-1-

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{indistinguishable}} = |f(\theta) + f(\pi - \vartheta)|^2 \xRightarrow[\vartheta = \pi/2]{} \left(\frac{d\sigma}{d\Omega}\right)_{ind} = |f(\pi/2) + f(\pi/2)|^2$$

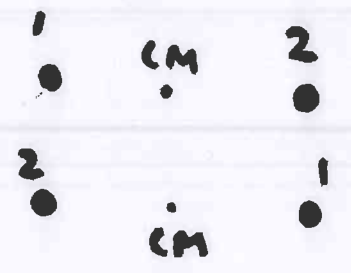
$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{ind} = |2f(\pi/2)|^2 = 4\,|f(\pi/2)|^2$$

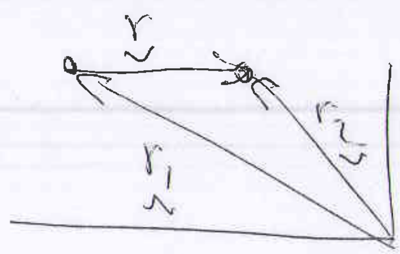
If the particles are distinguishable

$$\left(\frac{d\sigma}{d\Omega}\right)_{distin} = |f(\theta)|^2 + |f(\pi - \vartheta)|^2 = |f(\pi/2)|^2 + |f(\pi/2)|^2$$

$$= 2\,|f(\pi/2)|^2$$

When the only forces acting on two particles result from their mutual interaction, the overall motion can be separated into motion of the center of mass of the two particles and motion of the particles relative to each other. It is apparent that an interchange of two identical particles does not affect the position vector of the center of mass [which is $\frac{1}{2}(\underset{\sim}{r}_1 + \underset{\sim}{r}_2)$, since the particles have equal masses] but changes the sign of the relative position vector $\underset{\sim}{r} = (\underset{\sim}{r}_1 - \underset{\sim}{r}_2)$.





If the beams are randomly polarized then a fraction of $\frac{(s+1)\cancel{(2s+1)}}{(2s+1)^{\cancel{2}}} = (s+1)/(2s+1)$ of the collisions will be described by a eq.(3) of question (2) with (+) sign and a fraction of $\frac{s\cancel{(2s+1)}}{(2s+1)^{\cancel{2}}} = s(2s+1)$ will be described by eq(3) of question (2) with (-) sign if 2s is even. These and the similar result, ~~for~~ if 2s is odd, can be summarized:

$$\sigma(\theta) = |f(\theta)|^2 + |f(\pi-\theta)|^2 + \frac{(-1)^{2s}}{(2s+1)} 2\,\mathrm{Re}\left[f(\theta) f^*(\pi-\theta)\right]$$

where f is assummed to be independent of φ.

The asymptotic form of the scattering wave function in the center of mass system is given by

$$\psi(\underset{\sim}{r}) \xrightarrow[r \to \infty]{} e^{ikz} + \frac{1}{r} f(\theta, \varphi) e^{ikr} \quad (1)$$

Since the polar coordinates of the vector $-\underset{\sim}{r}$ are, $r, \pi-\theta, \varphi+\pi$ then the asymptotic form of the total scattering wave-function will be

$$\left(e^{ikz} \pm e^{-ikz}\right) + \frac{\left[f(\theta, \varphi) \pm f(\pi-\theta, \varphi+\pi)\right]}{r} e^{ikr} \quad (2)$$

Where $+$ is for a symmetric and $-$ for an antisymmetric wave function

Thus $\sigma(\theta, \varphi) = |f(\theta, \varphi) \pm f(\pi-\theta, \varphi+\pi)|^2$ (3)

Symmetric spin states (# $(s+1)(2s+1)$)

- if s integer $\Rightarrow$ symmetric space state
- if s (half-integer) $\Rightarrow$ anti-symmetric space state

Anstisymmetric spin states (# $s(2s+1)$)

- if $2s$ even $\Rightarrow$ anti-symmetric space state
- if $2s$ odd $\Rightarrow$ symmetric space state

or $$\sigma(\theta,\varphi) = |f(\theta,\varphi)|^2 + |f(\pi-\vartheta,\varphi+\pi)|^2 \pm 2\,\mathrm{Re}\left[f(\theta,\varphi)\, f^*(\pi-\vartheta,\varphi+\pi)\right] \quad (3)$$

Now we can take into account the spin of the particles if the interaction between the particles does not involve the spin.

Each particl has $2s+1$ spin eigenstates, Thus there are $(2s+1)^2$ independent spin functions for the pair, each of which is a product of one-particle spin functions

From these $(2s+1)^2$ we have two classes of states:

α) $(s+1)(2s+1)$ symmetric states

β) $s(2s+1)$ anti-symmetric states

4. Using the results from questions 2 and 3 find the differential cross section area in terms of the scattering amplitude in the case of the collision of two fermions with spin $s = 1/2$.

As we have shown in problem 2, when $s = 1/2$, there are $(2s+1)^2 \underbrace{=}_{s=1/2} 4$ different states, where $(s+1)(2s+1) \underbrace{=}_{s=1/2} 3$ of them are symmetric states, and 1 of them antisymmetric. These are the well-known "triplet" and "singlet" states:

$$\text{triplet: } \begin{cases} |1,1\rangle = |\uparrow\rangle|\uparrow\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right) \\ |1,-1\rangle = |\downarrow\rangle|\uparrow\rangle \end{cases}$$

$$\text{singlet: } |0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

From the discussion in question 3 we have since the spin is half-integer the symmetric spin states correspond to an anti-symmetric space state. In this case

$$d\sigma_a / d\Omega = \left| f(\theta) - f(\pi - \theta) \right|^2$$

Also since $2s = 1$ (odd) the antisymmetric spin state corresponds to a symmetric space state. In this case

$$d\sigma_s / d\Omega = \left| f(\theta) + f(\pi - \theta) \right|^2 .$$

If the beams of particles are randomly oriented the probability of being in a triplet state is 3/4 and in a singlet state 1/4 in this case:

$$d\sigma / d\Omega = \frac{3}{4}\left(d\sigma_a / d\Omega\right) + \frac{1}{4}\left(d\sigma_s / d\Omega\right) =$$

$$\left(\frac{3}{4}\right)\left| f(\theta) - f(\pi - \theta) \right|^2 + \left(\frac{1}{4}\right)\left| f(\theta) + f(\pi - \theta) \right|^2$$

Let $y(r,\vartheta) = \frac{1}{r}\left(1+\frac{i}{kr}\right)\exp(ikr)\cos\vartheta$ (1)

The angular dependence here is $\cos\vartheta$ but as we know $P_1(\cos\vartheta) = \cos\vartheta$ so the angular part has the correct form for a p-wave.

It remains ~~to~~ show what happens to the radial part.

We know that the differential equation

$$\frac{d^2\chi}{dr^2} + \left\{k^2 - \frac{\ell(\ell+1)}{r^2}\right\}\chi = 0 \quad (2)$$

We know that $y = \frac{\chi_\ell}{r}$ so $\chi_\ell(r) = ry = \left(1+\frac{i}{kr}\right)\exp(ikr)$

$$\chi'(r) = ik\left(1+\frac{i}{kr}\right)\exp(ikr) + \exp(ikr)\left(-\frac{i}{kr^2}\right) = \exp(ikr)\left[ik - \frac{1}{r} - \frac{i}{kr^2}\right] \quad (3)$$

$$\chi''(r) = ik\exp(ikr)\left[ik - \frac{1}{r} - \frac{i}{kr^2}\right] + \exp(ikr)\left[\frac{1}{r^2} + \frac{2i}{kr^3}\right] \quad (4)$$

Thus

$$ik\exp(ikr)\left[ik-\frac{1}{r}-\frac{i}{kr^2}\right]+\exp(ikr)\left[\frac{1}{r^2}+\frac{2i}{kr^3}\right]+k^2\left(1+\frac{i}{kr}\right)\exp(ikr)$$

$$-\frac{2}{r^2}\cdot\left(1+\frac{i}{kr}\right)\exp(ikr)=0 \Rightarrow$$

$$\Rightarrow \exp(ikr)\left\{\cancel{-k^2}\cancel{-\frac{ik}{r}}+\frac{1}{r^2}+\frac{1}{r^2}+\frac{2i}{kr^3}+\cancel{k^2}+\cancel{\frac{ik}{r}}-\frac{2}{r^2}-\frac{2i}{kr^3}\right\}=0$$

$\Rightarrow \quad 0=0 \quad$ and it is verified

b) Outside the range of the potential, the wave function for s and p waves has the form

$$\psi=\exp(ikz)+\underbrace{\frac{A}{r}\exp(ikr)}_{\text{s-wave}}+\frac{B}{r}\underbrace{\left(1+\frac{i}{kr}\right)\exp(ikr)\cos\theta}_{\text{p-wave}} \quad (5)$$

As $r\to\infty$ we know that

$$\psi\to\exp(ikz)+\frac{f(\theta)}{r}\exp(ikr) \quad (6)$$

Equating terms $(1/r)\exp(ikr)$ in (5) and (6) gives

$$f(\theta) = A + B\cos\vartheta$$

Since the scattering object is an impenetrable (hard) sphere, the wave funcion vanishes on the surface $r = a$. Thus

$$\exp(ika\cos\theta) + \frac{A}{a}\exp(ika) + \frac{B}{a}\left(1 + \frac{i}{ka}\right)\exp(ika)\cos\vartheta = 0$$

We expand $\exp(ika\cos\vartheta)$ in powers of ka and we have

$$\exp(ika\cos\vartheta) = 1 + ika\cos\vartheta - \frac{1}{2}\underbrace{(ka)^2\cos^2\vartheta}_{d\text{-wave}} + O(k^3a^3) + \cdots,$$

We have here a term which corresponds to the d-wave and we do not want to take into account the d-waves. So we take its average value $\overline{\cos^2\vartheta} = 1/3$, So

$$\exp(ika\cos\vartheta) = 1 + ika\cos\vartheta - \frac{1}{6}(ka)^2 + \cdots$$

Thus by inserting ( ) into ( ) we get:

$$1-\frac{1}{6}(k\alpha)^2+ik\alpha\cos\vartheta+\frac{A}{\alpha}\exp(ik\alpha)+\frac{B}{\alpha}\left(1+\frac{i}{k\alpha}\right)\exp(ik\alpha)\cos\vartheta=0$$

$$\Rightarrow\left[1-\frac{1}{6}(k\alpha)^2+\frac{A}{\alpha}\exp(ik\alpha)\right]+\cos\vartheta\left[ik\alpha+\frac{B}{\alpha}\left(1+\frac{i}{k\alpha}\right)\exp(ik\alpha)\right]=0$$

For this relation to hold we must have:

$$1-\frac{1}{6}(k\alpha)^2+\frac{A}{\alpha}\exp(ik\alpha)=0 \Rightarrow A=-\alpha\left[1-\frac{1}{6}(k\alpha)^2\right]\exp(-ik\alpha)$$

$$ik\alpha+\frac{B}{\alpha}\left(1+\frac{i}{k\alpha}\right)\exp(ik\alpha)=0 \Rightarrow B=\frac{-ik\alpha^2}{\left(1+\frac{i}{k\alpha}\right)\exp(ik\alpha)}$$

For $k\alpha \ll 1 \Rightarrow \left|\frac{i}{k\alpha}\right| \gg 1$

$$B=\frac{-ik\alpha^2}{1+\frac{i}{k\alpha}(1+ik\alpha)}=\frac{-ik\alpha^2}{\frac{i}{k\alpha}+1-1}=-k^2\alpha^3=-\alpha(k\alpha)^2$$

Thus $d\sigma/d\Omega = |f(\theta)|^2 = |A + B\cos\theta|^2 =$

$$= \left| -a\left\{1 - \tfrac{1}{6}(ka)^2\right\} e^{-ika} + a(ka)^2\cos\vartheta \right|^2$$

$$= a^2 \left| \left[1 - \tfrac{1}{6}(ka)^2\right] e^{-ika} + (ka)^2\cos\vartheta \right|^2$$

$$= a^2 \left\{ \sin^2(ka)\left(1 - \tfrac{1}{6}(ka)^2\right)^2 + \left[\cos\vartheta\,(ka)^2 + \cos ka \cdot \left(1 - \tfrac{1}{6}(ka)^2\right)\right]^2 \right\}$$

$$= \ldots = a^2\left\{1 - \tfrac{1}{3}(ka)^2 + 2(ka)^2\cos\vartheta\right\}$$

(up to terms in $(ka)^2 \ldots$)

$S = 1/2$

$$(2S+1)^2 = \left(2\cdot\frac{1}{2}+1\right)^2 = 4$$

$(S+1)(2S+1) = \left(\frac{1}{2}+1\right)\left(2\cdot\frac{1}{2}+1\right) = 2\cdot\frac{3}{2} = 3$ symmetric states

$S(2S+1) = \frac{1}{2}\left(2\cdot\frac{1}{2}+1\right) = \frac{2}{2} = 1$ anti-symmetric state

3 symmetric states $\xrightarrow{S=1/2}$ space state is anti-symmetric

1 anti-symmetric state $\xrightarrow[2S=1 \text{ odd}]{S=1/2}$ space state is symmetric

$$\sigma(\theta) = |f(\theta)|^2 + |f(\pi-\theta)|^2 + \frac{(-1)^{2S}}{(2S+1)} 2\,\mathrm{Re}\left[f(\theta)f^*(\pi-\theta)\right]$$

$$= |f(\theta)|^2 + |f(\pi-\theta)|^2 - \frac{1}{\cancel{2}}\cancel{2}\,\mathrm{Re}\left[f(\theta)f^*(\pi-\theta)\right]$$

probability to have anti-symmetric space state

$$\frac{3}{4}|f(\theta)-f(\pi-\theta)|^2+\frac{1}{4}|f(\theta)+f(\pi-\theta)|^2$$

$$=\frac{3}{4}|f(\theta)|^2+\frac{3}{4}|f(\pi-\theta)|^2-\frac{3\cdot 2}{4}\mathrm{Re}\,f^*(\theta)f(\pi-\theta)+\frac{1}{4}\cdot 2\,\mathrm{Re}\,f^*(\theta)f^*(\pi-\theta)$$

$$+\frac{1}{4}|f(\theta)|^2+\frac{3}{4}|f(\pi-\theta)|^2$$

$$=|f(\theta)|^2+|f(\pi-\theta)|^2-\frac{4}{4}\mathrm{Re}\,f^*(\theta)f(\pi-\theta)$$

$$=|f(\theta)|^2+|f(\pi-\theta)|^2-\mathrm{Re}\,f^*(\theta)f(\pi-\theta)$$

6 a) From eq. (6.8) of ~~handout~~ lecture-6 we have $\tan\delta_\ell = \dfrac{k j_\ell'(ka) - \gamma_\ell j_\ell(ka)}{k n_\ell'(ka) - \gamma_\ell n_\ell(ka)}$ (1)

where $\gamma_\ell = \dfrac{1}{R_\ell} \dfrac{dR_\ell}{dr}\Big|_{r=a}$ with $R_\ell = \chi_\ell(r)/r$. (2)

From eq.(6.3) we have $\left[\dfrac{d}{dr^2} + k^2 - V(r) - \dfrac{\ell(\ell+1)}{r^2}\right]\chi_\ell(r) = 0 \xRightarrow{V(r)=0}$

$$\Rightarrow \left[\frac{d}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right]\chi_\ell(r) = 0 \xRightarrow[\chi_\ell = rR_\ell]{} \frac{d}{dr^2}(rR_\ell) + k^2(rR_\ell) - \frac{\ell(\ell+1)}{r}R_\ell = 0$$

$$\Rightarrow r^2R'' + 2rR' + R\left[r^2k^2 - \ell(\ell+1)\right] = 0 \Rightarrow$$

$R(r) = A j_\ell(kr) + B n_\ell(kr)$ (3) j_ℓ, n_ℓ spherical Bessel functions

(3) $\to$ (2) $\quad \gamma_\ell = \dfrac{1}{A j_\ell(kr) + B n_\ell(kr)} \left[A j_\ell'(kr) + B n_\ell'(kr)\right]\Big|_{r=a}$ (4)

But we know that at $r=a$ $\chi_\ell(a)=0 \Rightarrow a\cdot R_\ell(a)=0 \Rightarrow R_\ell(a)=0$

$\Rightarrow A j_\ell(ka) + B n_\ell(ka) = 0$ (5)

This means that $\gamma_\ell \to \infty$ (from eq. 4). Thus (1) becomes

$$\tan\delta_\ell = \frac{k j_1'(ka) - \gamma_\ell j_\ell(ka)}{k n_\ell'(ka) - \gamma_\ell n_\ell(ka)} \Longrightarrow \tan\delta_\ell = \frac{-\gamma_\ell j_\ell(ka)}{-\gamma_\ell n_\ell(ka)} \Rightarrow$$

$$\Rightarrow \tan\delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)} \quad (6)$$

(b) The s-wave partial cross-section is

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\left(\frac{1}{1+\cot^2\delta_0}\right) \quad (7)$$

$j_0(x) = \frac{\sin x}{x}$

$n_0(x) = -\frac{\cos x}{x}$

But from (6) $\tan\delta_0 = \frac{j_0(ka)}{n_0(ka)} = \frac{\sin(ka)/(ka)}{-\cos(ka)/(ka)} = -\tan(ka)$

so $\cot\delta_0 = -\cot(ka) \Rightarrow \cot^2\delta_0 = \cot^2(ka)$ (8)

(8) → (7) : $\sigma_0 = \frac{4\pi}{k^2} \frac{1}{(1+\cot^2 ka)}$ (9)

We know that $\cot(x)$ becomes infinite at $x = n\pi$ and zero at $x = (2n+1)\pi/2$, so σ_0 becomes maximum as

$ka = (2n+1)\pi/2 \Rightarrow k = \frac{(2n+1)\pi}{2a}$ (10)

(c). From the behaviour of the spherical Bessel functions near zero, we obtain

$\tan\delta_l \sim \frac{(ka)^{2l+1}}{(2l+1)!!\,(2l-1)!!} = -(2l+1)\frac{(ka)^{2l+1}}{[(2l+1)!!]^2}$ (11) $(ka << 1)$

Scattering is dominated by $l=0$ for which $\tan\delta_l$ becomes maximum

$$\tan\delta_0 \sim -\frac{(2\cdot 0+1)\,(k\alpha)}{\left[(2\cdot 0+1)!!\right]^2} \approx -k\alpha \qquad (12)$$

(12) $\rightarrow$ (8), (9)

$$\sigma_0 = \frac{4\pi}{k^2}\,\frac{1}{1+\frac{1}{(ka)^2}} \approx \frac{4\pi}{k^2}(k\alpha)^2 = 4\pi a^2$$

7

We know that $\frac{d\sigma(\theta)}{d\Omega} = |f(\theta)|^2$. Also from mathematics

$$|f|^2 = (\mathrm{Re} f)^2 + (\mathrm{Im} f)^2 \geq (\mathrm{Im} f)^2 . \quad (1)$$

Thus $\frac{d\sigma(\theta)}{d\Omega} \geq (\mathrm{Im} f)^2$. But from the optical theorem

$$\sigma = \frac{4\pi}{k} \mathrm{Im} f(0) \quad (2)$$

$$\frac{d\sigma(0)}{d\Omega} = |f(0)|^2 \Rightarrow \frac{d\sigma(0)}{d\Omega} \geq (\mathrm{Im} f(0))^2 \Rightarrow \sqrt{\frac{d\sigma(0)}{d\Omega}} \geq (\mathrm{Im} f(0))$$

$$\Rightarrow \frac{4\pi}{k} \mathrm{Im}(f(0)) \leq \frac{4\pi}{k} \sqrt{\frac{d\sigma(0)}{d\Omega}} \quad (3)$$

From (2), (3)

$$\sigma \leq \frac{4\pi}{k} \sqrt{\frac{d\sigma(0)}{d\Omega}} \quad (4).$$

$\alpha_1 + \alpha_2$

$(\alpha_1+\alpha_2)^2 = \alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2 + \alpha_2\alpha_1$

b) Now for a central potential the scattering amplitude is

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta)$$

$$\frac{d\sigma(\theta)}{d\Omega} = |f(\theta)|^2 = \frac{1}{k^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} (2l+1)(2l'+1) e^{i\delta_l - i\delta_{l'}} \sin\delta_l \sin\delta_{l'} P_l(\cos\theta) P_{l'}(\cos\theta)$$

using the fact $P_l(\cos(0)) = P_l(1) = 1$

$$\frac{d\sigma(0)}{d\Omega} = \frac{1}{k^2} \left| \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l \right|^2 = \frac{1}{k^2} \left| \sum_{l=0}^{\infty} (2l+1)(\sin\delta_l \cos\delta_l + i\sin^2\delta_l) \right|^2$$

$$= \frac{1}{k^2} \left[\sum_{l=0}^{\infty} (2l+1) \sin\delta_l \cos\delta_l \right]^2 + \frac{1}{k^2} \left[\sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l \right]^2$$

$$\Rightarrow \frac{d\sigma(0)}{d\Omega} \geq \frac{1}{k^2} \left[\sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l \right]^2$$

5

The total cross-section is given by $\sigma = \frac{4\pi^2}{k^2} \sum_{l=0}^{\infty} (2l+1)\sin^2\delta_l$ (6)

From (5) and (6) $\frac{d\sigma(0)}{d\Omega} \geq \frac{k^2\sigma^2}{16\pi^2} \Rightarrow \sigma^2 \leq \frac{16\pi^2}{k^2}\frac{d\sigma(0)}{d\Omega}$

$\Rightarrow \sigma \leq \frac{4\pi}{k}\sqrt{\frac{d\sigma(0)}{d\Omega}}$

9

a) The wavelength of the neutrons is $\lambda = h/mv$.

So if $v = \frac{h}{m\lambda} << \frac{h}{m\alpha}$ then $\lambda >> \alpha$, i.e the wavelength of the particles is large compared to the range of the potential. For this condition only the s-partial wave is scattered, which means that the scattering is spherically symmetric.

b) We know that for $l=0$ the function $\chi_0(r)$ satisfies the equation

$$\left[\frac{d^2}{dr^2} + k^2 - W\right]\chi_0(r) = 0 \qquad (r \leq a)$$

$$\left[\frac{d^2}{dr^2} + k^2\right]\chi_0(r) = 0 \qquad (r > a)$$

If we call $k^2 - W = j^2$ we get

$$\frac{d^2\chi_I}{dt^2} + j^2\chi_I = 0 \quad (r<a), \qquad \frac{d^2\chi_{II}}{dr^2} + k^2\chi_{II}(r) = 0 \quad (r>a)$$

For $(r<a)$ $\chi_I(r) = A\sin(jr) + B\cos(jr)$ But at $r \to 0$ $\chi_I(0) = 0$

so $B = 0$, thus $\chi_I(r) = A\sin(jr)$ $(r<a)$

For $r>a$ if there was no potential the solution would be just $\chi_{II}(r) = C\sin(kr)$ now the presence of the potential imposes just a phase shift δ_0 : thus : $\chi_{II}(r) = C\sin(kr+\delta_0)$ $(r>\alpha)$

But at $x=\alpha$ $\chi(r)$ and $\chi'(r)$ must be continuous so

$$\left.\begin{aligned} \chi_I(\alpha) &= \chi_{II}(\alpha) \\ \chi_I'(\alpha) &= \chi_{II}'(\alpha) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} A\sin(j\alpha) &= C\sin(k\alpha+\delta_0) \\ jA\cos(j\alpha) &= kC\cos(k\alpha+\delta_0) \end{aligned}$$

Dividing we get

$$j\tan(k\alpha+\delta) = k\tan j\alpha$$