Math 111
Solutions of 1st Midterm Exam
1st Semester 1434-1435H

1. **Question I** [**Total:** 3 marks, each part one mark].

   (a) If $\int f(x)\,dx = \tan^{-1} x + c$, then $f(x) =$
   Correct choice is (b) because

   $$\begin{aligned} f(x) &= \frac{d}{dx}\left(\tan^{-1} x + c\right) \\ &= \frac{1}{1+x^2} \end{aligned}$$

   (b) $\int_6^9 g(x)\,dx + \int_3^3 g(x)\,dx - \int_{11}^9 g(x)\,dx =$
   Correct choice is (a) because

   $$\begin{aligned} & \int_6^9 g(x)\,dx + \int_3^3 g(x)\,dx - \int_{11}^9 g(x)\,dx \\ = & \int_6^9 g(x)\,dx + 0 + \int_9^{11} g(x)\,dx \\ = & \int_6^{11} g(x)\,dx \end{aligned}$$

   (c) The best substitution to solve the integral $\int x^3\sqrt{x^2-4}dx$ is
   Correct choice is (c) because

   $$\begin{aligned} u &= x^2 - 4 \\ &\Rightarrow du = 2xdx \text{ and } x^2 = u + 4 \end{aligned}$$

   from which we get

   $$\begin{aligned} \int x^3\sqrt{x^2-4}dx &= \frac{1}{2}\int x^2\sqrt{x^2-4}2xdx \\ &= \frac{1}{2}\int (u+4)\sqrt{u}du \\ &= \frac{1}{2}\int u^{\frac{3}{2}} + 4u^{\frac{1}{2}}du \end{aligned}$$

   which can be easily computed!

2. **Question II** [**Total:** 5 marks. **Part a**: 3 marks, **Part b:** 2 marks].

   (a) Prove that if $f(x)$ is a continuous function on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$.
   Solution : See proof of "The Fundamental Theorem of Mathematics, Part II" in the book "Calculus", by Smith and Minton.

   (b) For $F(x) = \tan x \int_0^{\ln|x|} \sqrt{t^2+t+1}dt$, find $F'(x)$, then prove that $F'(1) = \tan 1$.
   Solution :

   $$\begin{aligned} F'(x) &= \frac{d}{dx}(\tan x)\int_0^{\ln|x|}\sqrt{t^2+t+1}dt \\ &\quad + \tan x \frac{d}{dx}\left(\int_0^{\ln|x|}\sqrt{t^2+t+1}dt\right) \\ &= \sec^2 x \int_0^{\ln|x|}\sqrt{t^2+t+1}dt \\ &\quad + \tan x\sqrt{(\ln|x|)^2 + \ln|x| + 1}\left(\frac{1}{x}\right) \end{aligned}$$

Thus,

$$\begin{aligned} F'(1) &= \sec^2 1 \int_0^{\ln 1} \sqrt{t^2+t+1}dt \\ &\quad + \tan 1 \sqrt{(\ln 1)^2 + \ln 1 + 1} \end{aligned}$$

but $\ln 1 = 0$, therefore we get

$$F'(1) = \sec^2 1 \int_0^0 \sqrt{t^2+t+1}dt + \tan 1 = \tan 1$$

3. **Question III** [**Total**: 4.5 marks. **Part a**: 3 marks, **Part b**: 1.5 marks].

   (a) Find the area under the curve $y = x^2 + 1$ on $[0, 2]$ using the limit of Riemann sum.
   Solution:
   We have,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$
$$x_i = 0 + i\Delta x = \frac{2}{n}i, \quad i = 1, 2, ..., n.$$
$$f(x) = x^2 + 1$$

   Therefore,

$$\begin{aligned} \text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2}{n}i\right)\frac{2}{n} \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\left(\frac{2}{n}i\right)^2 + 1\right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[\sum_{i=1}^{n} \frac{4}{n^2}i^2 + \sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[\frac{4}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + n\right] \\ &= \lim_{n\to\infty} \left[\frac{8n(n+1)(2n+1)}{6n^3} + 2\right] \\ &= \frac{16}{6} + 2 \\ &= \frac{14}{3} \end{aligned}$$

   (b) Without solving the integral prove that

$$\int_1^4 \frac{x}{x+6}dx \leq \int_1^4 \frac{x}{7}dx$$

   Solution :

$$\begin{aligned} x &\geq 1 \\ &\Rightarrow x + 6 \geq 7 \\ &\Rightarrow \frac{1}{x+6} \leq \frac{1}{7} \end{aligned}$$

Since $x \geq 1 > 0$,we have

$$\frac{x}{x+6} \leq \frac{x}{7}$$

for all $x \in [1,4]$. Therefore,

$$\int_1^4 \frac{x}{x+6} dx \leq \int_1^4 \frac{x}{7} dx$$

4. **Question IV** [**Total:** 7.5 marks. **Part a:** 2.5 marks, **Part b:** 1 (2marks), 2 (1.5 marks), 3 (1.5 marks)].

   (a) Find the value of c that satisfies the conclusion of the Integral Mean Value Theorem

   $$\int_0^1 \left(2x^2 - 2\right) dx$$

   Solution: Since $f(x) = 2x^2 - 2$ is a continuous function on $[0,1]$, then $\exists c \in (0,1)$ such that

   $$f(c) = \frac{1}{1-0} \int_0^1 \left(2x^2 - 2\right) dx.$$

   That is,

   $$\begin{aligned} f(c) &= \int_0^1 \left(2x^2 - 2\right) dx \\ \Rightarrow \quad 2c^2 - 2 &= \left[2\frac{x^3}{3} - 2x \right] \Big|_0^1 \\ \Rightarrow \quad 2c^2 - 2 &= \left(\frac{2}{3} - 2 \right) - 0 \\ \Rightarrow \quad 2c^2 &= \frac{2}{3} \\ \Rightarrow \quad c^2 &= \frac{1}{3} \\ \Rightarrow \quad c &= \pm \frac{1}{\sqrt{3}} \end{aligned}$$

   but only $\frac{1}{\sqrt{3}} \in (0,1)$. Therefore, $c = \frac{1}{\sqrt{3}}$.

   (b) Evaluate the following integrals:

      1. $\int_0^3 f(x)\, dx$, where

      $$f(x) = \begin{cases} 2x^{\frac{1}{3}}, & x \geq 1 \\ \sin x, & x < 1 \end{cases} .$$

      Solution:

      $$\begin{aligned} \int_0^3 f(x)\, dx &= \int_0^1 f(x)\, dx + \int_1^3 f(x)\, dx \\ &= \int_0^1 \sin x dx + \int_1^3 2x^{\frac{1}{3}} d \\ &= -\cos x|_0^1 + 2\frac{x^{\frac{4}{3}}}{\frac{4}{3}} \Bigg|_1^3 \\ &= (-\cos 1) - (-\cos 0) + \frac{6}{4}\left(3^{\frac{4}{3}} - 1^{\frac{4}{3}}\right) \\ &= -\cos 1 + 1 + \frac{6}{4}\left(3^{\frac{4}{3}} - 1\right). \end{aligned}$$

2. $\int \left(\sec^2 x + \tan x \cos x\right) dx$
Solution:

$$\begin{aligned}
& \int \left(\sec^2 x + \tan x \cos x\right) dx \\
= & \int \sec^2 x dx + \int \frac{\sin x}{\cos x} \cos x dx \\
= & \tan x + \int \sin x dx \\
= & \tan x - \cos x + C.
\end{aligned}$$

3. $\int \frac{1}{x(1+\ln x)} dx$
Solution:

$$u = 1 + \ln x \Rightarrow du = \frac{1}{x} dx$$

Therefore,

$$\begin{aligned}
\int \frac{1}{x\left(1+\ln x\right)} dx &= \int \frac{1}{u} du \\
&= \ln |u| + C \\
&= \ln |1 + \ln x| + C.
\end{aligned}$$