HANDOUT-4

Show in the expression for C-G coefficients that $m = m_1 + m_2$

$$\langle j_1 j_2 m_1 m_2 | J_z | j_1 j_2 j m\rangle = \hbar m \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

Similarly $\quad \| \quad$ $m = m_1 + m_2$

$$\langle j_1 j_2 m_1 m_2 | \underbrace{J_{1z} + J_{2z}}_{\text{acting to the lef}} | j_1 j_2 , j m\rangle = \hbar (m_1 + m_2) \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

2.

Show the orthonormality relation: $\sum_{j,m} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m\rangle \cdot$
$\langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle = \delta_{m_1' m_1} \delta_{m_2' m_2}$

We know that $|j_1, j_2 m_1 m_2\rangle = \sum_{\substack{j \\ m = m_1 + m_2}} |j_1 j_2 j m\rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle$

Multiplying by $\langle j_1 j_2 m_1' m_2' |$ we get

$$\underbrace{\langle j_1 j_2 m_1' m_2' | j_1 j_2 m_1 m_2\rangle}_{\delta_{m_1' m_1} \delta_{m_2' m_2}} = \sum_{\substack{j \\ m = m_1 + m_2}} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m\rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle$$

$$\delta_{m_1' m_1} \delta_{m_2' m_2} = \sum_{\substack{j \\ m = m_1 + m_2}} \cdots\cdots$$

) Show that a Clebsch-Gordon coefficient is non-zero if $j_1' = j_1$ and $j_2' = j_2$

α) We know that $J_1^2 |j_1 j_2 j m\rangle = \hbar^2 j_1(j_1+1) |j_1 j_2 j m\rangle$

Thus $\langle j_1' j_2' m_1 m_2 | J_1^2 | j_1 j_2 j m\rangle = \hbar^2 j_1 (j_1+1) \langle j_1' j_2' m_1 m_2 | j_1 j_2 j m\rangle$ ①

b) $\langle j_1' j_2' m_1 m_2 | J_1^2 = \hbar^2 j_1'(j_1'+1) \langle j_1' j_2' m_1 m_2 |$

$\Rightarrow \langle j_1' j_2' m_1 m_2 | J_1^2 | j_1 j_2 j m\rangle = \hbar^2 j_1'(j_1'+1) \langle j_1' j_2' m_1 m_2 | j_1 j_2 j m\rangle$ ②

① = ② it must be $j_1' = j_1$

④ Show the orthonormality relation $\sum_{m_1 m_2} \langle j_1 j_2 j m | j_1 j_2 m_1 m_2 \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j' m' \rangle = \delta_{jj'} \delta_{mm'}$

We know that $|j_1 j_2 j m\rangle = \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$

$$\Rightarrow \langle j_1 j_2 j m | = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \langle j_1 j_2 m_1 m_2 |$$

Multiplying by $|j_1 j_2 j' m'\rangle$ we get

$$\underbrace{\langle j_1 j_2 j m | j_1 j_2 j' m' \rangle}_{\delta_{jj'} \delta_{mm'}} = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j' m'\rangle$$

If $j = j'$ $\quad m = m'$

$$\sum_{m_1 + m_2 = m} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle = 1$$

$$\Rightarrow \sum_{m_1 + m_2 = m} |\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle|^2 = 1$$

5 Construct the eigenstates of total angular momentum of a hydrogen atom at the excited state 2p

α) In this state $\ell = 1$ and $s = 1/2$. In this case j could be

$|\ell - s| \ldots |\ell + s|$ either $j = 3/2$ and $j = 1/2$.

In the case where $j = 3/2$ and $m_j = 3/2$ could be written as

$$|3/2, 3/2\rangle = \underbrace{|1, 1\rangle}_{|\ell, m_\ell\rangle} \underbrace{|+\rangle}_{\text{spin up}}$$

To obtain $|3/2, 1/2\rangle$ you just apply the lowering operator $j_- = \ell_- + s_-$.

$$j_- |3/2, 3/2\rangle = (\ell_- + s_-)|1, 1\rangle|+\rangle = (\ell_-|1,1\rangle)|+\rangle + |1,1\rangle(s_-|+\rangle)$$

$$\Rightarrow \sqrt{j(j+1) - m_j(m_j - 1)}\Big|_{\substack{j=3/2 \\ m_j = 3/2}} \hbar\, |3/2, 1/2\rangle = \sqrt{\ell(\ell+1) - m_\ell(m_\ell - 1)}\Big|_{\substack{\ell=1 \\ m_\ell = 1}} \hbar\, |1, 0\rangle|+\rangle$$

$$+ |1, 1\rangle|-\rangle\hbar \Rightarrow \sqrt{3}\, |3/2, 1/2\rangle = \sqrt{2}\, |1, 0\rangle|+\rangle + |1, 1\rangle|-\rangle$$

$$\Rightarrow |3/2, 1/2\rangle = \sqrt{\frac{2}{3}}\, |1, 0\rangle|+\rangle + \frac{1}{\sqrt{3}}\, |1, 1\rangle|-\rangle$$

b) $J_- |3/2, 1/2\rangle = (l_- + S_-)\left(\sqrt{\frac{2}{3}}\,|1,0\rangle|+\rangle + \frac{1}{\sqrt{3}}\,|1,1\rangle|-\rangle\right)$

$$= \sqrt{\frac{2}{3}}\left(l_-|1,0\rangle\right)|+\rangle + \sqrt{\frac{2}{3}}\,|1,0\rangle\left(S_-|+\rangle\right) + \frac{1}{\sqrt{3}}\left(l_-|1,1\rangle\right)|-\rangle + \frac{1}{\sqrt{3}}|1,1\rangle\underbrace{\left(S_-|-\rangle\right)}_{0}$$

$$\Rightarrow \sqrt{j(j+1)-m_j(m_j-1)}\,\Big|_{\substack{j=3/2\\ m_j=1/2}} |3/2,-1/2\rangle = \sqrt{\frac{2}{3}}\sqrt{l(l+1)-m_l(m_l-1)}\,\Big|_{\substack{l=1\\ m_l=0}} |1,-1\rangle|+\rangle$$

$$+ \sqrt{\frac{2}{3}}\,|1,0\rangle|-\rangle + \frac{1}{\sqrt{3}}\sqrt{l(l+1)-m_l(m_l-1)}\,\Big|_{\substack{l=1\\ m_l=1}} |1,0\rangle|-\rangle$$

$$\Rightarrow 2|3/2,-1/2\rangle = \sqrt{\frac{2}{3}}\sqrt{2}\,|1,-1\rangle|+\rangle + \sqrt{\frac{2}{3}}\,|1,0\rangle|-\rangle + \frac{1}{\sqrt{3}}\sqrt{2}\,|1,0\rangle|-\rangle$$

$$\Rightarrow 2|3/2,-1/2\rangle = \frac{2}{\sqrt{3}}\,|1,-1\rangle|+\rangle + 2\sqrt{\frac{2}{3}}\,|1,0\rangle|-\rangle$$

$$\Rightarrow |3/2,-1/2\rangle = \frac{1}{\sqrt{3}}\,|1,-1\rangle|+\rangle + \sqrt{\frac{2}{3}}\,|1,0\rangle|-\rangle$$

Similarly we can get $|3/2,-3/2\rangle = |1,-1\rangle|-\rangle$

c) For the state $|1/2, 1/2\rangle$ we notice that we can get it only if $(m_\ell = 1, m_s = -1/2)$ and $(m_\ell = 0, m_s = 1/2)$.

Thus you can write $|1/2, 1/2\rangle = \alpha\, |1,0\rangle |+\rangle + \beta\, |1,1\rangle \langle -|$

But $\langle 3/2, 3/2 | 1/2, 1/2\rangle = 0 \Rightarrow \alpha \sqrt{\frac{2}{3}} + \beta \frac{1}{\sqrt{3}} = 0$ (1)

And also $\alpha^2 + \beta^2 = 1$ (2). From (1) and (2) we get

$\alpha = \frac{1}{\sqrt{3}} \quad \beta = -\sqrt{\frac{2}{3}}$

Thus $|1/2, 1/2\rangle = \frac{1}{\sqrt{3}} |1,0\rangle |+\rangle - \sqrt{\frac{2}{3}} |1,1\rangle \langle -|$ (3)

With the action of J_- in (3) we get

$|1/2, -1/2\rangle = \frac{1}{\sqrt{3}} |1,0\rangle |-\rangle - \sqrt{\frac{2}{3}} |1,-1\rangle |+\rangle$

(6). Find the C-G coefficients for 2 p electrons. (no spin taken into account)

Here we find that the electrons have $j_1 = 1$, $j_2 = 1$, $m_1 = -1, 0, 1$, $m_2 = -1, 0, 1$. The total angular momentum is $J = 0, 1, 2$ $(|J_1 - J_2| \cdots |J_1 + J_2|)$ and for the total projection $M = 0, \pm 1, \pm 2$.

(1.) We find the state with maximum M : $M_{max} = 2$

$$|j_1 = 1, j_2 = 1, J = 2, M = 2\rangle \quad (1)$$

- We apply the lowering operator L_- on this state

$$L_- |j_1 = 1, j_2 = 1, J = 2, M = 2\rangle = \sqrt{2(2+1) - 2(2-1)} = 2\,|j_1 = 1, j_2 = 1, J = 2, M = 1\rangle \quad (2)$$

But the state (1) is also given by

$$|j_1 = 1, j_2 = 1, J = 2, M = 2\rangle = |j_1 = 1, m_1 = 1\rangle\,|j_2 = 1, m_2 = 1\rangle \quad (3)$$

- We apply the lowering operator $L_- = L_-^1 + L_-^2$ on this state (3)

$$\left(L_-^1 + L_-^2\right)|j_1=1, m_1=1\rangle|j_2=1, m_2=1\rangle = L_-^1 |j_1=1, m_1=1\rangle|j_2=1, m_2=1\rangle$$

$$+ L_-^2 |j_1=1, m_1=1\rangle|j_2=1, m_2=1\rangle = \sqrt{1(1+1)-1(1-1)}\,|j_1=1, m_1=0\rangle|j_2=1, m_2=1\rangle$$

$$+ \sqrt{1(1+1)-1(1-1)}\,|j_1=1, m_1=1\rangle|j_2=1, m_2=0\rangle = \sqrt{2}\,|j_1=1, m_1=0\rangle|j_2=1, m_1=1\rangle +$$

$$+ \sqrt{2}\,|j_1=1, m_1=1\rangle|j_2=1, m_2=0\rangle \quad (4)$$

Comparing (2) and (4) we get

$$\boxed{|j_1=1, j_2=1, J=2, M=1\rangle = \frac{1}{\sqrt{2}}\left\{|j_1=1, m_1=0\rangle|j_2=1, m_2=1\rangle + |j_1=1, m_1=1\rangle|j_2=1, m_2=0\rangle\right\}}$$

(5)

Now we go on by applying the lowering operator

$$L_-|j_1=1, j_2=1, J=2, M=1\rangle = \sqrt{2(2+1)-1(1-1)}\,|j_1=1, j_2=1, J=2, M=0\rangle =$$

$$= \sqrt{6}\,|j_1=1, j_2=1, J=2, M=0\rangle \quad (6)$$

But if we take $|j_1=1, j_2=1, J=2, M=1\rangle$ from (5)

$$\left(L_-^1 + L_-^2\right)\frac{1}{\sqrt{2}}\left\{|j_1=1, m_1=0\rangle|j_2=1, m_2=1\rangle + |j_1=1, m_1=1\rangle|j_2=1, m_2=0\rangle\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{L_-^1|j_1=1, m_1=0\rangle|j_2=1, m_2=1\rangle + L_-^1|j_1=1, m_1=1\rangle|j_2=1, m_2=0\rangle\right.$$

$$\left. + L_-^2|j_1=1, m_1=0\rangle|j_2=1, m_2=1\rangle + L_-^2|j_1=1, m_1=1\rangle|j_2=1, m_2=0\rangle\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\sqrt{1(1+1)-0}\,|j_1=1, m_1=-1\rangle|j_2=1, m_2=1\rangle + \sqrt{1(1+1)-1(1-1)}\,|j_1=1, m_1=0\rangle|j_2=1, m_2=0\rangle\right.$$

$$\left. + \sqrt{1(1+1)-1(1-1)}\,|j_1=1, m_1=0\rangle|j_2=1, m_2=0\rangle + \sqrt{1(1+1)-0}\,|j_1=1, m_1=1\rangle|j_2=1, m_2=-1\rangle\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\sqrt{2}\,|j_1=1, m_1=-1\rangle|j_2=1, m_2=1\rangle + \sqrt{2}\,|j_1=1, m_1=0\rangle|j_2=1, m_2=0\rangle\right.$$

$$\left. + \sqrt{2}\,|j_1=1, m_1=0\rangle|j_2=1, m_2=0\rangle + \sqrt{2}\,|j_1=1, m_1=1\rangle|j_2=1, m_2=-1\rangle\right\} \quad (7)$$

Comparing (6) and (7) we get

$$\boxed{|j_1=1, j_2=1, J=2, M=0\rangle = \frac{1}{\sqrt{6}}\left\{|j_1=1, m_1=-1\rangle|j_2=1, m_2=1\rangle + 2|j_1=1, m_1=0\rangle|j_2=1, m_2=0\rangle + |j_1=1, m_1=1\rangle|j_2=1, m_2=-1\rangle\right\}} \quad (8)$$

Similarly you can show that

$$\boxed{|j_1=1, j_2=1, J=2, M=-1\rangle = \frac{1}{\sqrt{2}}\left\{|j_1=1, m_1=0\rangle|j_2=1, m_2=-1\rangle + |j_1=1, m_1=-1\rangle|j_2=1, m_2=0\rangle\right\}} \quad (9)$$

$$\boxed{|j_1=1, j_2=1, J=2, M=-2\rangle = |j_1=1, m_1=-1\rangle|j_2=1, m_2=-1\rangle} \quad (10)$$

- The next step is to find the states with $J=1$ and $M=1,0,-1$

We start from the state $|j_1=1, j_2=1, J=1, M=1\rangle$

This state could arise as a linear combination as follows:

$$|j_1=1, j_2=1, J=1, M=1\rangle = \alpha|j_1=1, m_1=1\rangle|j_2=1, m_2=0\rangle + b|j_1=1, m_1=0\rangle|j_2=1, m_2=1\rangle \quad (11)$$

The first thing is to apply normalization

$$\langle j_1=1, j_2=1, J=1, M=1|j_1=1, j_2=1, J=1, M=1\rangle = 1 \Longrightarrow \boxed{|\alpha|^2+|b|^2=1} \quad (12)$$

Then state (11) is orthogonal to previous states like the state (5)

$$\langle j_1=1, j_2=1, J=2, M=1|j_1=1, j_2=1, J=1, M=1\rangle = 0 \quad \Longrightarrow$$

$$\frac{1}{\sqrt{2}}\left\{ \langle j_1=1, m_1=0 | \langle j_2=1, m_2=1 | + \langle j_1=1, m_1=1 | \langle j_2=1, m_2=0 | \right\} \cdot$$

$$\cdot \left\{ \alpha |j_1=1, m_1=1\rangle |j_2=1, m_2=0\rangle + b |j_1=1, m_1=0\rangle |j_2=1, m_2=1\rangle \right\} = 0$$

$$\Rightarrow \alpha \langle j_1, m_1=1 | j_1=1, m_1=1\rangle \langle j_2=1, m_2=0 | j_2=1, m_2=0\rangle$$

$$+ b \langle j_1=1, m_1=0 | j_1=1, m_1=0\rangle \langle j_2=1, m_2=0 | j_2=1, m_2=0\rangle = 0$$

$$\Rightarrow \alpha + b = 1 \quad (13)$$

Solving (12), (13) we get $2|\alpha|^2 = 1 \Rightarrow \alpha = \pm 1/\sqrt{2}$ but we keep $\alpha = +\frac{1}{\sqrt{2}}$ (because it corresponds to a state with maximum m_1)!

Thus $b = -1/\sqrt{2}$

So from (11)

$$\boxed{|j_1=1, j_2=1, J=1, M=1\rangle = \frac{1}{\sqrt{2}}\left\{ |j_1=1, m_1=1\rangle |j_2=1, m_2=0\rangle - |j_1=1, m_1=0\rangle |j_2=1, m_2=1\rangle \right\}}$$

With the use of lowering operators we can find

$$\boxed{|j_1=1, j_2=1, J=1, M=0\rangle = \frac{1}{\sqrt{2}}\left\{|j_1=1, m_1=1\rangle\,|j_2=1, m_2=-1\rangle - |j_1=1, m_1=-1\rangle\,|j_2=1, m_2=1\rangle\right\}} \quad (14)$$

$$\boxed{|j_1=1, j_2=1, J=1, M=-1\rangle = \frac{1}{\sqrt{2}}\left\{|j_1=1, m_1=0\rangle\,|j_2=1, m_2=-1\rangle - |j_1=1, m_1=-1\rangle\,|j_2=1, m_2=0\rangle\right\}} \quad (15)$$

- The final step is the calculation of the state $J=0$, $M=0$

This state can be written as:

$$|j_i=1, j_2=1, J=0, M=0\rangle = \alpha\,|j_1=1, m_1=0\rangle\,|j_2=1, m_2=0\rangle + b\,|j_1=1, m_1=-1\rangle\,|j_2=1, m_2=1\rangle + c\,|j_1=1, m_1=1\rangle\,|j_2=1, m_2=-1\rangle \quad (16)$$

From normalization $|a|^2 + |b|^2 + |c|^2 = 1$ (17)

From orthogonality with other states we get:

$$\langle j_1=1, j_2=1, J=1, M=0\,|\,j_1=1, j_2=1, J=0, M=0\rangle = 0 \Rightarrow \boxed{b=c} \quad (18)$$

$$\langle j_1=1, j_2=1, J=2, M=0\,|\,j_1=1, j_2=1, J=0, M=0\rangle = 0 \Rightarrow \boxed{\alpha=-b} \quad (19)$$

We assign to c the positive value since it corresponds to maximum value of m_1 and we end up to

$$\boxed{|j_1=1, j_2=1, J=0, M=0\rangle = \frac{1}{\sqrt{3}}\Big\{|j_1=1, m_1=-1\rangle|j_2=1, m_2=1\rangle - |j_1=1, m_1=0\rangle|j_2=1, m_2=0\rangle + |j_1=1, m_1=1\rangle|j_2=1, m_2=-1\rangle\Big\}}$$

(20)

But what about our aim? What are the C-G coefficients?

If you recall $|j_1, j_2, J, M\rangle = \sum_{m_1, m_2} |j_1, m_1, j_2, m_2\rangle \underbrace{\langle j_1\, m_1\, j_2\, m_2 | J M\rangle}_{C-G}$

If you consider eq. 5 then

$$\langle 1\,0\,1\,1|2\,1\rangle = \langle j_1=1, m_1=0, j_2=1, m_2=1 | J=2, M=1\rangle =$$

$$= \langle j_1=1, m_1=0 | \langle j_2=1, m_2=1 | J=2, M=1\rangle =$$

$$= \langle j_1=1, m_1=0 | \langle j_2=1, m_2=1 | \frac{1}{\sqrt{2}} |j_1=1, m_1=0\rangle |j_2=1, m_2=1\rangle$$

$$+ \langle j_1=1, m_1=0 | \langle j_2=1, m_2=1 | \frac{1}{\sqrt{2}} \cancel{|j_1=1, m_1=0\rangle |j_2=1, m_2=0\rangle}_{0}$$

$$= \frac{1}{\sqrt{2}}$$

(7) When $l_1 = 1$, $l_2 = 1$ the possible values of the quantum number L of the total angular momentum are:

$$L = |l_1 - l_2|, \dots, |l_1 + l_2| = 0, \dots, 2 = 0, 1, 2$$

Similarly the quantum number S of the total spin is

$$S = |s_1 - s_2|, \dots, |s_1 + s_2| = 0, \dots, 1 = 0, 1$$

By combining now each value of L with each value of S

a) $L = 0$ and $S = 0 \Rightarrow J = 0$

b) $L = 0$ and $S = 1 \Rightarrow J = 1$

c) $L = 1$ and $S = 0 \Rightarrow J = 1$

d) $L = 1$ and $S = 1 \Rightarrow J = |1 - 1|, \dots |1 + 1| = 0, 1, 2$

e) $L = 2$ and $S = 0 \Rightarrow J = 2$

f) $L = 2$ and $S = 1 \Rightarrow J = |2 - 1|, \dots |2 + 1| = 1, 2, 3$

We see that we have states with same J. But we must be careful: these states are different.

Now as far as we concern the number of states before and after the composition:

Before: $(2l_1+1)(2l_2+1)(2s_1+1)(2s_2+1) = 3\cdot 3\cdot 2\cdot 2 = 36$

After: With $J=0$ we have 2

With $J=1$ we have 4 (but $m_j=-1,0,1$) so 12

With $J=2$ we have 3 (but $m_j=-2,-1,0,1,2$) so 15

With $J=3$ we have 1 (but $m_j=-3,-2,-1,0,1,2,3$) so 7

Total 36 states.

8 We know that $\underset{\sim}{j} = \underset{\sim}{\ell} + \underset{\sim}{s} \Rightarrow \underset{\sim}{j}^2 = (\underset{\sim}{\ell} + \underset{\sim}{s})^2 \Rightarrow$

$$\Rightarrow \underset{\sim}{j}^2 = \underset{\sim}{\ell}^2 + \underset{\sim}{s}^2 + 2\underset{\sim}{\ell}\cdot\underset{\sim}{s} \Rightarrow \underset{\sim}{\ell}\cdot\underset{\sim}{s} = \frac{1}{2}\left(\underset{\sim}{j}^2 - \underset{\sim}{\ell}^2 - \underset{\sim}{s}^2\right) \quad (1)$$

We see that $\underset{\sim}{\ell}\cdot\underset{\sim}{s}$ is just a function of $\underset{\sim}{j}^2, \underset{\sim}{\ell}^2, \underset{\sim}{s}^2$

These quantities can be measured simultaneously (remember that $[\underset{\sim}{j}, \underset{\sim}{\ell}] = [\underset{\sim}{\ell}, \underset{\sim}{s}] = [\underset{\sim}{j}, \underset{\sim}{s}] = 0$)

Thus from (1) we can write:

$$\underset{\sim}{\ell}\cdot\underset{\sim}{s}\Big|_{\text{eigenvalues}} = \frac{\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right] \quad (2)$$

(9) We know that for the total spin $\underset{\sim}{S} = \underset{\sim}{S_1} + \underset{\sim}{S_2}$, thus

$$\underset{\sim}{S}^2 = (\underset{\sim}{S_1} + \underset{\sim}{S_2})^2 \Rightarrow \underset{\sim}{S}^2 = \underset{\sim}{S_1}^2 + \underset{\sim}{S_2}^2 + 2\underset{\sim}{S_1}\cdot\underset{\sim}{S_2} \Rightarrow 2\underset{\sim}{S_1}\cdot\underset{\sim}{S_2} = \underset{\sim}{S}^2 - \underset{\sim}{S_1}^2 - \underset{\sim}{S_2}^2$$

$$\Rightarrow \underset{\sim}{S_1}\cdot\underset{\sim}{S_2} = \frac{1}{2}\left[\underset{\sim}{S}^2 - \underset{\sim}{S_1}^2 - \underset{\sim}{S_2}^2\right] \quad (1)$$

Using (1) the Hamiltonian of the problem becomes: $(H = A\,\underset{\sim}{S_1}\cdot\underset{\sim}{S_2})$

$$H = \frac{A}{2}\left(\underset{\sim}{S}^2 - \underset{\sim}{S_1}^2 - \underset{\sim}{S_2}^2\right) \quad (2).$$

The eigenvalues of (2) are:

$$E_S = \frac{A}{2}\Big(S(S+1) - S_1(S_1+1) - S_2(S_2+1)\Big) \quad (3)$$

In our case $S_1(S_1+1) = \frac{3}{2}\left(\frac{3}{2}+1\right) = \frac{15}{4}$ $\quad S_2(S_2+1) = 5/2\,(5/2+1) = \frac{35}{4}$

$$E_S = \frac{A}{2}\left(S(S+1) - \frac{15}{4} - \frac{35}{4}\right) = \frac{A}{2}\left(S(S+1) - \frac{25}{2}\right)$$

For the values of S we have: $S = \left|\frac{5}{2} - \frac{3}{2}\right| \dots \left|\frac{5}{2} + \frac{3}{2}\right| = 1, \dots, 4 = 1, 2, 3, 4$

$E_1 = \frac{A}{2}\left(1(1+1) - \frac{25}{2}\right) = -\frac{21A}{4}$, $\quad E_2 = \frac{A}{2}\left(2(2+1) - \frac{25}{2}\right) = -\frac{13}{4}A$

$E_3 = \frac{A}{2}\left(3(3+1) - \frac{25}{2}\right) = -\frac{A}{4}$, $\quad E_4 = \frac{A}{2}\left(4(4+1) - \frac{25}{2}\right) = \frac{15}{4}A$

Each of these energies has a degeneracy given by $d = 2S+1$

So $d_1 = 2\cdot1+1 = 3$, $d_2 = 2\cdot2+1 = 5$, $d_3 = 2\cdot3+1 = 7$, $d_4 = 2\cdot4+1 = 9$

3 states with energy E_1 , 5 states with E_2 , 7 states with E_3 , 9 states with E_4