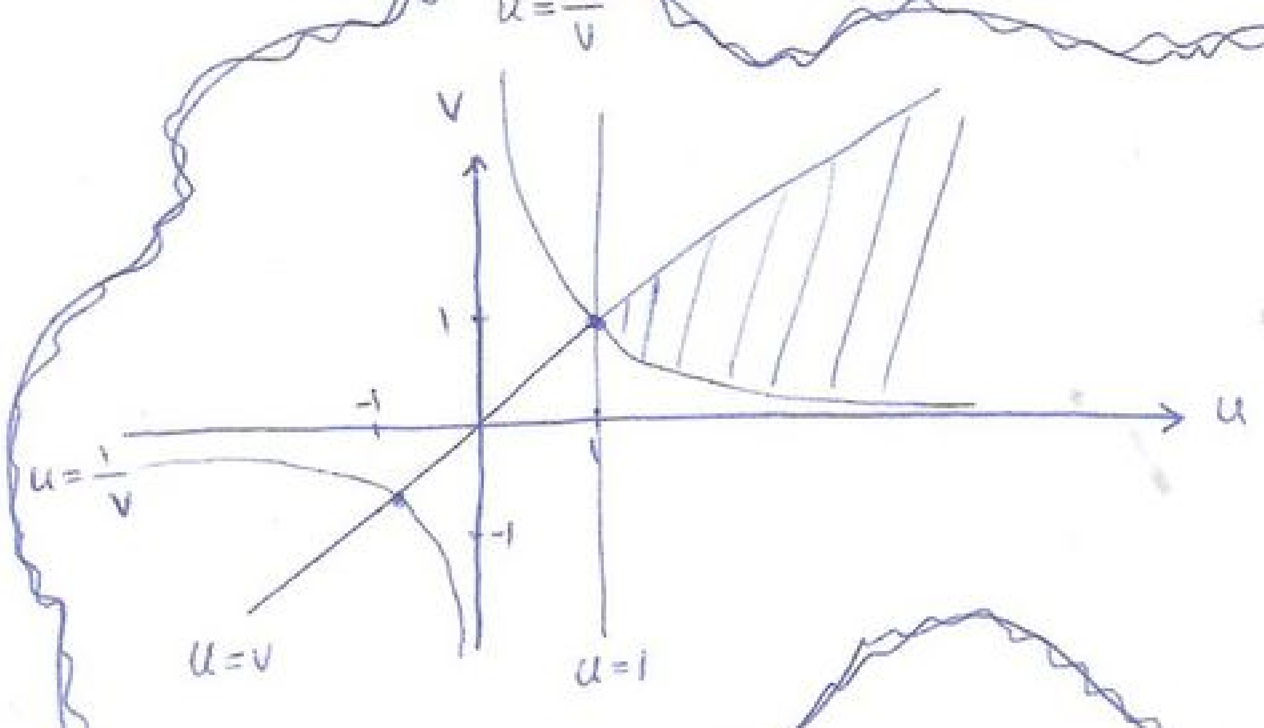
* $U = XY \Rightarrow \frac{U}{X} = Y$

$V = \frac{X}{Y} \Rightarrow VY = X \Rightarrow V\frac{U}{X} = X \Rightarrow X^2 = VU \Rightarrow X = \sqrt{VU}$

$\therefore \frac{U}{X} = Y \Rightarrow \frac{U}{\sqrt{VU}} = Y \Rightarrow Y = \sqrt{\frac{U}{V}}$

* $\left.\begin{matrix} x > 1 \\ y > 1 \end{matrix}\right\} \Rightarrow \begin{matrix} x \cdot y > 1 \Rightarrow U > 1 \\ \frac{x}{y} > 0 \Rightarrow V > 0 \end{matrix}$

$\left.\begin{matrix} x > 1 \\ y > 1 \end{matrix}\right\} \Rightarrow \begin{matrix} \sqrt{VU} > 1 \Rightarrow VU > 1 \\ \sqrt{\frac{U}{V}} > 1 \Rightarrow \frac{U}{V} > 1 \Rightarrow U > V \end{matrix}$



$\lim_{u \to \infty} \frac{1}{u} = 0$

$\left\{ \frac{1}{u} < V < u \Rightarrow 1 < u < \infty \right.$

$\left\{ \begin{matrix} \frac{1}{V} < u < \infty \Rightarrow 0 < V < 1 \\ V < u < \infty \Rightarrow 1 < V < \infty \end{matrix} \right.$

$$* \; J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x}u & \frac{\partial}{\partial y}u \\ \frac{\partial}{\partial x}v & \frac{\partial}{\partial y}v \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -\frac{x}{y} - \frac{x}{y} = \frac{-2x}{y}$$

$$\Rightarrow |J(x,y)|^{-1} = \frac{1}{2}\frac{y}{x} = \frac{1}{2}\frac{1}{v}$$

$$* \; f(u,v) = f(x,y)\,|J(x,y)|^{-1} = \frac{1}{(uv)(\frac{u}{v})}\,\frac{1}{2v} = \frac{1}{2u^2 v}$$

b)

$$f(u) = \int_{1/u}^{u} \frac{1}{2u^2 v}\,dv = \frac{1}{2u^2}\int_{1/u}^{u}\frac{1}{v}\,dv = \frac{1}{2u^2}\ln v \Big|_{1/u}^{u} = \frac{1}{2u^2}\left[\ln u - \ln\frac{1}{u}\right] = \frac{1}{u^2}\ln u \quad , \; 1<u<\infty$$

$$f(v) = \begin{cases} \displaystyle\int_{1/v}^{\infty}\frac{1}{2u^2 v}\,du = \frac{1}{2v}\int_{1/v}^{\infty}\frac{1}{u^2}\,du = \frac{-1}{2v}\,\frac{1}{u}\Big|_{1/v}^{\infty} = \frac{1}{2} & , \; 0<v<1 \\[2ex] \displaystyle\int_{v}^{\infty}\frac{1}{2u^2 v}\,du = \frac{1}{2v}\int_{v}^{\infty}\frac{1}{u^2}\,du = \frac{-1}{2v}\,\frac{1}{u}\Big|_{v}^{\infty} = \frac{1}{2v^2} & , \; 1<v<\infty \end{cases}$$

15 $* \; X_1, X_2 \sim \exp(\lambda) \Rightarrow f(x_1) = \lambda e^{-\lambda x_1}, \; x_1>0$

$f(x_2) = \lambda e^{-\lambda x_2}, \; x_2>0$ } X_1, X_2 indep. $\therefore f(x_1,x_2) = f(x_1)f(x_2)$

$= \lambda^2 e^{-\lambda(x_1+x_2)}, \; x_1>0, \; x_2>0$

$* \; Y_1 = X_1+X_2 \Rightarrow X_2 = Y_1 - X_1$

$Y_2 = e^{X_1} \Rightarrow \underline{X_1 = \ln Y_2}$

$\therefore X_2 = Y_1 - X_1 = Y_1 - \ln Y_2 \Rightarrow \underline{X_2 = Y_1 - \ln Y_2}$

$$* \; \left.\begin{array}{l} 0<x_1<\infty \\ 0<x_2<\infty \end{array}\right\} \Rightarrow \begin{array}{l} 0<x_1+x_2<\infty \Rightarrow \underline{0<y_1<\infty} \\ 0<e^{x_2}<\infty \Rightarrow \underline{0<y_2<\infty} \end{array}$$

$$\left.\begin{array}{l} 0<x_1<\infty \\ 0<x_2<\infty \end{array}\right\} \Rightarrow \begin{array}{l} \underline{0<\ln y_2<\infty} \\ 0<y_1-\ln y_2<\infty \Rightarrow \underline{\ln y_2<y_1<\infty} \end{array}$$

$$* \; J(x_1,x_2) = \begin{vmatrix} \frac{\partial}{\partial x_1} y_1 & \frac{\partial}{\partial x_2} y_1 \\ \frac{\partial}{\partial x_1} y_2 & \frac{\partial}{\partial x_2} y_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^{x_1} & 0 \end{vmatrix} = 0 - e^{x_1} = -e^{x_1} = -e^{\ln y_2} = -y_2$$

$$\Rightarrow \quad |J(x_1,x_2)|^{-1} = \frac{1}{y_2}$$

$$* \; f(y_1,y_2) = f(x_1,x_2)\,|J(x_1,x_2)|^{-1} = \lambda^2 e^{-\lambda(\ln y_2 + y_1 - \ln y_2)} \frac{1}{y_2}$$

$$= \lambda^2 e^{-\lambda y_1} \frac{1}{y_2}$$

16 a)

$$* \; X_1 \sim exp(\lambda_1) \Rightarrow f(x_1) = \lambda_1 e^{-\lambda_1 x_1}, \; x_1 > 0$$

$$X_2 \sim exp(\lambda_2) \quad f(x_2) = \lambda_2 e^{-\lambda_2 x_2}, \; x_2 > 0$$

X_1, X_2 indep.

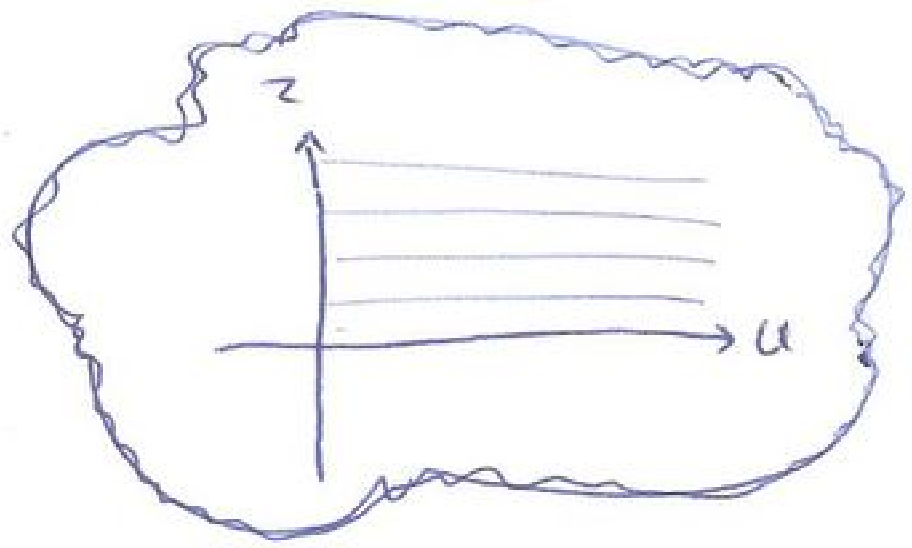
$$f(x_1,x_2) = f(x_1) f(x_2) = \lambda_1 \lambda_2 e^{-(\lambda_1 x_1 + \lambda_2 x_2)}, \; x_1 > 0, \; x_2 > 0$$

$$* \; Z = \frac{X_1}{X_2} \Rightarrow X_2 = \frac{X_1}{Z} = \frac{U}{Z} \Rightarrow X_2 = \frac{U}{Z}$$

$$U = X_1 \Rightarrow X_1 = U$$

$$* \; \left.\begin{array}{l} x_1 > 0 \\ x_2 > 0 \end{array}\right\} \Rightarrow \begin{array}{l} u > 0 \\ z > 0 \end{array}$$

$$\left.\begin{array}{l} x_1 > 0 \\ x_2 > 0 \end{array}\right\} \Rightarrow \begin{array}{l} u > 0 \\ \frac{u}{z} > 0 \Rightarrow u > 0, \; z > 0 \text{ or } u < 0, \; z < 0 \end{array}$$



$$* \; J(x_1,x_2) = \begin{vmatrix} \frac{\partial}{\partial x_1} z & \frac{\partial}{\partial x_2} z \\ \frac{\partial}{\partial x_1} u & \frac{\partial}{\partial x_2} u \end{vmatrix} = \begin{vmatrix} \frac{1}{x_2} & \frac{-x_1}{x_2^2} \\ 1 & 0 \end{vmatrix} = \frac{+x_1}{x_2^2} = \frac{u}{(u/z)^2} = \frac{z^2}{u}$$

$$\Rightarrow \quad |J(x_1,x_2)|^{-1} = \frac{u}{z^2}$$

$$* \; f(z,u) = f(x_1,x_2)\,|J(x_1,x_2)|^{-1} = \lambda_1 \lambda_2 e^{-(\lambda_1 u + \lambda_2 \frac{u}{z})} \frac{u}{z^2} = \lambda_1 \lambda_2 \frac{u}{z^2} e^{-u(\lambda_1 + \lambda_2 \frac{1}{z})}$$

$$* \; f(z) = \int_0^{\infty} f(z,u)\,du = \lambda_1\lambda_2 \frac{1}{z^2} \int_0^{\infty} u\, e^{-u(\lambda_1 + \lambda_2 \frac{1}{z})}\,du$$

$$= \lambda_1\lambda_2 \frac{1}{z^2} \frac{\Gamma 2}{(\lambda_1+\lambda_2\frac{1}{z})^2} = \lambda_1\lambda_2\frac{1}{z^2}\frac{1}{(\frac{\lambda_1 z+\lambda_2}{z})^2} = \frac{\lambda_1\lambda_2}{(\lambda_1 z+\lambda_2)^2} \quad , \; z>0$$

$$\Rightarrow F(z) = \begin{cases} 0 & , \; z<0 \\ \int_0^z \frac{\lambda_1\lambda_2}{(t\lambda_1+\lambda_2)^2}\,dt = \lambda_1\lambda_2 \int_0^z (t\lambda_1+\lambda_2)^{-2}\,dt & \\ = \lambda_1\lambda_2 \frac{(t\lambda_1+\lambda_2)^{-1}}{-\lambda_1}\Big|_0^z = 1 - \frac{\lambda_2}{z\lambda_1+\lambda_2} = \frac{z\lambda_1}{z\lambda_1+\lambda_2} & , \; z>0 \end{cases}$$

b)

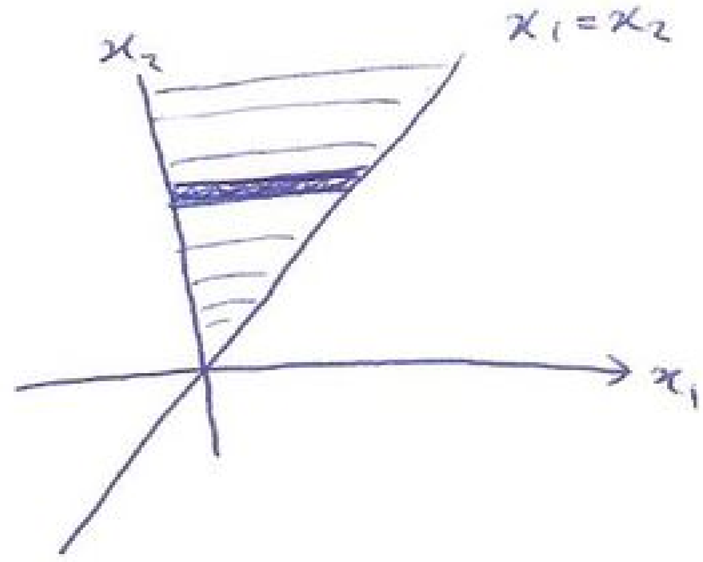
$$P(X_1<X_2) = \int_0^{\infty}\int_0^{x_2} f(x_1,x_2)\,dx_1\,dx_2$$

$$= \int_0^{\infty} \lambda_2 e^{-\lambda_2 x_2}\left[-e^{-\lambda_1 x_1}\Big|_0^{x_2}\right]dx_2$$

$$= \int_0^{\infty} \lambda_2 e^{-\lambda_2 x_2}(1-e^{-\lambda_1 x_2})\,dx_2$$

$$= \int_0^{\infty} \lambda_2 e^{-\lambda_2 x_2}\,dx_2 - \frac{\lambda_2}{\lambda_1+\lambda_2}\int_0^{\infty}(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x_2}\,dx_2$$

$$= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$



Exercise 8

(17) $f(x,y) = \frac{e^{-y}}{y}$; $0 < x < y$, $0 < y < \infty$

$$E(X) = \int_0^\infty \int_0^y x \cdot \frac{1}{y} e^{-y} \, dx \, dy$$

$$= \int_0^\infty \frac{1}{y} e^{-y} \left(\frac{1}{2} x^2 \Big]_0^y \right) dy = \int_0^\infty \frac{1}{2} y e^{-y} \, dy$$

Let $W = ye^{-y} \sim Exp(1)$

$$\therefore E(X) = \frac{1}{2} E[W] = \frac{1}{2}$$

$$E[Y] = \int_0^\infty \int_0^y y \cdot \frac{1}{y} e^{-y} \, dx \, dy = \int_0^\infty y e^{-y} \, dy = 1$$

$$E[X^2] = \int_0^\infty \int_0^y x^2 \, \frac{1}{y} e^{-y} \, dx \, dy$$

$$= \int_0^\infty \frac{1}{y} e^{-y} \left(\frac{1}{3} x^3 \Big]_0^y \right) dy = \int_0^\infty \frac{1}{3} y^2 e^{-y} \, dy$$

$$= \frac{1}{3} E[W^2] = \frac{1}{3} \times 2 = \frac{2}{3}$$

$$\therefore V(X) = \frac{2}{3} - \left(\frac{1}{2}\right)^2 = \frac{5}{12}$$

$$E[Y^2] = \int_0^\infty y^2 e^{-y} \, dy = 2 \qquad \therefore V(Y) = 2 - 1 = 1$$

$$E[XY] = \int_0^\infty \int_0^y xy \, \frac{1}{y} e^{-y} \, dx \, dy = 1$$

$$\therefore Cov(X,Y) = 1 - \frac{1}{2}(1) = \frac{1}{2}$$

a)

(18) * $X, Y \sim \text{unif}(0,1) \Rightarrow f(x) = 1, \quad 0<x<1$

$f(y) = 1, \quad 0<y<1$

X, Y indep.

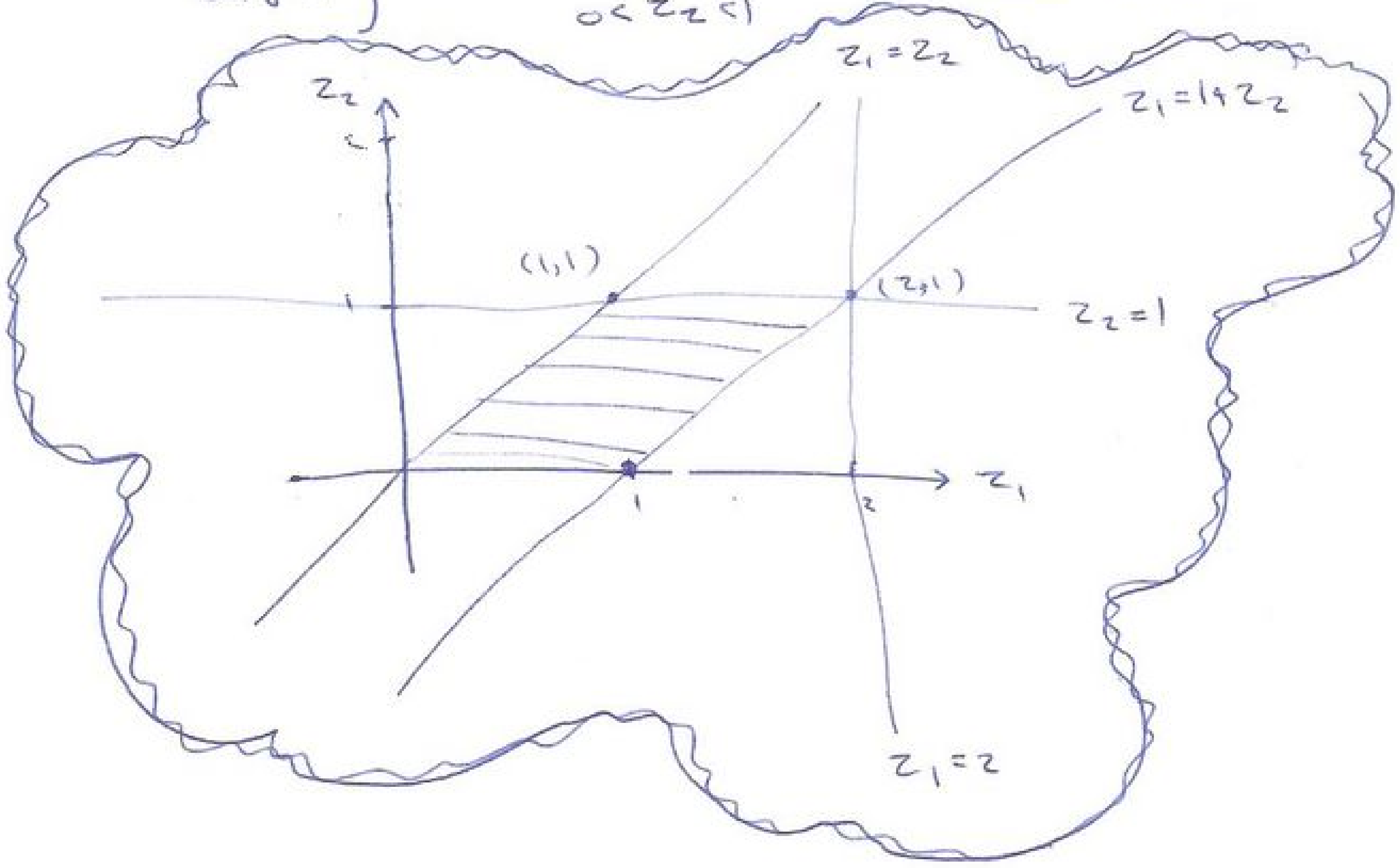
$f(x,y) = f(x)f(y) = 1, \quad 0<x<1, \; 0<y<1$

* $Z_1 = X+Y \Rightarrow X = Z_1 - Y \Rightarrow \underline{X = Z_1 - Z_2}$

$\underline{Z_2 = Y}$

* $\left.\begin{array}{l} 0<x<1 \\ 0<y<1 \end{array}\right\} \Rightarrow \begin{array}{l} 0<x+y<2 \Rightarrow \underline{0<z_1<2} \\ 0<y<1 \Rightarrow \underline{0<z_2<1} \end{array}$

$\left.\begin{array}{l} 0<x<1 \\ 0<y<1 \end{array}\right\} \Rightarrow \begin{array}{l} 0<z_1-z_2<1 \Rightarrow \underline{z_2<z_1<1+z_2} \\ 0<z_2<1 \end{array}$



* $J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x} z_1 & \frac{\partial}{\partial y} z_1 \\ \frac{\partial}{\partial x} z_2 & \frac{\partial}{\partial y} z_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \Rightarrow |J(x,y)|^{-1} = 1$

* $f(z_1, z_2) = f(x,y)\,|J(x,y)|^{-1} = 1$

b) $f(z_1) = \int f(z_1, z_2)\, dz_2 = \begin{cases} \int_0^{z_1} 1\, dz_2 = z_1, & 0<z_1<1 \\ \int_{z_1-1}^{1} 1\, dz_2 = 2-z_1, & 1<z<2 \end{cases}$

19 a)

* $X, Y \sim exp(1)$ which they are indep. $\Big\}$ $f(x,y) = f(x) f(y) = e^{-x} e^{-y} = e^{-(x+y)}$, $x>0, y>0$

* $Z = X+Y \Rightarrow X = Z-Y$

$U = \frac{X}{X+Y} \Rightarrow U = \frac{X}{Z} = \frac{Z-Y}{Z} \Rightarrow UZ = Z-Y \Rightarrow Y = Z - ZU = Z(1-U) \Rightarrow \underline{Y = Z(1-U)}$

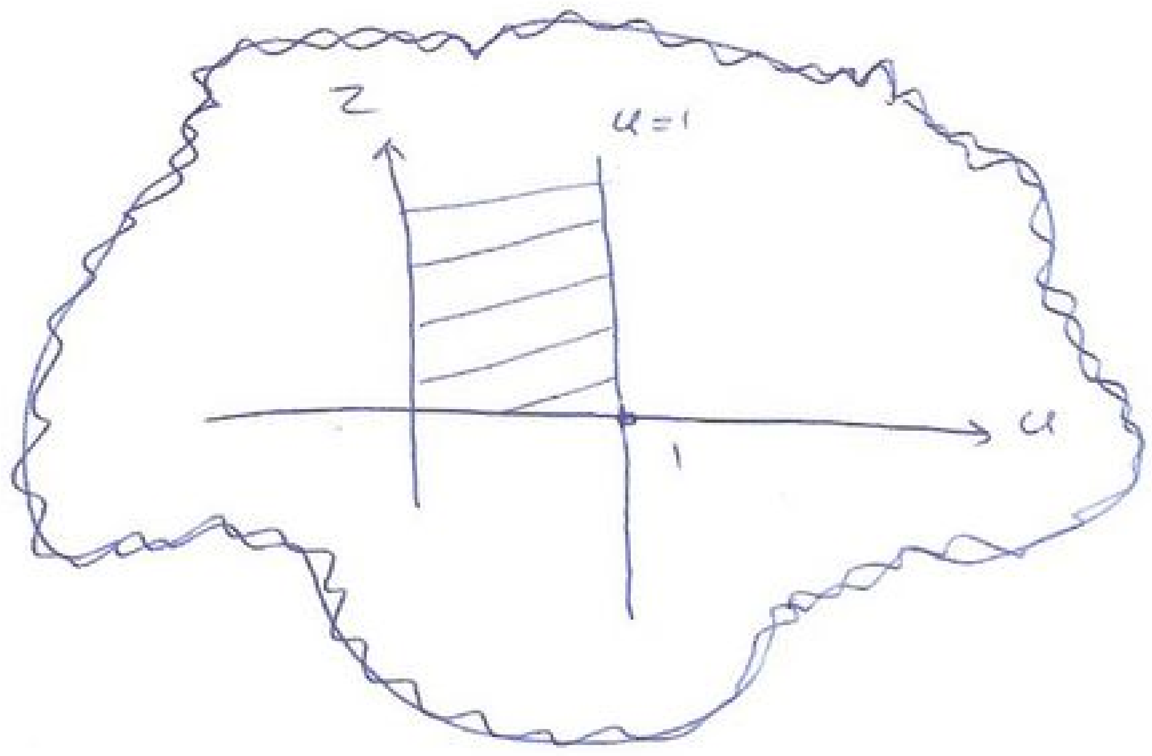
$\therefore X = Z-Y = Z - Z(1-U) = ZU \Rightarrow \underline{X = ZU}$

*
$\left.\begin{array}{l} x>0 \\ y>0 \end{array}\right\} \Rightarrow$ $x+y>0 \Rightarrow \underline{z>0}$

$\frac{x}{x+y} > \frac{0}{0}$ (X), but $\frac{x}{x+y} > 0 \Rightarrow \underline{u>0}$

$\left.\begin{array}{l} x>0 \\ y>0 \end{array}\right\} \Rightarrow$ $zu>0 \Rightarrow z>0$ or $u>0$

$z(1-u)>0 \Rightarrow z>0$ or $(1-u)>0 \Rightarrow z>0$ or $\underline{1>u}$



* $J(x,y) = \begin{vmatrix} \frac{\partial}{\partial x} z & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} u & \frac{\partial}{\partial y} u \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{(x+y)-x}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix} = \frac{-1}{(x+y)} = \frac{-1}{z}$

$\Rightarrow |J(x,y)|^{-1} = z$

* $f(z,u) = f(x,y) \, |J(x,y)|^{-1} = e^{-(zu + z(1-u))} (z) = z e^{-z}$

b) pdf of u:

$f(y) = \int_0^\infty z e^{-z} \, dz = \frac{\Gamma 2}{1^2} = 1$, $0<u<1$

(20) deleting

* $f(x,y) = 24xy, \quad 0<x<1, \quad 0<y<1, \quad x+y<1$

* $Z = XY^2 \Rightarrow Y = \sqrt{\frac{Z}{X}}$

$\underline{U = X}$

$\therefore\ Y = \sqrt{\frac{Z}{X}} = \sqrt{\frac{Z}{U}} \Rightarrow \underline{Y = \sqrt{\frac{Z}{U}}}$

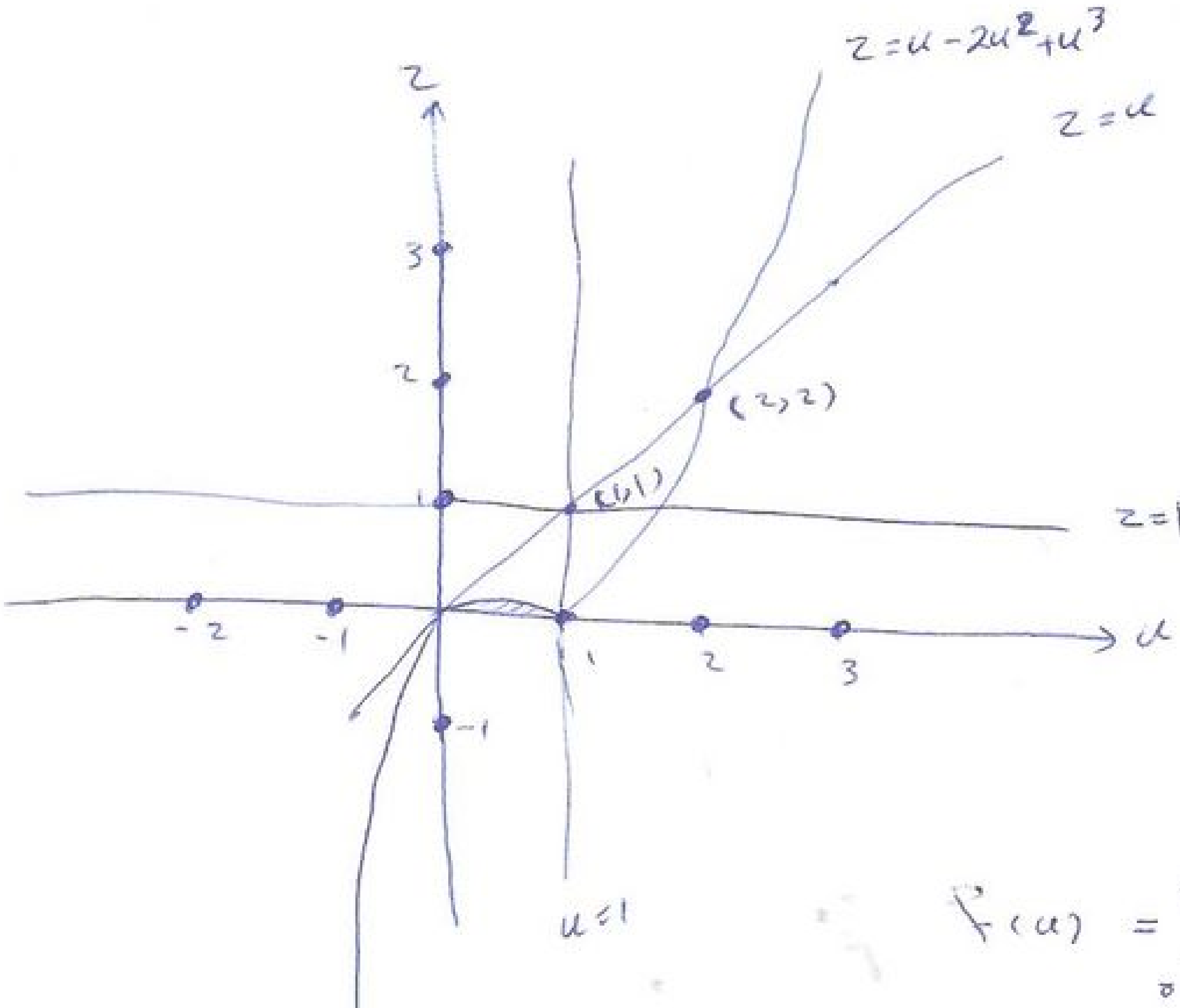
*
$$\left.\begin{array}{l} 0<x<1 \\ 0<y<1 \\ x+y<1 \end{array}\right\} \Rightarrow \begin{array}{l} 0<xy^2<1 \Rightarrow \underline{0<z<1} \\ \underline{0<u<1} \end{array}$$

$$\left.\begin{array}{l} 0<x<1 \\ 0<y<1 \\ x+y<1 \end{array}\right\} \begin{array}{l} 0<u<1 \\ 0<\sqrt{\frac{z}{u}}<1 \Rightarrow 0<\frac{z}{u}<1 \Rightarrow \underline{0<z<u} \\ u+\sqrt{\frac{z}{u}}<1 \Rightarrow \frac{u^{3/2}+\sqrt{z}}{\sqrt{u}}<1 \Rightarrow u^{3/2}+\sqrt{z}<\sqrt{u} \end{array}$$

$\Rightarrow \sqrt{z} < \sqrt{u} - u^{3/2}$

$\Rightarrow z < (u^{1/2} - u^{3/2})^2$

$\Rightarrow z < u - 2u^2 + u^3$



$\lim\limits_{u\to\infty}(u-2u^2+u^3) = \infty - 2\infty + \infty = \infty - \infty + \infty = \infty$

$\lim\limits_{u\to-\infty}(u-2u^2+u^3) = -\infty - 2\infty - \infty$
$= -\infty - \infty - \infty = -3\infty = -\infty$

$$\left.\begin{array}{l} z = u-2u^2+u^3 \\ z = u \end{array}\right\} \Rightarrow \begin{array}{l} u-2u^2+u^3 = u \\ \Rightarrow u = 2 \\ \Rightarrow z = 2 \end{array}$$

$f(u) = \int_0^{u-2u^2+u^3} f(z,u)\,dz, \quad 0<u<1$

$f(z) = \int_{\bigcirc}^{\bigcirc} f(z,u)\,du, \quad 0<z<\bigcirc$

هذا السؤال إلي ~~هـ~~ ما صعب، بغاله تركيز...
لذا هذا ملغي بالنسبة لي

a)

(21) * $X \sim Gamma(\alpha, \lambda)$

$Y \sim Gamma(\alpha, \lambda)$

$\therefore f(x) = \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x}, \quad x > 0$

$f(y) = \frac{\lambda^\alpha}{\Gamma\alpha} y^{\alpha-1} e^{-\lambda y}, \quad y > 0$

as X, Y are indep. so,

$f(x,y) = f(x) f(y)$

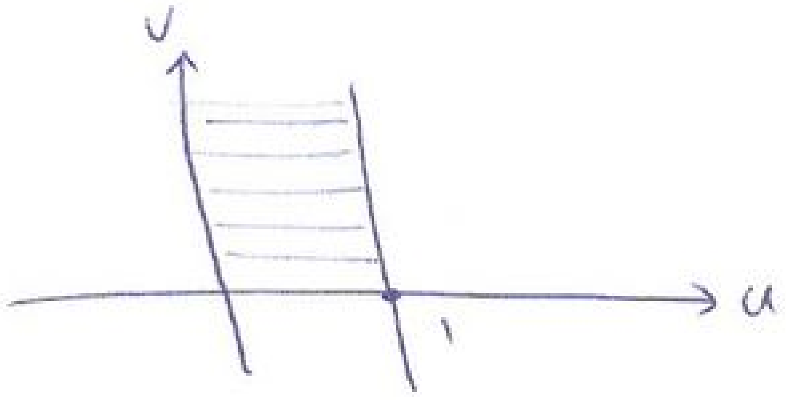
$= \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 x^{\alpha-1} y^{\alpha-1} e^{-\lambda(x+y)}, \quad x > 0, y > 0$

From (19) we can see that:

* $X = VU$

$Y = V(1-U)$

* $V > 0, \quad U > 0, \quad 1 > U$



* $|J(x,y)|^{-1} = V$

but:

* $f(V,U) = f(x,y)\,|J(x,y)|^{-1} = \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 (VU)^{\alpha-1} \left(V(1-U)\right)^{\alpha-1} e^{-\lambda(VU+V(1-U))} (V)$

$= \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 V^{2\alpha-1} U^{\alpha-1} (1-U)^{\alpha-1} e^{-\lambda(V)}$

b) $f(U) = \int f(V,U)\, dV = \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 U^{\alpha-1}(1-U)^{\alpha-1} \int_0^\infty V^{2\alpha-1} e^{-\lambda V} dV$

$= \left(\frac{\lambda^\alpha}{\Gamma\alpha}\right)^2 U^{\alpha-1}(1-U)^{\alpha-1} \frac{\Gamma 2\alpha}{\lambda^{2\alpha}} \quad \left(\text{from } \int_0^\infty x^{n-1} e^{-kx} dx = \frac{\Gamma n}{k^n}\right)$

$= \frac{1}{(\Gamma\alpha)^2} \frac{\Gamma 2\alpha}{1} (U)^{\alpha-1}(1-U)^{\alpha-1}$

$= \frac{\Gamma 2\alpha}{(\Gamma\alpha)^2} U^{\alpha-1}(1-U)^{\alpha-1} = \frac{\Gamma(\alpha+\alpha)}{\Gamma\alpha\,\Gamma\alpha} U^{\alpha-1}(1-U)^{\alpha-1}, \quad 0 < U < 1$

$\left(X \sim Beta(a,b) \Rightarrow f(x) = \frac{\Gamma a+b}{\Gamma a\,\Gamma b} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1 \right)$

$\therefore U \sim Beta(\alpha, \alpha)$