King Saud University
Department Of Mathematics
M-203
(Differential and Integral Calculus)
First Mid-Term Examination
(II-Semester 1430/1431)

Max. Marks: 20 Time: 90 Minutes

Q. No: 1 Use the **nth partial sum** of the series $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ to discuss its convergence or the divergence. [3]

Q. No: 2 Use the **integral test** to determine whether the following series is convergent or divergent $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. [4]

Q. No: 3 Find the **interval of convergence** and **radius** of convergence of the power series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{n^2 4^n}$. [5]

Q. No: 4 Find the **Taylor series** for the function $f(x) = \frac{1}{x^2}$ at $x = 1$. [4]

Q. No: 5 Find the **Maclaurin series** for the function $f(x) = \sin x$ and use its **first three non-zero** terms to approximate the integral $\int_0^1 \sin(x^2)dx$ to four decimal places. [4]

M-203

I Mid-term Exam. (II Sem. 1430/1431)

Q # 1(a) Use the nth partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ to discuss its convergence or divergence

[Mark: 3]

Soln. $\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$

$= \frac{A(2n+1) + B(2n-1)}{(2n-1)(2n+1)}$

If $n = \frac{1}{2}$, $A = \frac{1}{2}$

If $n = -\frac{1}{2}$, $B = -\frac{1}{2}$

$\therefore \frac{1}{4n^2-1} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$

$\therefore \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$

$= \frac{1}{2}\left[\left(1-\cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}}-\frac{1}{7}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right]$

$= \lim_{n\to\infty} \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] = \frac{1}{2}$; Conv.

(2)

Q # 2. Use the Integral test to determine whether the following Series is convergent or divergent: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

[Mark: 4]

Soln. Let $f(x) = \frac{1}{x(\ln x)^2}$; $x \geq 2$

$$f'(x) = -\frac{2(\ln x) + (\ln x)^2}{[x(\ln x)^2]^2} < 0 \Rightarrow f \downarrow$$

and f is continuous on $[2, \infty)$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{t \to \infty} \int_2^t \frac{dx}{x(\ln x)^2}$$

Put $\ln x = z \Rightarrow \frac{dx}{x} = dz$

$$\int \frac{dz}{z^2} = -\frac{1}{z}$$

$$\therefore \lim_{t \to \infty} \int_2^t \frac{dx}{x(\ln x)^2}$$

$$= \lim_{t \to \infty} \left[-\frac{1}{\ln x}\right]_{x=2}^{x=t}$$

$$= \lim_{t \to \infty} \left[-\frac{1}{\ln t} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2} \text{ ; Conv.}$$

(3)

Q # 3. Find the interval of convergence and the radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{n^2 4^n}$.

Soln. $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{(n+1)^2 4^{n+1}} \times \frac{n^2 4^n}{(-1)^n (x-3)^n} \right|$ [Mark: 5]

$= \lim_{n\to\infty} \left| \frac{(x-3)}{4} \right| < 1$

Abs. Conv $\Leftrightarrow$ $-1 < \frac{x-3}{4} < 1$

$\Leftrightarrow$ $-4 < x-3 < 4$

$\Leftrightarrow$ $-1 < x < 7$

For $x = -1$, we have $\sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{n^2 4^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which

is conv. by p-series test as $p = 2$

For $x = 7$, we have $\sum_{n=1}^{\infty} (-1)^n \frac{4^n}{n^2 4^n} = \sum (-1)^n \frac{1}{n^2}$

which is also conv by Alternating Series test.

Hence interval of conv: $[-1, 7]$

Radius $r = 4$

Q#4. Find the Taylor Series for the function
$f(x) = \frac{1}{x^2}$ at $x=1$. [Marks: 4]

Soln. Here $f(x) = \frac{1}{x^2} \Rightarrow f(1) = 1$

$f'(x) = -\frac{2}{x^3} \Rightarrow f'(1) = -2$

$f''(x) = \frac{6}{x^4} \Rightarrow f''(1) = 6$

$f'''(x) = -\frac{24}{x^5} \Rightarrow f'''(1) = -24$

$f^{(iv)}(x) = +\frac{24\times5}{x^6} \Rightarrow f^{(iv)}(1) = 120$

We substitute these values in the Taylor series:

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{(iv)}(1) + \cdots + \frac{(x-1)^n}{n!}f^{(n)}(1) + \cdots$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 + \cdots + (-1)^n(n+1)(x-1)^n + \cdots$$

(5)

Q #5. Find the Maclaurin Series for the function $f(x) = \sin x$ and use its first three non-zero terms to approximate the integral $\int_0^1 \sin(x^2)\,dx$ to four decimal places

[Mark: 4]

Soln. we have $f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{(iv)}(x) = \sin x \Rightarrow f^{(iv)}(0) = 0$

$f^{(v)}(x) = \cos x \Rightarrow f^{(v)}(0) = 1$

Substituting these values in the Maclaurin Series

$$f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

we have $f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$

Now, $\int_0^1 \sin(x^2)\,dx$

$$= \int_0^1 \left[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)} - \cdots\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \cdots$$

$$= 0.3333 - 0.02380 + 0.00075 - \cdots$$

$$= 0.33408 - 0.02380$$

$$= 0.310287 \simeq 0.31029$$