PHYS 454
1st Midterm Exam – Fall 2016
Wednesday 7th December 2016

Instructor: Dr. V. Lempesis

Student Name: ……………………………………………….

Student ID Number……………………………..

Student Grade: ……./20

Please answer all questions

1. The state of a particle is described at a given moment from the wave-function

$$|\psi\rangle = N\left(|\psi_1\rangle + (1+i)|\psi_2\rangle\right)$$

where ψ_1 and ψ_2 are eigenfuntions of the Hamiltonian with eigenvalues E_1 and E_2 respectively (which are considered as known). What is the average energy of the particle?

(5 marks)

Solution:
The first step is to find the normalization constant *N*:

$$\langle\psi|\psi\rangle = 1 \Rightarrow N^*\left(\langle\psi_1| + (1-i)\langle\psi_2|\right)N\left(|\psi_1\rangle + (1+i)|\psi_2\rangle\right) = 1 \Rightarrow$$

$$|N|^2\langle\psi_1|\psi_1\rangle + |N|^2(1+i)\langle\psi_1|\psi_2\rangle + |N|^2(1-i)\langle\psi_2|\psi_1\rangle + |N|^2(1+i)(1-i)\langle\psi_2|\psi_2\rangle = 1 \Rightarrow$$

$$|N|^2 + 2|N|^2 = 1 \Rightarrow 3|N|^2 = 1 \Rightarrow |N|^2 = 1/3 \Rightarrow$$

$$|N| = 1/\sqrt{3}$$

Then the average value of the energy is given by:

$$\bar{E} = \langle\psi|\hat{H}|\psi\rangle = \frac{1}{\sqrt{3}}\left(\langle\psi_1| + (1-i)\langle\psi_2|\right)\hat{H}\frac{1}{\sqrt{3}}\left(|\psi_1\rangle + (1+i)|\psi_2\rangle\right) =$$

$$\frac{1}{3}\left\{\langle\psi_1|\hat{H}|\psi_1\rangle + (1+i)\langle\psi_1|\hat{H}|\psi_2\rangle + (1-i)\langle\psi_2|\hat{H}|\psi_1\rangle + (1+i)(1-i)\langle\psi_2|\hat{H}|\psi_2\rangle\right\} =$$

$$\frac{1}{3}\left\{E_1 + (1+i)E_2\langle\psi_1|\psi_2\rangle + (1-i)E_1\langle\psi_2|\psi_1\rangle + (1+i)(1-i)E_2\right\} =$$

$$\frac{1}{3}\left\{E_1 + (1+i)(1-i)E_2\right\} = \frac{1}{3}\left\{E_1 + 2E_2\right\}$$

2. Consider the one dimensional normalized wavefuncions $\psi_0(x)$ and $\psi_1(x)$ with the following properties:

$$\psi_0(x) = \psi_0^*(x) = \psi_0(-x), \quad \psi_1(x) = N\frac{d}{dx}\psi_0 .$$

a) Show that $\psi_0(x)$ and $\psi_1(x)$ are orthogonal.

b) Find the average values of position x and momentum p at the states $\psi_0(x)$ and $\psi_1(x)$.

Assume N is real.

(5 marks)

Solution:

a)

$$\int_{-\infty}^{+\infty}\psi_0^*(x)\psi_1(x) = \int_{-\infty}^{+\infty}\psi_0(x)N\frac{d}{dx}\psi_0(x)dx = N\int_{-\infty}^{+\infty}\psi_0(x)d\psi_0(x) =$$

$$\frac{N}{2}\int_{-\infty}^{+\infty}d\psi_0^2(x) = \frac{N}{2}\left\{\psi_0^2(x)\Big|_{-\infty}^{+\infty}\right\} = \frac{N}{2}\left\{\psi_0^2(\infty)-\psi_0^2(-\infty)\right\} = \frac{N}{2}\{0-0\} = 0$$

Thus the two wave-functions are orthogonal

b)

i) $\bar{x} = \int_{-\infty}^{+\infty}\psi_0^*(x)x\psi_0(x) = \int_{-\infty}^{+\infty}\psi_0(x)x\psi_0(x)dx = \int_{-\infty}^{+\infty}x\psi_0^2(x)dx$.

But the integrand function is an odd function of x thus the integral will be zero. So $\bar{x} = 0$.

ii) $\bar{p} = 0$, because the wave-function is a real one.

iii)

$$\bar{x} = \int_{-\infty}^{+\infty}\psi_1^*(x)x\psi_1(x) = |N|^2\int_{-\infty}^{+\infty}\left(\frac{d}{dx}\psi_0^*(x)\right)\left(\frac{d}{dx}\psi_0(x)\right)dx = |N|^2\int_{-\infty}^{+\infty}\left(\frac{d}{dx}\psi_0(x)\right)\left(\frac{d}{dx}\psi_0(x)\right)dx =$$

$$|N|^2\left\{\psi_0(x)\frac{d}{dx}\psi_0(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\psi_0(x)\left(\frac{d}{dx}\psi_0(x)\right)dx\right\} = |N|^2\left\{\psi_0(x)\frac{d}{dx}\psi_0(x)\Big|_{-\infty}^{+\infty} - \frac{1}{2}\int_{-\infty}^{+\infty}d\psi_0^2(x)\right\} =$$

$$|N|^2\left\{\psi_0(x)\frac{d}{dx}\psi_0(x)\Big|_{-\infty}^{+\infty} - \frac{1}{2}\int_{-\infty}^{+\infty}d\psi_0^2(x)\right\} =$$

$$|N|^2\left\{\psi_0(+\infty)\frac{d}{dx}\psi_0(+\infty) - \psi_0(+\infty)\frac{d}{dx}\psi_0(+\infty) - \frac{1}{2}\left[\psi_0^2(+\infty)-\psi_0^2(-\infty)\right]\right\} = 0$$

iv) $\bar{p} = 0$, because the wave-function is a real one.

3. A particle is at the eigenstate $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$. Calculate the following quantities: a) $\langle x\rangle$, b) $\langle x^2\rangle$, c) $\langle p\rangle$, d) $\langle p^2\rangle$ and e) $\Delta x\cdot\Delta p$.

(10 marks)

Solution:

The wavefunction of the body is

$$\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

a) The average position is given by

$$\langle x\rangle = \int_0^L x\left|\psi_2(x)\right|^2 dx = \frac{2}{L}\int_0^a x\sin^2\left(\frac{2\pi x}{L}\right)dx = \frac{L}{2}.$$

b)

$$\langle x^2\rangle = \int_0^L x^2\left|\psi_2(x)\right|^2 dx = \frac{2}{L}\int_0^L x^2\sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{1}{24}L^2\left(\frac{8\pi^2-3}{\pi^2}\right) = 0.32L^2$$

c) The average momentum is zero because the wavefunction is real.

d)

$$\langle \mathrm{p}^2\rangle = \int_0^L \psi_2^*(x)p^2\psi_2(x)\,dx = (-i\hbar)^2\int_0^L \psi_2^*(x)\frac{d^2}{dx^2}\psi_2(x)\,dx =$$

$$\frac{2\hbar^2}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)\left(\frac{2\pi}{L}\right)^2\sin\left(\frac{2\pi x}{L}\right)dx = \frac{2\hbar^2}{L}\left(\frac{2\pi}{L}\right)^2\int_0^L \sin^2\left(\frac{2\pi x}{L}\right)dx =$$

$$\frac{2\hbar^2}{L}\left(\frac{2\pi}{L}\right)^2\left\{\frac{x}{2}-\frac{L}{8\pi}\sin\left(\frac{4\pi x}{L}\right)\right\}_0^L = \frac{2\hbar^2}{L}\left(\frac{2\pi}{L}\right)^2\frac{L}{2} = \frac{4\pi^2\hbar^2}{L^2}$$

e) $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{0.32L^2-(0.5L)^2} = 0.26L$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{4\pi^2\hbar^2}{L^2}-0} = \frac{2\pi\hbar}{L}.$$

Thus

$$\Delta x\cdot\Delta p = 0.26L\frac{2\pi\hbar}{L} = 6.28\cdot 0.26\hbar = 1.63\hbar$$

Physical constants and formulas

$h = 6.63 \times 10^{-34} J \cdot s$, $\hbar = h/2\pi = 1.055 \times 10^{-34} J \cdot s$, $1\overset{o}{A} = 10^{-10} m$, $m_e = 9.1 \times 10^{-31}$ kg, $1eV = 1.6 \times 10^{-19}$ J

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

For an infinite square well extending from 0 to a:

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2, \qquad n = 1,\ 2,\ \ldots, \infty$$

Useful mathematics:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\int \sin^2(kx)\,dx = \frac{x}{2} - \frac{1}{4k}\sin(2kx)$$

$$\int x\sin^2(kx)\,dx = \frac{x}{2k}\left[kx - \cos(kx)\sin(kx)\right] + \frac{\sin^2(kx)}{4k^2} - \frac{x^2}{4}$$

$$\int x^2\sin^2(kx)\,dx = \frac{1}{k^3}\left\{k^2x^2\left[\frac{1}{2}kx - \frac{1}{2}\cos(kx)\sin(kx)\right] - \frac{1}{2}kx\cos^2(kx) + \frac{1}{4}\cos(kx)\sin(kx) + \frac{1}{4}kx - \frac{1}{3}k^3x^3\right\}$$

$$\int \sin(kx)\sin(2kx)\,dx = \frac{1}{6k}\left(3\sin(kx) - \sin(3kx)\right)$$

$$\int x\sin(kx)\sin(2kx)\,dx = \frac{1}{2k^2}\left[\cos(kx) + kx\sin(kx)\right] - \frac{1}{18k^2}\left[\cos(3kx) + 3kx\sin(3kx)\right]$$

$$\left\langle A^n \right\rangle = \int_{-\infty}^{\infty} \psi^*(x)\left(A^n\psi(x)\right)dx$$

$$\Delta A = \sqrt{\left\langle A^2 \right\rangle - \left\langle A \right\rangle^2}$$