PHYS 404
2nd FINAL Exam
Saturday 26th November 2015

Instructor: Dr. V. Lempesis

Student Name: ………………………………………………….

Student ID Number……………………………..

Student Grade: ……./40

Please answer <u>only</u> 4 of the following questions

1. Show that the Legendre polynomials satisfy the relation: $P_n(1)=1$. Hint: Use the relation for the generating function of the Legendre polynomials: $\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n$. You will also need the relation: $\frac{1}{1-t}=\sum_{n=0}^{\infty}t^n$.

(10 marks)

Solution:

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \underset{x=1}{\Rightarrow} \frac{1}{\sqrt{1-2t+t^2}}=\sum_{n=0}^{\infty}P_n(1)t^n \Rightarrow$$

$$\frac{1}{\sqrt{(1-t)^2}}=\sum_{n=0}^{\infty}P_n(1)t^n \Rightarrow \frac{1}{1-t}=\sum_{n=0}^{\infty}P_n(1)t^n \qquad (1)$$

But

$$\frac{1}{1-t}=\sum_{n=0}^{\infty}t^n \qquad (2)$$

From (1) and (2) we see that:

$$\sum_{n=0}^{\infty}t^n=\sum_{n=0}^{\infty}P_n(1)t^n \Rightarrow \sum_{n=0}^{\infty}t^n-\sum_{n=0}^{\infty}P_n(1)t^n=0 \Rightarrow$$

$$\sum_{n=0}^{\infty}\left(1-P_n(1)\right)t^n=0 \Rightarrow \left(1-P_n(1)\right)=0 \Rightarrow P_n(1)=1$$

2. Show that: $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

Hint: You are given that,

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \quad J_{n-1}(x) - J_{n+1}(x) = 2J_n^{'}(x)$$

(10 marks)

Solution:

$$\left.\begin{array}{r} J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} J_n(x) \\ J_{n-1}(x) - J_{n+1}(x) = 2J_n^{'}(x) \end{array}\right\} \underset{+}{\Rightarrow} J_{n-1}(x) + J_{n+1}(x) + J_{n-1}(x) - J_{n+1}(x) = \frac{2n}{x} J_n(x) + 2J_n^{'}(x) \Rightarrow$$

$$2J_{n-1}(x) = \frac{2n}{x} J_n(x) + 2J_n^{'}(x) \Rightarrow J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n^{'}(x) \Rightarrow$$

$$x^n J_{n-1}(x) = \frac{n}{x} x^n J_n(x) + x^n J_n^{'}(x) \Rightarrow x^n J_{n-1}(x) = n x^{n-1} J_n(x) + x^n J_n^{'}(x) \Rightarrow$$

$$x^n J_{n-1}(x) = \left(x^n\right)^{'} J_n(x) + x^n J_n^{'}(x) \Rightarrow x^n J_{n-1}(x) = \frac{d}{dx}\left[x^n J_n(x)\right]$$

3. a) If $f(x) = \sum_{n=0}^{\infty} A_n H_n(x)$ show that $A_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} f(x) H_n(x)\, dx$. You are given that: $\int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0 & m \neq n \\ 2^n n! \sqrt{\pi} & m = n \end{cases}$

(5 marks)

Solution:

a)

$$f(x) = \sum_{n=0}^{\infty} A_n H_n(x) \Rightarrow f(x) H_m(x) e^{-x^2/2} = \sum_{n=0}^{\infty} A_n H_n(x) H_m(x) e^{-x^2/2} \Rightarrow$$

$$\int_{x=-\infty}^{x=\infty} f(x) H_m(x) e^{-x^2/2}\, dx = \sum_{n=0}^{\infty} A_n \int_{x=-\infty}^{x=\infty} H_n(x) H_m(x) e^{-x^2/2}\, dx \Rightarrow$$

$$\int_{x=-\infty}^{x=\infty} f(x) H_n(x) e^{-x^2/2}\, dx \underset{m=n}{=} A_n \int_{x=-\infty}^{x=\infty} \left[H_n(x)\right]^2 e^{-x^2/2}\, dx \Rightarrow$$

$$\int_{x=-\infty}^{x=\infty} f(x) H_n(x) e^{-x^2/2}\, dx \underset{m=n}{=} A_n \sqrt{\pi} 2^n n! \Rightarrow$$

$$A_n = \frac{1}{\sqrt{\pi} 2^n n!} \int_{x=-\infty}^{x=\infty} f(x) H_n(x) e^{-x^2/2}\, dx$$

b) Use the relation $H_n^{'}(x) = 2nH_{n-1}\left(x\right)$ to find the integral $\int_0^2 H_2(x)dx$ if you are given that $H_3 = 8x^3 - 12x$.

(5 marks)

Solution:

$$H_n^{'}(x) = 2nH_{n-1}\left(x\right) \underset{n=3}{\Rightarrow} \frac{d}{dx}H_3\left(x\right) = 6H_2\left(x\right) \Rightarrow$$

$$H_2\left(x\right) = \frac{1}{6}\frac{d}{dx}H_3\left(x\right)$$

$$\int_0^2 H_2(x)dx = \frac{1}{6}\int_0^2 \frac{d}{dx}H_3\left(x\right)dx = \frac{1}{6}\int_0^2 dH_3\left(x\right) = \frac{1}{6}H_3\left(x\right)\Big|_0^2$$

$$= \frac{1}{6}\left(8x^3 - 12x\right)\Big|_0^2 = \frac{1}{6}\left(8\cdot 2^3 - 12\cdot 2\right)\Big| = \frac{1}{6}\left(64 - 24\right) = \frac{40}{6} = \frac{20}{3}$$

4. a) Find the associated Laguerre polynomial $L_1^1(x)$. You are given that $L_n^k(x) = \left(-1\right)^k\left(\frac{d}{dx}\right)^k L_{n+k}(x)$ and that $L_2 = \frac{1}{2}\left(x^2 - 4x + 2\right)$.

(5 marks)

Solution:

$$L_n^k(x) = \left(-1\right)^k\left(\frac{d}{dx}\right)^k L_{n+k}(x) \underset{n=1,k=1}{\Rightarrow} L_1^1(x) = \left(-1\right)^1\left(\frac{d}{dx}\right)^1 L_2(x) \Rightarrow$$

$$L_1^1(x) = -\frac{d}{dx}L_2(x) \Rightarrow L_1^1(x) = -\frac{d}{dx}\left[\frac{1}{2}\left(x^2 - 4x + 2\right)\right] \Rightarrow$$

$$L_1^1(x) = -\frac{1}{2}\left(2x - 4\right) = 2 - x$$

b) Show that the associated Laguerre polynomial $L_1^1(x)$, satisfies the differential equation:

$$xL_n^{k''}\left(x\right) + \left(k + 1 - x\right)L_n^{k'}\left(x\right) + nL_n^k\left(x\right) = 0.$$

(5 marks)

Solution:

$$xL_n^{k''}(x)+(k+1-x)L_n^{k'}(x)+nL_n^{k}(x)\underset{n=1,k=1}{=}$$

$$xL_1^{1''}(x)+(2-x)L_1^{1'}(x)+L_1^{1}(x)=$$

$$x\frac{d^2}{dx^2}L_1^1(x)+(2-x)\frac{d}{dx}L_1^1(x)+L_1^1(x)=$$

$$x\frac{d^2}{dx^2}L_1^1(x)+(2-x)\frac{d}{dx}L_1^1(x)+L_1^1(x)=$$

$$x\frac{d^2}{dx^2}(2-x)+(2-x)\frac{d}{dx}(2-x)+(2-x)=$$

$$0-(2-x)+(2-x)=0$$

5. a) Find the associated Legendre functions $P_2^1(x)$ and $P_3^1(x)$ starting from the Legendre polynomials $P_2(x)$ and $P_3(x)$.

(5 marks)

Solution:

$$P_n^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)\underset{n=2,m=1}{\Rightarrow}P_2^1(x)=(1-x^2)^{1/2}\frac{d}{dx}P_2(x)\Rightarrow$$

$$P_2^1(x)=(1-x^2)^{1/2}\frac{d}{dx}\frac{1}{2}(3x^2-1)\Rightarrow P_2^1(x)=\frac{1}{2}(1-x^2)^{1/2}6x\Rightarrow$$

$$P_2^1(x)=3x(1-x^2)^{1/2}$$

$$P_n^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)\underset{n=3,m=1}{\Rightarrow}P_3^1(x)=(1-x^2)^{1/2}\frac{d}{dx}P_3(x)\Rightarrow$$

$$P_3^1(x)=(1-x^2)^{1/2}\frac{d}{dx}\frac{1}{2}(5x^3-3x)\Rightarrow P_3^1(x)=\frac{1}{2}(1-x^2)^{1/2}(15x^2-3)$$

b) Show that the functions $P_2^1(x)$ and $P_3^1(x)$ are orthogonal i.e. $\int_{-1}^{1}P_2^1(x)P_3^1(x)dx=0$.

(5 marks)

Solution:

$$\int_{-1}^{1}P_2^1(x)P_3^1(x)\,dx=\frac{3}{2}\int_{-1}^{1}x(1-x^2)^{1/2}(1-x^2)^{1/2}(15x^2-3)\,dx=$$

$$\frac{3}{2}\int_{-1}^{1}x(1-x^2)(15x^2-3)\,dx=\frac{3}{2}\int_{-1}^{1}(x-x^3)(15x^2-3)\,dx=$$

$$\frac{3}{2}\int_{-1}^{1}(-15x^5+18x^3-3x)\,dx=0$$

You are given that: $P_n^m(x)=(1-x^2)^{m/2}\dfrac{d^m}{dx^m}P_n(x)$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) \quad \text{and} \quad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$